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PREFACE

THE book completely covers the whole course in Algebra for the Matriculation examination of the Indian Universities. It has been divided into two parts: Part I for pre-matriculation, and Part II for the matriculation class. Each part is divided into three sections, each of which covers one term's work. At the end of each section Sectional Revision in the form of Test Papers has been added.

The elementary portions have been treated in the first three sections and the more advanced in the last three.

The main features of the book are:

(i) The subject is introduced in the form of *Literal Arithmetic*, with special emphasis on *generalisation*.

(ii) A separate and comprehensive chapter on *Directed Numbers* forms the basis of Algebra.

(iii) *Formulæ* and *Factors* have been exhaustively treated at various stages and their application in other rules is also worked out in a separate chapter.

(iv) *Equations* have been properly emphasised and they are treated at various stages, as (i) simple equations, (ii) fractional equations, (iii) simultaneous equations, (iv) exponential equations, (v) equations involving surds, and (vi) quadratic equations.

(v) The application of the method of *cross-multiplication* is fully worked out in (*a*) simultaneous equations, (*b*) elimination and (*c*) in finding the necessary relations to satisfy a given condition.

(vi) The fractional and negative *Indices* have been properly introduced and due attention has been paid to *Surds*.

(vii) An exhaustive chapter on *Elimination* has been added.

(viii) *Conditional Identities* with their application have been included.

(ix) In order to apply the principles of proportion to other rules, *Ratio* and *Proportion* have been treated quite early in Part II.

(x) There is a separate chapter on *linear, statistical* and *quadratic* graphs but graphic illustrations have been employed wherever necessary and useful.

(xi) One chapter on *Homogeneity, Symmetry, Cyclic Order* and *Indeterminate Co-efficients,* another on the *Remainder Theorem* with its application in factors and a third on *Conditional Identities, Harder Factors, Σ—notation, Fractions with denominators in cyclic order, etc.*, have been added for those students who possess special aptitude for Mathematics.

(xii) 100 selected questions covering the whole course have been given at the end as miscellaneous exercises.

*—marked articles, examples and exercises are meant for brighter students only and not for the whole class.

In preparing the Fifth Edition of this book an opportunity has been taken to thoroughly revise it, with a view to increase its practical utility as a text-book. It is hoped that this revised edition will meet with the same good appreciation from all its users as was accorded to the previous four editions.

Late Professor of Mathematics,
Central Training College, Lahore.

G. M. Vohra,
P.E.S. (Retd.)

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Part I [For Class IX]

Part II [For Class X]

PART I

CHAPTER I

LITERAL ARITHMETIC

1. Signs and symbols. In Arithmetic figures 0, 1, 2, 3, ... 9 are used to represent numbers and these figures can have only *one value*, whereas in Literal Arithmetic, in addition to figures, letters such as a, b, c . . . . . x, y, z are used, to which *any value* may be assigned. For instance, when we speak of 12 pencils, we have a particular number of pencils in our mind, but when we speak of x pencils, we may be thinking of 5 pencils, 7 pencils, or in fact *any number* of pencils.

NOTE. One particular letter has only one particular value in the same problem.

In Arithmetic as well as in Literal Arithmetic, the signs $+$, $-$, $\times$, $\div$, $=$, $>$, $<$, $\because$ and $\therefore$ have got the same meaning.

In Literal Arithmetic we use $\sim$ as the *sign of difference*, as distinguished from the sign of subtraction; for example $a \sim b$ means the difference between a and b, where a may or may not be greater than b.

In Arithmetic 3×4 is sometimes written as 3.4, so in Literal Arithmetic $a \times b$ is written as $a.b$; but in Literal Arithmetic this dot is generally omitted and $a \times b$ is written as ab. Similarly, $5 \times a \times b$ is written as $5ab$.

In Arithmetic the sign of multiplication cannot be omitted without changing the meaning; for example $3 \times 4 = 12$ or $10 + 2$ and $34 = 30 + 4$.

NOTE. The dot used as the sign of multiplication is written near the bottom, whereas the dot used as a decimal point is written near the top, as $3.4 = 3 \times 4$ and $3{\cdot}4 = 3 + \frac{4}{10}$.

Just as in Arithmetic $3 + 3 + 3 + 3 + 3$ is 5 times 3 or 5.3, so in Literal Arithmetic $a + a + a + a + a$ is 5 times a or $5.a$ or $5a$. Similarly, $ab + ab + ab + ab = 4.ab$ or $4ab$.

Definitions. In $5a$ and $4ab$ the numbers 5 and 4 are called the **numerical co-efficients** of a and ab respectively.

NOTE. (i) When the numerical co-efficient is 1, it is not expressed; thus in ab which is really $1 \times ab$, the numerical co-efficient is 1.

(ii) 0×4 means 4 times 0 and is equal to 0; similarly, $0 \times a$ means a times 0 and is equal to 0.

Again, 4×0 means 0 times 4 which is clearly equal to 0; similarly $a \times 0$ means 0 times a and is equal to 0.

EXERCISE 1. (Oral)

1. Find the total number of runs scored by Rashid, Hamid and Saeed :—

(i) If Rashid scores 12, Hamid 7 and Saeed r.
(ii) ,, ,, 12 ,, q ,, ,, r.
(iii) ,, ,, p ,, q ,, ,, r.

2. What is the total length of three lines L, M, and N:

(i) If L is 3″, M is 9″ and N is z″ ?
(ii) ,, L ,, 3″, M ,, y″ and N ,, z″ ?
(iii) ,, L ,, x″, M ,, y″ and N ,, z″ ?

3. Write in symbols :

(i) 5 added to x. (ii) x increased by 6.
(iii) x increased by y.
(iv) 5 added to x, and y added to the result.

4. Express in words :

(i) $x + 7$. (ii) $p + q$.
(iii) $x + 11 + y$. (iv) $a + b + c$.

5. Find the value of $x + y + z$:

(i) When $x = 3$, $y = 2$, $z = 4$.
(ii) When $x = \frac{2}{3}$, $y = \frac{3}{4}$, $z = \frac{1}{6}$.
(iii) When $x = \cdot 24$, $y = \cdot 16$, $z = \cdot 15$.
(iv) When $x = 0$, $y = 5$, $z = 13$.

6. How many rupees are left with me :
(i) If I spend Rs. 7 out of Rs. x ?
(ii) „ „ Rs. y „ „ Rs. x ?
(iii) „ „ Rs. y and then Rs. z out of Rs. x ?

7. Write in symbols :
(i) 9 subtracted from 15. (ii) 9 subtracted from m.
(iii) 11 diminished by x. (iv) a taken from b.
(v) The difference between m and n.
(vi) The sum of 11 and b diminished by c.
(vii) y subtracted from x and z added to the result.

8. Express in words :
(i) $p-7$. (ii) $m \sim n$.
(iii) $l+m-n$. (iv) $p-q+r$.
(v) $x-y-z$. (vi) $a+b-c-d$.

9. Find the value of $a+b-c-d$:
(i) When $a = 2$, $b = 7$, $c = 1$, $d = 3$.
(ii) When $a = \frac{1}{2}$, $b = \frac{3}{4}$, $c = \frac{1}{4}$, $d = \frac{1}{8}$.
(iii) When $a = \cdot 15$, $b = \cdot 8$, $c = \cdot 02$, $d = \cdot 46$.

10. Find the area of a rectangle :
(i) If its length is 14 ft. and breadth is 12 ft.
(ii) „ „ 14 ft. „ „ b ft.
(iii) „ „ l ft. „ „ b ft.

11. (i) How many feet are there in 6 yds. ?
(ii) „ „ „ „ „ x yds. ?
(iii) „ „ inches „ „ m ft. ?
(iv) „ „ annas „ „ p rupees ?
(v) „ „ centimetres „ „ n metres ?

12. If 1 horse costs Rs. r, what will n horses cost ?

13. If a person travels at the rate of m miles an hour, how far will he go in t hours ?

14. Write in symbols:
(i) 5 multiplied by x. (ii) m multiplied by 5.
(iii) The product of m and n.
(iv) 7 multiplied by a and the product by b.

15. Write down *briefly* the following and state the numerical co-efficient in each case:
(i) $x+x+x+x$. (ii) $a+a+a+a+a$.
(iii) $pq+pq+pq$. (iv) $abc+abc+abc+abc$.

16. Express in words:
(i) $8\times x$. (ii) xy. (iii) $3xy$.
(iv) $5xyz$. (v) $7abc$.

17. Distinguish between:
(i) 39, 3.9, 3·9. (ii) 47, 4×7, 4.7, 4·7. (iii) mn, $m\times n$, $m.n$.

18. Find the value of $5mn+3pq$:
(i) When $m=6$, $n=2$, $p=4$, $q=1$.
(ii) When $m=\cdot 7$, $n=\cdot 4$, $p=\cdot 5$, $q=\cdot 24$.
(iii) When $m=\frac{1}{2}$, $n=\frac{7}{10}$, $p=\frac{5}{8}$, $q=\frac{2}{3}$.
(iv) When $m=7$, $n=1$, $p=0$. $q=2$.

19. Write in symbols:
(i) 15 divided by 7. (ii) 15 divided by x.
(iii) m divided by 7. (iv) p divided by q.
(v) The sum of p and q divided by m.
(vi) The difference between a and b divided by c.

20. Express in words:
(i) $\dfrac{x}{y}$. (ii) $p\times r\div s\times q$. (iii) $\dfrac{5mn}{2n}$.
(iv) $\dfrac{3p+q}{5}$. (v) $\dfrac{7p+2q}{p-q}$.

21. (i) How many yards are there in $18x$ inches?
(ii) ,, feet ,, in $30ab$ inches?
(iii) ,, rupees ,, in $48m$ annas?
(iv) ,, metres ,, in $250x$ centimetres?

22. If the price of 7 chairs is Rs. $28m$, what is the price of 1 chair? The price of x chairs?

23. Find the value of $\dfrac{3ab+4cd}{2ad}$:

(i) If $a=2$, $b=5$, $c=1$, $d=4$.
(ii) If $a=\cdot4$, $b=\cdot5$, $c=\cdot16$, $d=2\cdot5$.
(iii) If $a=\frac{1}{3}$, $b=\frac{3}{4}$, $c=\frac{2}{3}$, $d=\frac{1}{2}$.

24. Write in symbols:

(i) Five added to 6 is equal to 11. [$6+5=11$]
(ii) Four and five make nine.
(iii) Two and two make four.
(iv) Four times three equals three times four.
(v) Twenty divided by five is equal to eighty divided by twenty.
(vi) Half of p is equal to q.
(vii) Since five times x is equal to forty, therefore x is equal to eight. [$\because\ 5x=40,\ \therefore\ x=8$.]
(viii) Since one-third of y is equal to fifteen, therefore y is equal to forty-five.
(ix) Since seven times m is greater than thirty-five, therefore m is greater than five.
(x) Since six times p is less than eighteen, therefore p is less than three.

25. Express in words:

(i) $x+7=16$. (ii) $x-11=5$.
(iii) $5x=30$. (iv) $\dfrac{2x}{9}=6$.
(v) $2a+3b=5$. (vi) $\dfrac{2a+3b}{7}=10$.
(vii) $\dfrac{3a-2b}{5}=12$. (viii) $\because\ 5x-3=12,\ \therefore x=3$.
(ix) $\because\ 7x-2>12,\ \therefore\ x>2$.
(x) $\because\ 2x+5<15,\ \therefore\ x<5$.

2. Definitions and Fundamental Laws. Just as in Arithmetic

5×5 is written briefly 5^2

$5 \times 5 \times 5$,, ,, ,, 5^3

$5 \times 5 \times 5 \times 5$,, ,, ,, 5^4, and so on ;

similarly, in Literal Arithmetic

$a \times a$ is written briefly a^2 { Read as '*a squared*' or '*the square of a*'.

$a \times a \times a$,, ,, ,, a^3 { Read as '*a cubed*' or '*the cube of a*'.

$a \times a \times a \times a$,, ,, ,, a^4 { Read as '*a to the fourth*' or '*the fourth power of a*'.

If in a product the same factor is repeated a certain number of times, as above, the *product* is called the **power** of that factor and the small figure placed above the factor to indicate *the number of times* it is repeated as a factor in that product is called its **index.**

Thus in 7^2, 6^4, a^5, b^3 the indices are, ^{2}, ^{4}, ^{5}, ^{3} respectively.

An **expression** is a collection of symbols—that is, of letters, figures and signs. Thus $3a$, $7a^2b$, $4a+5b$, $3a^2+2ab-b^2$ are all expressions.

The parts of an expression connected by the sign + or – are called the **terms** of the expression.

$3a$ is an expression of *one term* or a **monomial.**

$4a+5b$ is an expression of *two terms* or a **binomial.**

$3a^2+2ab-b^2$ is an expression of *three terms* or a **trinomial.**

Terms having the same letters are called **like terms** and terms having different letters are called **unlike terms,** *e.g.*, $3abc$ and $7abc$ are like terms, and $3abc$ and $7bcd$ are unlike terms.

An expression containing only one term is also called a **Simple Expression.**

An expression containing more than one term is called a **Compound Expression.**

Example 1. Distinguish between $3x$ and x^3.

$3x = x + x + x$ and $x^3 = x \times x \times x$.

Example 2. If $a = 4$, $b = 3$ and $c = 2$, find the value of:

(i) $5ab$. (ii) a^2b^3. (iii) $3a^b$. (iv) $4a^3 - 3b + 5^c$.

(i) $5ab = 5 \times a \times b = 5 \times 4 \times 3 = 60$.

(ii) $a^2b^3 = a \times a \times b \times b \times b = 4 \times 4 \times 3 \times 3 \times 3 = 432$.

(iii) $3a^b = 3 \times a^3 = 3 \times a \times a \times a = 3 \times 4 \times 4 \times 4 = 192$.

(iv) $4a^3 - 3b + 5^c = 4 \times a \times a \times a - 3 \times b + 5^2$

$= 4 \times 4 \times 4 \times 4 - 3 \times 3 + 5 \times 5$

$= 256 - 9 + 25 = 272$.

EXERCISE 2.

1. Write briefly:

(i) $7 \times 7 \times 7 \times 7$. (ii) $a \times a \times a \times a \times a$.

(iii) $a \times p \times p \times p$. (iv) $1 \times a \times b \times b$.

(v) $\cdot 01 \times \cdot 01 \times \cdot 01$. (vi) $\frac{1}{a} \times \frac{1}{a} \times \frac{1}{a} \times \frac{1}{a}$

(vii) $3 \times m \times m + 4 \times n \times n$. (viii) $a \times a \times n \times n \times n$.

(ix) $pppp + ppp$. (x) $a \times x \times x \times x + b \times y \times y \times y$.

2. Read the following:

(i) p^3. (ii) x^4. (iii) r^6. (iv) n^{15}

3. State the value of:

(i) 5^4. (ii) 7^2. (iii) 2^5. (iv) 3^4.

4. Distinguish between:

(i) $5a$ and a^5. (ii) p^4 and $4p$.

(iii) Three times 5 and 5 cubed.

5. If $a = 3$, $b = 5$, $c = 2$, $x = 1$, $y = 4$, find the value of:

(i) $2a^3 + 3b^2$. (ii) $5a^4 - 2b^2 + 4x^3$.

(iii) $11a^3 + 6b^3 + 2x^c + y^3$. (iv) $2ab^3 + 3a^2c^2 + 4xy^3$.

(v) $a^c + b^x$. (vi) $y^a + x^b + c^x$.

6. Find the value of $x^3+x^2y+xy^2+y^3$ from the values of x and y given in the table :

x	3	2	4	1	4
y	1	3	2	3	1

Just as in Arithmetic, we have

(i) $4\times3=3\times4$,

(ii) $5\times4\times3=5\times3\times4=4\times5\times3=4\times3\times5$
$=3\times5\times4=3\times4\times5$, and so on;

similarly, in Literal Arithmetic, we have

(i) $a\times b=b\times a$,

(ii) $a\times b\times c=a\times c\times b=b\times c\times a=b\times a\times c$
$=c\times a\times b=c\times b\times a$,

or $abc=acb=bca=bac=cab=cba$.

Example 3. Simplify $3a\times5b$.

$$3a\times5b=3\times a\times5\times b$$
$$=3\times5\times a\times b$$
$$=15ab.$$

First Law of Indices.

(i) $\because$ $a^3=a\times a\times a$ and $a^2=a\times a$,

$\therefore$ $a^3\times a^2=a\times a\times a\times a\times a$
$=a^5=a^{3+2}$.

(ii) $\because$ $a^3=a\times a\times a$ and $a^4=a\times a\times a\times a$

$\therefore$ $a^3\times a^4=a\times a\times a\times a\times a\times a\times a$
$=a^7=a^{3+4}$.

Similarly, $a^4 \times a^5 = a^{4+5}$
and $a^5 \times a^6 = a^{5+6}$
or in general

$$\boldsymbol{a^m \times a^n = a^{m+n}},$$

where m and n are *ordinary arithmetical integers*.

Law. *The index of the product of two factors which are powers of the same quantity is the sum of their indices.*

Example 4. Multiply $5a^3b^2 \times 3a^2b^4$.

$$\begin{aligned} 5a^3b^2 \times 3a^2b^4 &= 5 \times a^3 \times b^2 \times 3 \times a^2 \times b^4 \\ &= 5 \times 3 \times a^3 \times a^2 \times b^2 \times b^4 \\ &= 15a^{3+2}b^{2+4} \\ &= 15a^5b^6. \end{aligned}$$

7. Prove that:
   (i) $3^4 \times 3^2 = 3^6$. (ii) $6^4 \times 6^7 = 6^{11}$.

8. Write down the values of the following:
   (i) $5^3 \times 5^2 \times 5^2$. (ii) $7^2 \times 7^3 \times 7$.
   (iii) $a^4 \times a^2 \times a^7$. (iv) $x^6 \times x^3 \times x^4$.

9. Multiply:
   (i) $2ab$ by $3cd$. (ii) $6x^3y$ by $2xy^2$.
   (iii) a^2bc by ab^2c. (iv) $a^2b \times ab \times ac^2$.
   (v) $3a^2b^3c^2$ by $5a^3b^4c$. (vi) $2p^3q \times pq^2 \times 3pq$.
   (vii) $4m^2n \times 3mn^3 \times 5mnp^3 \times n^3p^2$.

Just as in Arithmetic, we have

$$\frac{12}{4} = \frac{4 \times 3}{4} = 3$$

so in Literal Arithmetic, we have

$$\frac{ab}{a} = \frac{a \times b}{a} = b.$$

Example 5. Simplify $\dfrac{12xy}{4y}$.

$$\frac{12xy}{4y} = \frac{12 \times x \times y}{4 \times y} = 3x.$$

Second Law of Indices.

(i) $a^5 \div a^3 = \dfrac{a^5}{a^3} = \dfrac{a \times a \times a \times a \times a}{a \times a \times a} = a^2 = a^{5-3}$.

(ii) $a^7 \div a^4 = \dfrac{a^7}{a^4} = \dfrac{a \times a \times a \times a \times a \times a \times a}{a \times a \times a \times a} = a^3 = a^{7-4}$.

Similarly, $a^9 \div a^5 = a^{9-5}$ and $a^{11} \div a^6 = a^{11-6}$,
or in general $\boldsymbol{a^m \div a^n = a^{m-n}}$,
where m and n are *ordinary arithmetical integers and* $m > n$.

Law. *When a power of a quantity is divided by a lower power of the same quantity, the index of the quotient is equal to the index of the dividend diminished by the index of the divisor.*

Example 6. Simplify $\dfrac{18x^5y^3}{6x^4y}$.

$$\frac{18x^5y^3}{6x^4y} = \frac{18 \times x^5 \times y^3}{6 \times x^4 \times y}$$
$$= 3x^{5-4}y^{3-1}$$
$$= 3xy^2.$$

10. Prove that:

(i) $5^7 \div 5^3 = 5^{7-3}$. (ii) $7^{12} \div 7^4 = 7^{12-4}$.

11. Write down the values of:

(i) $a^{17} \div a^{12}$. (ii) $p^{24} \div p^{15}$.
(iii) $m^{13} \div m^7$. (iv) $x^{24} \div x^{12}$.

12. Divide:

(i) $12x^4y^6$ by $6x^2y^2$. (ii) $10m^7n^5$ by $2m^4n^4$.
(iii) $8p^5q^4r^3$ by $4p^3q^2r^3$. (iv) $4m^6n^3p^5$ by $2m^2n^3p^4$.
(v) $15l^6m^2n^9$ by $5l^4m^2n^2$.
(vi) $24x^5y^6z^3w^4$ by $4w^2x^2y^2z$.

Third Law of Indices.

(i) $(a^3)^2 = a^3 \times a^3 = a^6 = a^{3\times 2}$.

(ii) $(a^2)^3 = a^2 \times a^2 \times a^2 = a^6 = a^{2\times 3}$.

(iii) $(a^2)^5 = a^2 \times a^2 \times a^2 \times a^2 \times a^2 = a^{10} = a^{2\times 5}$.

From these and similar examples it follows that

$$(a^m)^r = a^{mr},$$

where m and r are *ordinary arithmetical integers.*

Law. *When a quantity, raised to a power, is again raised to a certain power, the index of the result is the product of the indices.*

13. Prove that:

(i) $(7^2)^5 = 7^{10}$. (ii) $(8^3)^4 = 8^{12}$.

(iii) $(a^3)^3 = a^9$. (iv) $(x^4)^4 = x^{16}$.

14. Write down the values of the following:

(i) $(2^3)^2$. (ii) $(3^2)^3$. (iii) $(a^m)^3$.

(iv) $(x^3)^m$. (v) $(p^4)^n$. (vi) $(y^3)^5$.

$\sqrt[2]{a}$ or $\sqrt{a}$ is read as *the square root of a.*

$\sqrt[3]{a}$ is read as *the cube root of a.*

$\sqrt[4]{a}$ is read as *the fourth root of a.*

Example 7. Prove that:

(i) $\sqrt{a^{12}} = a^6$. (ii) $\sqrt[3]{a^{12}} = a^4$. (iii) $\sqrt[4]{a^{12}} = a^3$.

(i) Since $a^{12} = a^6 \times a^6 = (a^6)^2$,

$$\therefore \quad \sqrt{a^{12}} = \sqrt{(a^6)^2} = a^6.$$

(ii) Since $a^{12} = a^4 \times a^4 \times a^4 = (a^4)^3$,

$$\therefore \quad \sqrt[3]{a^{12}} = \sqrt[3]{(a^4)^3} = a^4.$$

(iii) Since $a^{12} = a^3 \times a^3 \times a^3 \times a^3 = (a^3)^4$,

$$\therefore \quad \sqrt[4]{a^{12}} = \sqrt[4]{(a^3)^4} = a^3.$$

15. Prove that:

(i) $\sqrt{10^4}=10^2$. (ii) $\sqrt{x^{10}}=x^5$.

(iii) $\sqrt{9a^2b^2}=3ab$. (iv) $\sqrt{36^3}=6^3$.

(v) $\sqrt[3]{a^6}=a^2$. (vi) $\sqrt[3]{8x^3y^3}=2xy$.

(vii) $\sqrt[4]{16}=2$. (viii) $\sqrt[4]{81}=3$.

16. Write down the square root of :

(i) 5^4. (ii) 4^6. (iii) x^{12}.

(iv) $4x^6$. (v) $9y^8$. (vi) $25a^2b^4$.

17. Write down the cube root of :

(i) $64a^9$. (ii) $216x^{12}$. (iii) $125y^{27}$.

18. Write down the fourth root of :

(i) $16a^8$. (ii) $81b^{12}$. (iii) $256c^{20}$.

19. If $y=3x^2+5x$ and $x=2$, find the value of y.

20. If x is equal to the square of y, $y=3z$ and $z=2$, what is the value of xyz?

3. **Addition and Subtraction.** Just as in Arithmetic,

$8\times15+5\times15+7\times15=(8+5+7)\times15$ or 20×15 and

$8\times15-5\times15=(8-5)\times15=3\times15$,

similarly, in Literal Arithmetic,

$8\times a+5\times a+7\times a=(8+5+7)\times a$ or $20a$

and $8\times a-5\times a\quad=(8-5)\times a$ or $3a$.

EXERCISE 3.

Write down the value of :

1. $4a^2+a^2+6a^2$.
2. $3ab+4ab+2ab$.
3. $p+p+p+p+\ldots$ to 15 terms.
4. $9xy+7xy+12xy$.
5. $13ab-7ab$.
6. $17abc-12abc$.
7. $9a^2-4a^2$.
8. $15x^2-11x^2$.
9. (i) $7a+5a$.
   (ii) $7\times13+5\times13$.
10. (i) $8x-2x$.
    (ii) $8\times5-2\times5$.
11. (i) $13x+x$.
    (ii) $13\times4+4$.
12. (i) $15y-y$.
    (ii) $15\times7-7$.

Example 1. Simplify $3a+4a+b+5b$.

Since $3a+4a=7a$ and $b+5b=6b$,

$\therefore \quad 3a+4a+b+5b=7a+6b$.

Example 2. Simplify $5x-2x+3y-y+1$.

Taking the like terms, we have

$5x-2x=3x$ and $3y-y=2y$,

$\therefore \quad 5x-2x+3y-y+1=3x+2y+1$

Example 3. Simplify $4xy+3x+2y-x-y$.

Taking the like terms together, we have

$$4xy+3x+2y-x-y=4xy+\underline{3x-x}+\underline{2y-y}=4xy+2x+y.$$

Example 4. Simplify $5a^2+a+7+3a-2a^2-6$.

Taking the like terms together, we have

$$5a^2+a+7+3a-2a^2-6=\underline{5a^2-2a^2}+\underline{a+3a}+\underline{7-6}$$
$$=3a^2+4a+1.$$

Simplify :

13. $a+a+b+b$.
14. $x+x+y+y+y$.
15. $a+b+a+b$.
16. $x+y+x+y+x$.
17. $x+5+x+2+x$.
18. $x+y+3+x+1$.
19. $a+a+b-a$.
20. $5x+7y-2x+4$.
21. $a+b-c+a-b+c$.
22. $5a+b-2c+b-2a$.
23. $4xy+3xz-2xy$.
24. a^2+5a^2-3a.
25. $x^3+2x^2+4x-x^2$.
26. $p^2-3p+2+p^2+4p+1$.
27. $a^4+11a^2+1-7a^2$.
28. $9-a^2+4a-a^2-3a+1$.
29. $a^3+a+a^2+1+a+a^2$.
30. $7x^3+2x^2+2x+4-5x^2-4x$.

An expression such as $3x^5+2x+7x^3+x^2+5x^4+1$ cannot be simplified, for all its terms are unlike, but it is useful to arrange its terms, either (i) in *descending powers* of x, *i.e.*, beginning with the highest power of x, then the next highest, and so on; for example, $3x^5+5x^4+7x^3+x^2+2x+1$, or (ii) in *ascending powers* of x, *i.e.*, beginning with the term independent of x, then the x term, then the x^2 term, and so on; for example, $1+2x+x^2+7x^3+5x^4+3x^5$.

31. Arrange $4+5x^2+x^3+7x^4+3x$ in (i) descending powers of x, (ii) ascending powers of x.

32. Arrange $a^2+a^7+5a^3+1+4a^6$ in (i) descending powers of a, (ii) ascending powers of a.

33. Simplify $x^4+3x^2+4x+5-x+2x^3-x^2$ and arrange the answer in ascending powers of x.

34. Simplify $8+x^2+3x^4+2x^2+7x-4x^3+5x^2-3x+1$ and arrange the answer in descending powers of x.

35. The number 435 can be written $4.10^2+3.10+5$. Write similarly 374 and in the answer substitute x for 10 and x^2 for 10^2 or 100.

Can you simplify $3x^2+7x+4$?

36. The number 3048 can be written $3.10^3+4.10+8$. Write similarly 5704 and in the answer substitute x^2 for 10^2 and x^3 for 10^3.

Can you simplify $5x^3+7x^2+4$?

4. Highest Common Factor, Lowest Common Multiple, Fractions.

Of the two expressions $6a^3b^2$, $8a^2b^3$, there are several common factors, viz., 2, $2a$, $2a^2$, $2b$, $2b^2$, $2ab$ and $2a^2b^2$. But $2a^2b^2$ is the highest of all these factors in power and contains in it every other common factor.

$2a^2b^2$ is called the **Highest Common Factor** (H. C. F.) of $6a^3b^2$ and $8a^2b^3$.

The **Lowest Common Multiple** (L. C. M.) of $6a^3b^2$ and $8a^2b^3$ is the lowest expression which is exactly divisible by $6a^3b^2$ as well as $8a^2b^3$; therefore it must contain the L.C.M. of 6 and 8, *i.e.*, 24 as a factor and also it must be divisible by a^3b^2 and a^2b^3, *i.e.*, it must contain a^3b^9 as a factor, for it is the lowest in power of all the common multiples like a^3b^4, a^4b^3, a^4b^4 &c. Hence $24a^3b^3$ is the L.C.M. of the two given expressions.

Example 1. Find (i) the H.C.F. and (ii) the L.C.M. of $9x^2y^3$ and $12xy^2z$.

(i) 3 is the H.C.F. of the co-efficients; of the powers of x, *i.e.*, x^2 and x, we take the common factor, x; of the powers of y, *i.e.*, y^3 and y^2, we take the common factor, y^2; we neglect z, as it is not a factor of $9x^2y^3$.

Multiplying the common factors we have taken, *i.e.*, 3, x and y^2, we get the H.C.F. $3xy^2$.

(ii) 36 is the L.C.M. of 9 and 12; of the powers of x we take the highest x^2; of the powers of y we take the highest y^3 and of the powers of z, the highest z. Multiplying these together we get the L.C.M. $36x^2y^3z$.

EXERCISE 4.

1. Of the expressions $3a$, $6ab$, $4a^2$, $2ab$, $12a^3b^3$, $12a^2b^2$, $6a^2b$, $16a^4$, ab, $3a^2b$, $2ac$, $6a^2c$, $6abc^2$, which are factors of: (*i*) $6a^2$, (*ii*) factors of $3ab$, (*iii*) common factors of $6ab$ and $4a^2$, (*iv*) multiples of $4a^2$, (*v*) multiples of $3ab$ (*vi*) common multiples of $2a^2$ and $3ab$? (*Oral*).

2. Find the H.C.F. and the L.C.M. of:

   (i) $4a^2b^2$, $6a^3b^3$. (ii) $12a^3b$, $18ab^3$.
   (iii) $12ab$, $3a^2b$. (iv) $39a^3b^4$, $52a^2b^5$.
   (v) $4a^2b$, $6a^3b^3$, $3abc^2$. (vi) $6ab^2$, $8a^2b^2$, $12a^2bc$, $18abc^2$,

3. Find by factors the H.C.F. and the L.C.M. of:

   (i) $2^3.7^2$, $7^3.2$. (ii) $3^2.5^3$, $3^3.5^2$.
   (iii) 864, 720. (iv) 1050, 840.

Example 2. Simplify $\dfrac{6x^3y^2}{4xy^3z}$.

The H.C.F. of the numerator and the denominator is $2xy^2$. Divide the numerator and the denominator by $2xy^2$.

$$\therefore \frac{6x^3y^2}{4xy^3z}=\frac{3x^2}{2yz}.$$

Example 3. Express $\frac{4x}{3y}$ as an equivalent fraction with denominator $6ay$.

If $6ay$ be divided by $3y$, the quotient is $2a$.

Multiplying the numerator and denominator by $2a$, we have

$$\frac{4x}{3y}=\frac{2a\times4x}{2a\times3y}=\frac{8ax}{6ay}.$$

Example 4. Express $\frac{2a^2x^2}{3b^2y^2}$ as an equivalent fraction with numerator $12a^2x^2z^2$.

If $12a^2x^2z^2$ be divided by $2a^2x^2$, the quotient is $6z^2$. Multiplying the numerator and the denominator by $6z^2$, we have

$$\frac{2a^2x^2}{3b^2y^2}=\frac{6z^2\times2a^2x^2}{6z^2\times3b^2y^2}=\frac{12a^2x^2z^2}{18b^2y^2z^2}.$$

Example 5. Reduce $\frac{5}{3x}$ and $\frac{7}{4x}$ to fractions having the least common denominator.

The L.C.M. of $3x$ and $4x=12x$.

$$\therefore\frac{5}{3x}=\frac{4\times5}{4\times3x}=\frac{20}{12x}\text{ and }\frac{7}{4x}=\frac{3\times7}{3\times4x}=\frac{21}{12x}.$$

Example 6. (i) Add $\frac{3}{4x}$ and $\frac{1}{10x}$.

(ii) Subtract $\frac{1}{10x}$ from $\frac{3}{4x}$.

The L.C.M. of the denominators is $20x$.

Expressing each fraction with denominator $20x$, we have

(i) $$\frac{3}{4x}+\frac{1}{10x}=\frac{15}{20x}+\frac{2}{20x}$$
$$=\frac{15+2}{20x}=\frac{17}{20x}.$$

(ii) $$\frac{3}{4x}-\frac{1}{10x}=\frac{15}{20x}-\frac{2}{20x}$$
$$=\frac{15-2}{20x}=\frac{13}{20x}.$$

4. Simplify:

(i) $\frac{6ab^3c^4}{10a^2b^2c^3}$. (ii) $\frac{15a^2x^2y}{5axy^2}$. (iii) $\frac{48a^2x^2y^5}{12a^2x^3y^4}$.

(iv) $\frac{13a^2b^2c^4}{39a^3b^3c^5}$. (v) $\frac{51x^5y^3z^2}{17x^4y^2z}$. (vi) $\frac{21a^7x^7y^3}{7a^8x^6y^5}$.

5. Fill in the gaps in the following :—

(i) $\frac{5x}{2y}=\frac{\quad}{14y^2}$. (ii) $\frac{4a^2b}{6ab^2}=\frac{\quad}{18a^2b^2}=\frac{\quad}{3b}$.

(iii) $a^3=\frac{\quad}{b^2}$. (iv) $\frac{1}{a}=\frac{b^2}{\quad}$. (v) $\frac{m^2}{a}=\frac{m^3n^3}{\quad}$.

6. Reduce the following to fractions having the least common denominator:

(i) $\frac{5}{6a}, \frac{4}{3x}, \frac{3c}{ax}$. (ii) $\frac{3}{4x}, \frac{x}{6y^2}, \frac{y}{3x^2}$.

(iii) $\frac{x^2}{2ab}, \frac{y^2}{3ac}, \frac{z^2}{4bc}$. (iv) $\frac{x^2y}{a^3}, \frac{2x^3}{3a^2b}, \frac{3y^3}{4ab^2}, \frac{4xy^2}{5b^3}$.

7. Simplify the following fractions:

(i) $\frac{2x}{3}+\frac{3x}{4}$. (ii) $\frac{3a}{2x}+\frac{b}{x}$. (iii) $\frac{1}{2x}+\frac{1}{3x}$.

(iv) $\frac{x}{y}+\frac{y}{x}$. (v) $1+\frac{x}{y}$. (vi) $\frac{1}{x}+3$.

(vii) $1-\frac{1}{3x}$. (viii) $\frac{a}{b}-\frac{2a}{3b}$. (ix) $1-\frac{a}{b}$.

(x) $1-\frac{1}{xy}$. (xi) $\frac{2}{xy}-\frac{xy}{z}$. (xii) $\frac{x}{2}+\frac{x}{3}+\frac{x}{4}$.

(xiii) $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}$. (xiv) $\frac{2x}{3}-\frac{x}{4}+\frac{3x}{5}$. (xv) $\frac{a}{2x}+\frac{a}{3x}+\frac{a}{4x}$.

(xvi) $\frac{2a}{3x}+\frac{5b}{12y}+\frac{3c}{4z}$. (xvii) $\frac{x^2}{12}-\frac{3xy}{15}+\frac{2y^2}{18}$.

(xviii) $\frac{a}{bc}+\frac{b}{ca}+\frac{c}{ab}$. (xix) $\frac{b-c}{bc}+\frac{c-a}{ca}+\frac{a-b}{ab}$.

8. Simplify the following:

(i) $\frac{a^2}{bc}\times\frac{b^2}{ac}$. (ii) $xz\div\frac{x}{z}$. (iii) $\frac{a}{b}\times\frac{b}{c}\times\frac{c}{a}$.

(iv) $\frac{a^2b^2}{c}\times\frac{c^2}{ab}$. (v) $\frac{12bc}{25a^2}\times\frac{21ab^2}{30b^3}$. (vi) $\frac{36x^2y^2}{15z^3}\div\frac{27xy^2}{10z^2}$.

9. What fraction is $4m$ inches of m feet ?

10. If in the fraction $\frac{4x}{5x}$ different values are given to x, is its value changed ?

11. ABC is a straight line, its part AB measures x units and BC measures y units. What fraction is (i) AB of AC, (ii) AC of BC ?

5. Generalisation, Substitution, Problems.

It is important to remember that the *letters* stand for *numbers* and not for the *number of things*. Thus it is wrong to say 'Let l stand for the length of a rectangle'; l can stand for the number of linear units in the length and not for the length itself.

The term *quantity* means *number of things*. Thus x *rupees* is a quantity, but x is a number. When a quantity is to be indicated, the unit must be stated.

EXERCISE 5.

Criticise the following statements :

1. If one chair costs x, then four chairs cost $4x$.

2. If the length of a rectangle is l ft. and breadth b ft., its area is lb.

3. The price of silk per yd. is x.

4. The temperature of a place was x and then it rose 5 degrees; thus it was then $x+5$.

5. Let a chair cost x and a table y, then since the price of the table is Rs. 5 more than that of the chair, $\therefore\ y-x$ = Rs. 5.

Example 1. Suppose we know that

the sum of the first 2 natural numbers $= \dfrac{2(1+2)}{2}$;

,, ,, 3 ,, $= \dfrac{3(1+3)}{2}$;

,, ,, 4 ,, $= \dfrac{4(1+4)}{2}$;

and so on.

From the similarity of the form of these expressions, we can deduce a general form:

$$\textit{the sum of the first } n \textit{ natural numbers} = \frac{n(1+n)}{2},$$

which includes all the above three statements.

The process by which we arrive at the general form is called **generalisation.**

Generalise:

6. 5 exceeds 3 by $5-3$
   7 ,, 4 by $7-4$
   9 ,, 5 by $9-5$.

7. $4 + 5 = 5 + 4$
   $7 + 3 = 3 + 7$
   $9 + 4 = 4 + 9$.

8. $5 \times 4 = 4 \times 5$
   $7 \times 9 = 9 \times 7$
   $12 \times 8 = 8 \times 12$.

9. $7^2 \times 7 = 7^3$
   $9^2 \times 9 = 9^3$
   $11^2 \times 11 = 11^3$.

10. $9 \times \frac{1}{9} = 1$
    $7 \times \frac{1}{7} = 1$
    $12 \times \frac{1}{12} = 1$.

11. $0 \times 3 = 0$
    $0 \times 4 = 0$
    $0 \times 5 = 0$.

12. $3 \times 0 = 0$
    $4 \times 0 = 0$
    $5 \times 0 = 0$

13. 2×1 is an even number
    2×2 ,, ,,
    2×3 ,, ,,

14. $2 \times 1 + 1$ is an odd number
    $2 \times 2 + 1$,, ,,
    $2 \times 3 + 1$,, ,,

15. The sum of the first 2 odd numbers is 2^2
,, ,, 3 ,, ,, ,, 3^2
,, ,, 4 ,, ,, ,, 4^2

16. 4 times $6+6=5$ times 6
4 times $7+7=5$ times 7
4 times $9+9=5$ times 9.

17. 5% of a number is $\frac{5}{100}$ of that number
7% ,, ,, ,, $\frac{7}{100}$,, ,,
9% ,, ,, ,, $\frac{9}{100}$,, ,,

18. In a rectangle, if
length $=16$ ft., breadth $=12$ ft., area $=16\times12$ sq. ft.
,, $=24$ ft., ,, $=18$ ft., ,, $=24\times18$,, ,,
,, $=15$ ft., ,, $=14$ ft., ,, $=15\times14$,, ,,

19. $9-1$, 9, $9+1$ are three consecutive numbers
$7-1$, 7, $7+1$,, ,, ,, ,,
$28-1$, 28, $28+1$,, ,, ,, ,,

20. One number is p. (*a*) Give the next higher consecutive number. (*b*) Give the next lower consecutive number.

21. Out of three consecutive numbers the middle one is m; give the other numbers.

Substitution. It is extremely useful to derive a few particular cases from a general statement by substituting the given numerical values for the letters, as illustrated in the next example.

Example 2. If a rectangular room is l ft. long, b ft. broad and h ft. high, the area of its four walls is $2(l+b)\times h$ sq. ft. Derive from this statement the area of the four walls of a room, 16 ft. long, 14 ft. wide and 12 ft. high.

Since in this case $l=16$, $b=14$ and $h=12$,

$\therefore$ the area of the four walls

$$= 2\ (16+14)\times 12 \text{ sq. ft.}$$
$$= 2\times 30\times 12 \text{ sq. ft.}$$
$$= 720 \text{ sq. ft.}$$

22. If the parallel sides of a trapezium are a and b and its altitude is h, the area$=\frac{1}{2}\ (a+b)\times h$. Derive from this the area of another trapezium whose parallel sides are 18 ft. and 12 ft. and the altitude is 8 ft.

23. From the general statement "area of a circle$=\pi r^2$", where r stands for the radius of the circle, derive the area of a circle whose radius is 15 ft.

24. If the interest on a certain sum p @ the rate r% per annum for t years is $\frac{p\times r\times t}{100}$, find the interest on Rs. 450 @ 6% per annum for 3 years.

25. If $a+1$ is the square root of a^2+2a+1, show that it is true when $a=5, 7, 9, 11$.

26. The temperature of a place is x degrees Centigrade or $\frac{9x}{5}+32$ degrees Fahrenheit. What will be its temperature in Fahrenheit, when it is 40 degrees Centigrade ?

27. If a figure has n sides, the sum of its interior angles is $(2n-4)$ right angles. What is the sum of the angles of (i) a figure of 5 sides, (ii) a figure of 8 sides, (iii) a figure of 10 sides and (iv) a figure of 12 sides ?

28. If a figure is regular and has n sides, each of its interior angles is $\frac{2n-4}{n}$ right angles. What is the magnitude of an interior angle of (i) a six-sided regular figure, (ii) an eight sided regular figure ?

Hint. If you fail to solve a literal problem, frame for yourself a similar problem using numbers for letters and solve it ; then do the similar process for the given problem, as illustrated further in examples 3 and 4.

Example 3. The telegraph posts are fixed d ft. apart along a railway line and a train passes n poles per minute. Find the speed of the train per hour.

Let us frame a similar problem, putting numbers instead of the letters d and n.

The telegraph posts are fixed 220 ft. apart along a railway line and a train passes 9 poles per minute. Find the speed of the train per hour.

In one minute the train passes 9 poles or covers $(9-1)$ or 8 distances between two consecutive poles.

$\therefore$ in 1 minute it goes 220×8 ft.

$\therefore$,, 60 minutes ,, $220\times 8\times 60$ ft.

$\therefore$,, 1 hour ,, $\dfrac{220\times 8\times 60}{3\times 1760}$ miles.

Now we can work out the original problem:

In one minute the train passes n poles or covers $(n-1)$ distances between two consecutive poles.

$\therefore$ in 1 minute it goes $d\times(n-1)$ ft.

$\therefore$,, 60 minutes ,, $d\times(n-1)\times 60$ ft.

$\therefore$,, 1 hour ,, $\dfrac{d\times(n-1)\times 60}{3\times 1760}$ miles

$\therefore$,, 1 ,, ,, $\dfrac{d(n-1)}{88}$ miles.

Example 4. A cow is bought for x rupees and sold at a profit for y rupees; find the profit per cent.

Let us take a similar problem in Arithmetic. A cow is bought for Rs. 57 and sold for Rs. 72; find the profit per cent.

The profit is Rs. $(72-57)$,

The cost price is Rs. 57,

$\therefore$ the profit is $\left[\dfrac{(72-57)}{57}\times 100\right]\%$

Now in the original problem,

the profit is	Rs. $(y-x)$,
the cost price is	Rs x,
$\therefore$ the profit is	$\left[\frac{(y-x)}{x}\times 100\right]\%$
i.e.	$\frac{100(y-x)}{x}\%$.

29. Reduce Rs. x, y annas to annas.

30. How many hours does it take to walk:

(i) x miles at the rate of 5 miles an hour ?

(ii) x miles at the rate of y miles an hour ?

31. In t hours a man walks m miles:

(i) At what rate does he walk in miles per hour ?

(ii) How many yards does he walk in 15 minutes ?

32. A train runs with the speed of m miles per hour; what is its speed in yards per minute ?

33. A clock loses s seconds in 24 hours; in how many days will it be x minutes too slow?

34. If 1*s*.$=$10 as., how many shillings can we have for Rs. 25 ? for Rs. x ?

35. If three-fourths of the distance between Delhi and Simla is x miles, what is the distance between the two towns ?

36. If a yds. of silk cost Rs. 15; how much of it can be had for Rs. x?

37. For x miles the railway fare is Rs. y; how much will a journey of

(i) $8x$ miles, (ii) m miles,

cost at the same rate ?

38. If x men can do a piece of work in 14 days, how many days will $6x$ men take to do it ? How long will y men take to do it ?

39. A garrison has provisions for x men for y days. If m more men come in, how long will these last ?

40. If 15 srs. of sugar cost Rs. 4 as. 10, how much of it can be bought for Rs. x as. y ?

41. A merchant bought sugar worth Rs. m and sold it at a profit for Rs. n. Find his profit per rupee.

42. A motor car runs l miles in a hours on the first day, m miles in b hours on the second day and n miles in c hours on the third day. Find its average speed per hour.

43. What is the result of increasing x by 16 per cent. ?

44. How much per cent. is x of y ?

45. Express as a percentage (i) $\frac{1}{a}$, (ii) $\frac{a}{b}$.

46. Find the simple interest on Rs. 300 at r% per annum for 5 years.

47. Find the simple interest on Rs. p at r% per annum for 5 years.

48. Find the simple interest on Rs. p at r% per annum for t years.

49. If x% of a property is spent every year, how long will the whole property last ?

50. If x seeds are sown and 84% of them take root, how many of them are wasted ?

51. Sugar is bought at a as. per seer and sold at a profit of $12\frac{1}{2}$%. Find the selling price of 6 seers.

52. The price of cycles is reduced by 5%. If the old price of a cycle were r rupees, what is its new price ?

53. A bookseller allows x% commission on the published price of his books sold for cash. How many annas in a rupee is this ?

54. The price of wheat has fallen $p\%$ in two years. It is now selling @ 32 seers per rupee. What was its rate two years ago ?

55. A rectangular path is l yds. long and 5 ft. wide; find its area in square feet.

56. Find the cost in rupees of a carpet for a room x ft. long and y ft. wide @ 4 as. per sq. ft.

57. The area of a wall is A sq. ft. and its height is 8 ft.; find its length.

58. A piece of cloth is x yds. long and y'' wide. How many pieces $10'' \times 8''$ can be cut out of it?

59. What is the volume of a brick a'' long, b'' wide and c'' thick ? What is the area of its surface ?

60. A reservoir is l ft. long and b ft. wide. If its capacity is x cu. ft., find its depth.

61. The cross section of a cylinder is x sq. inches and its length is $50''$ Find its volume.

62. Find the number of bricks $9'' \times 4\frac{1}{2}''$ required for a rectangular floor x ft. long and y ft. wide.

63. Find the number of bricks $9'' \times 4\frac{1}{2}'' \times 3''$ required for building a rectangular platform a ft. long, b ft. wide and $30''$ high.

64. Find the number of bricks $9'' \times 4\frac{1}{2}'' \times 3''$ required to build a wall h ft. high, w'' thick round a rectangular garden l ft. long and b ft. wide.

65. The thickness of a sheet of paper is $\frac{I}{k}$ in. How many sheets will be required to form a pile of paper h in. high?

66. Find the length of a wire whose cross-section is $\frac{1}{t}$ sq. in. and whose volume is 1 cu. ft.

CHAPTER II

DIRECTED NUMBERS, FUNDAMENTAL LAWS, FIRST FOUR RULES

1. Illustrations.

(i) If a tradesman invests in business a capital of Rs. 100, and from his first transaction he gains Rs. 5, and from his second transaction he loses Rs. 5, his capital or financial position remains unchanged.

Here the *loss* of Rs. 5 *destroys* the *gain* of Rs. 5 ; thus the loss of Rs. 5 and the gain of Rs. 5 are *opposite in character* but *equal in magnitude*.

(ii) If a man ascends 10 yds. and then descends 10 yds., his ultimate position remains unchanged.

Here the *descent* of 10 yds. *destroys* the *ascent* of 10 yds. ; thus the ascent of 10 yds. and the descent of 10 yds. are *opposite in character* but *equal in magnitude*.

(iii) If a man walks 3 miles east and then 3 miles west, his ultimate position remains unchanged.

Here the effect of *going* 3 *miles east* is *destroyed* by the effect of *going* 3 *miles west* ; thus going 3 miles east and going 3 miles west are *opposite in character* but *equal in magnitude*.

(iv) $7+3-3=7$.

Here the effect of *adding* 3 is *destroyed* by the effect of *subtracting* 3, and thus the subtraction of 3 and the addition of 3 are *opposite operations* with *equal magnitudes*.

These four examples are of *equal* and *opposite quantities* or *movements*.

It is evident that the gain of Rs. 8 does not *completely* destroy the loss of Rs. 12, but the effect of one on the

capital is opposite to the effect of the other ; hence *gain* and *loss* are two *opposite quantities.*

Similarly, credit and debit, income and expenditure, ascent and descent, rise and fall, the distance to the right-hand side and the distance to the left-hand side, the distance to the east and the distance to the west, the distance to the north and the distance to the south, the degrees north of the equator and the degrees south of the equator, the degrees above the freezing-point and the degrees below the freezing-point, the arrival of men and the departure of men, the time before a particular event and the time after that event, the operation of addition and the operation of subtraction, are pairs of opposite quantities or movements.

In Arithmetic the operations of addition and subtraction which are opposite to each other, are denoted by the signs + and − respectively ; similarly, it would be *convenient* to denote any pair of opposite quantities or movements by the same signs, + and −.

If one quantity or movement be represented by the sign +, the opposite one would be represented by the sign −.

EXERCISE 6. (Oral)

1. Name three pairs of equal and opposite quantities.

2. Name two pairs of equal and opposite movements.

3. If a loss of Rs. 5 be denoted by + Rs. 5, what would be the notation for a gain of Rs. 12 ?

4. If 30 miles towards east be represented by —30 miles, what would be the notation for 20 miles towards west ?

5. If 60 miles towards north be denoted by + 60 miles, what would be the notation for 15 miles towards south ?

6. If 23° north of the equator be denoted by +23°, what would be the notation for 17° south of the equator ?

7. If 500 yrs, before the birth of Christ be denoted by +500 yrs., what would be the meaning of −500 yrs. ?

8. If the arrival of 5 men be denoted by +5 men, what would be the meaning of −5 men ?

9. If − Rs. 50 mean a gain of Rs. 50, what would be the meaning of + Rs. 60 ?

10. If + Rs. 12 denote Rs. 12 received, what would − Rs. 12 denote ?

11. If 4000 ft. height above the sea-level be represented by + 4000 ft., what would be the meaning of − 2000 ft. ?

12. What must be taken as negative, if each of the following is taken as positive :

(i) Number of feet to the right-hand side.
(ii) Number of miles to the east.
(iii) Number of rupees lost.
(iv) Degrees of falling temperature.
(v) Degrees north of the equator.
(vi) Days to come.

2. **Convention.** We have seen that if *either* of the two opposite quantities is denoted by the sign +, the other is denoted by the sign − ; but, in general, it is *usual* to mark such quantities *positive* as *raise* or *strengthen* our position, and to mark such quantities *negative* as *lower* or *weaken* our position.

Thus, *gain*, *income*, *credit*, *ascent*, *rise*, etc., are taken as positive, and *loss*, *expenditure*, *debit*, *descent*, *fall*, etc., are taken as negative.

Similarly, motion and distance to the right-hand, motion and distance to the east, motion and distance to the north, degrees north of the equator, degrees above the freezing-point, the coming of men, etc., are taken as positive ; whereas motion and distance to the left-hand, motion and distance

to the west, motion and distance to the south, degrees south of the equator, degrees below the freezing point, the departure of men, etc., are taken as negative.

NOTE. The + sign is generally omitted before a positive quantity; hence when no sign is prefixed to a quantity, + sign is understood. Thus + Rs. 5 and Rs. 5, + 8 and 8 are positive.

EXERCISE 7. (Oral)

Write the following symbolically prefixing the *usual* signs:

1. An income of Rs. 50. **2.** A loss of Rs. 20.

3. A debt of Rs. 100.

4. An expenditure of Rs. 40.

5. A saving of Rs. 15. **6.** An ascent of 25 yds.

7. A fall of 125 yds. **8.** A promotion of Rs. 5.

9. 200 ft. below the sea level.

10. 500 yds. above the sea level.

11. 13 miles due east. **12.** 15 miles due north.

13. 17 miles due south. **14.** 20 miles due west.

15. 20° north of the equator.

16. 15° south of the equator.

17. 8 men arrived. **18.** 10 men gone away.

19. 13 yds. to the left-hand.

20. 19 yds. to the right-hand.

21. 35° above the freezing-point.

22. 23° below the freezing-point.

23. Rs. 40 received. **24.** Rs. 40 given away.

25. Express with proper signs all the following quantities as gains:

(i) gain of Rs. 12; (ii) loss of Rs. 9;
(iii) gain of Rs. 25; (iv) loss of Rs. 15.

26. Express all the following, first as northerly movements and then as southerly movements:

(i) $5°N.$; (ii) $7°S.$; (iii) $3°N.$; (iv) $8°S.$

27. If the dates are reckoned + from 0 A. D. onwards, express the following with proper signs:

(i) 420 *A.D.*; (ii) 75 *B.C.*;

(iii) 1625 *A.D.*; (iv) 1200 *B.C.*

3. **Definitions.** (i) Gain and loss, credit and debit, etc., are reckoned from the original financial position of a man.

(ii) Rise and fall, ascent and the descent, etc., are measured from the level of the ground.

(iii) The distance to the east and the distance to the west, the distance to the right-hand side and the distance to the left-hand side, etc., are measured from a particular position.

Similarly, in other cases as well, we always measure the magnitudes of opposite quantities from a particular position.

The position from which we measure the magnitude of opposite quantities is called the **starting-point** and numbers indicating their measures and direction are called **directed numbers.**

Example 1. A barrel floating on the surface of a sea is carried by successive tides as follows:

3 miles east, 4 miles west, 2 miles east, 5 miles west, 6 miles east. Represent these movements by means of the usual signs.

Since the distance to the east is represented by the positive sign and the distance to the west is represented by the negative sign, therefore we have:

$$(+3 \text{ miles})+(-4 \text{ miles})+(+2 \text{ miles})+(-5 \text{ miles}) \\ +(+6 \text{ miles}),$$

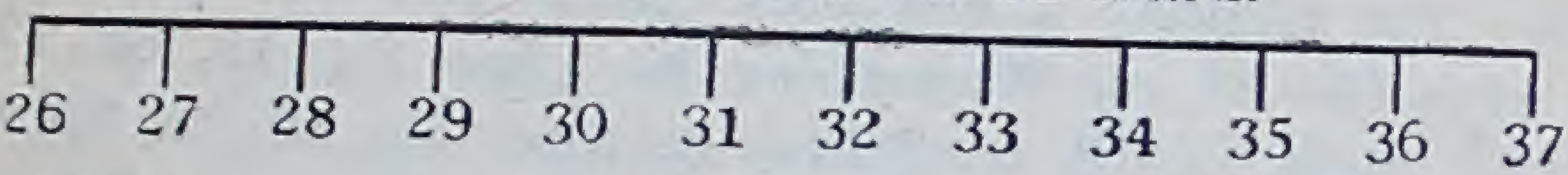
as the representation of the changes in the position of the barrel.

Example 2. A person gains Rs. 30, loses Rs. 25, gains Rs. 15, and then loses Rs. 10. Represent these changes symbolically, using a for Rs. 5.

Since a is to stand for Rs. 5, therefore a gain of Rs. 30 would be represented by $+6a$, loss of Rs. 25 would be represented by $-5a$, a gain of Rs. 15 would be represented by $+3a$, and a loss of Rs. 10 would be represented by $-2a$. Thus these changes, when represented symbolically, stand as follows :

$$(+6a)+(-5a)+(+3a)+(-2a).$$

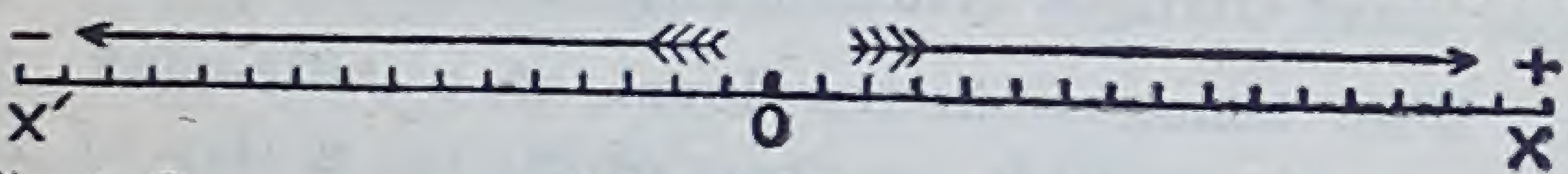
Relative Position. Suppose all measurements are calculated on the following scale from the 32 mark.



Since 35 is 3 steps forward from 32, it is said to be $(+3)$ *relative* to 32, and as 30 is 2 steps backward from 32, it is said to be (-2) *relative* to 32.

EXERCISE 8.

1. In the graduated scale XX', take O as the starting-point, and *mark* on it the positions of the following points :



(i) $+3$. (ii) -3. (iii) $+5$.
(iv) -8. (v) -10. (vi) $+12$.
(vii) -9. (viii) $+15$. (ix) $+7$.
(x) 0. (xi) $+14$. (xii) -15.

2. *Tell* from the scale the number of divisions between

(i) $+3$ and -3. (ii) $+5$ and -8.
(iii) -10 and $+10$. (iv) -5 and -8.
(v) -6 and $+4$. (vi) 0 and -7.
(vii) -9 and 0. (viii) $+8$ and 0.
(ix) 0 and $+5$.

How many divisions do I move, and in what direction

3. If I move from $+3$ to -3?
4. If I move from $+5$ to $+8$?
5. If I move from -8 to -5?
6. If I move from -5 to $+8$?
7. If I move from $+9$ to $+4$?
8. If I move from $+11$ to -3?
9. If I move from -10 to $+2$?
10. If I move from -4 to -10?
11. If I move from $+3$ to -9?

12. On Monday a merchant gains Rs. 60, on Tuesday he loses Rs. 25, on Wednesday he loses Rs. 15, on Thursday he gains Rs. 12, on Friday he loses Rs. 16, on Saturday he gains Rs. 24, and on Sunday he gains Rs. 40. Represent these changes by means of the usual signs.

13. The daily changes in the height of a barometer are: Monday a fall of ·6 in., Tuesday a rise of ·8 in., Wednesday a fall of ·4 in., Thursday a rise of ·2 in., Friday a rise of ·3 in., Saturday a fall of ·2 in., and Sunday a fall of ·5 in. Represent these changes symbolically, using x for ·1 in.

14. The changes in the level of a hilly-road are as follows: a rise of 80 ft., a drop of 50 ft., a rise of 110 ft., a drop of 60 ft., a rise of 130 ft., a drop of 70 ft., a rise of 150 ft. Represent the level symbolically, using x for 10 ft.

15. P, Q, R, S, T are five stations on a railway. From P to Q the level falls 50 ft., from Q to R it rises 70 ft., from R to S it falls 40 ft., and from S to T it rises 80 ft. Represent the whole level symbolically, using a for 5 ft.

16. An aeroplane rises to a height of 850 ft., then descends 350 ft., it again rises 260 ft., and then descends 145 ft. Express its ascent and descent symbolically.

17. The average temperature of a day was 80°. At certain hours of the day the temperature was 75°, 78°, 84°, 88°. State each of these, relative to the average temperature.

18. Reckoning times from 10 A.M., express each of the following times on a certain day by the directed numbers :

(i) 4 P.M., (ii) noon, (iii) 7-30 A.M., (iv) 3-20 P.M.

19. The freezing-point of a Fahrenheit thermometer is 32°. Express the following temperatures relative to the freezing-point :

(i) 78°, (ii) 24°, (iii) 32°, (iv) 0°.

20 If a goods train runs with a speed of 20 miles an hour and a passenger train with a speed of 35 miles an hour in the same direction, express :

(i) the speed of the passenger train relative to that of the goods train ;

(ii) the speed of the goods train relative to that of the passenger train ;

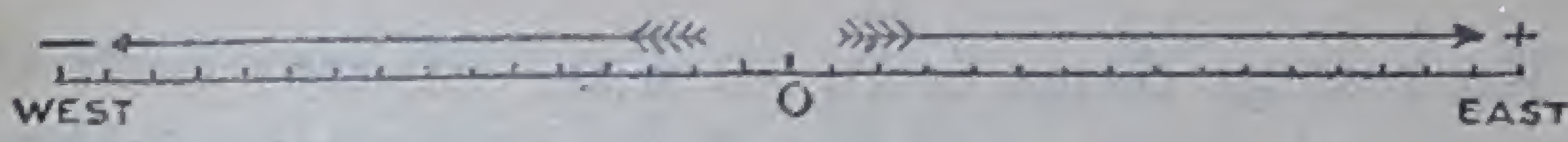
(iii) the speed of a pole fixed along the railway line relative to that of the goods train :

(iv) the speed of a pole fixed along the railway line relative to that of the passenger train.

ADDITION

1. **Graphic Illustrations.** The following graduated scale stands for a straight road from east to west or west to east ; each of its divisions stands for *one mile*. *O* is the starting-point. A man is supposed to be travelling along this road.

Mark the successive positions of the man and express in each of the following cases the whole movement by means of the *usual signs*:



(i) When he goes 8 miles to the east, then 5 miles to the east, and is thus 13 miles to the east.

Symbolic representation: (+8 miles)+(+5 miles) = +13 miles.

(ii) When he goes 4 miles to the east, 3 miles to the east, and then 5 miles in the same direction, and is thus 12 miles to the east.

Symbolic representation: (+4 miles)+(+3 miles) +(+5 miles)= +12 miles.

(iii) When he goes 8 miles to the west, then 5 miles to the west, and is thus 13 miles to the west.

Symbolic representation: (−8 miles)+(−5 miles) = −13 miles.

(iv) When he goes 4 miles to the west, 3 miles to the west, and then 5 miles in the same direction, and is thus 12 miles to the west.

Symbolic representation: (−4 miles)+(−3 miles) +(−5 miles)= −12 miles.

Money Illustrations. Let a stand for *one* rupee.

Arithmetical examples	Algebraic representation
(i) Rs. 4 gain+Rs. 3 gain =Rs. 7 gain.	$(+4a)+(+3a)=+7a.$
(ii) Rs. 4 gain+Rs. 3 gain +Rs. 5 gain=Rs. 12 gain.	$(+4a)+(+3a)+(+5a) =+12a.$
(iii) Rs. 4 loss +Rs. 3 loss =Rs. 7 loss.	$(-4a)+(-3a)=-7a.$
(iv) Rs. 4 loss +Rs. 3 loss +Rs. 5 loss=Rs. 12 loss.	$(-4a)+(-3a)+(-5a) =-12a.$

From these and similar examples it follows:

Rule 1. **The sum of a number of like terms, all of the same sign, is a single like term of the same sign, and its co-efficient is the sum of the numerical values of the several co-efficients.**

Example 1. Find the sum of $+2ab$, $+7ab$, $+8ab$.

Here all the terms are alike and have the same sign; hence the literal part of the sum is ab, the co-efficient is 17, *i.e.* $(2+7+8)$, and the sign to be prefixed is $+$.

$\therefore \qquad \text{sum} = +17ab.$

Example 2. Find the sum of $-3ab^2$, $-5ab^2$, $-7ab^2$.

Here all the terms are alike and have the same sign; hence the literal part of the sum is ab^2, the co-efficient is 15, *i.e.* $(3+5+7)$, and the sign to be prefixed is $-$.

$\therefore \qquad \text{sum} = -15ab^2,$

In the above scale, mark the successive positions of the man and express in each of the following cases the whole movement by means of the *usual signs*:

(v) When he goes 8 miles to the east, then 5 miles to the west, and is thus 3 miles to the east.

Symbolic representation: $(+8 \text{ miles}) + (-5 \text{ miles})$
$= +3 \text{ miles}.$

(vi) When he goes 8 miles to the west, then 5 miles to the east, and is thus 3 miles to the west.

Symbolic representation: $(-8 \text{ miles}) + (+5 \text{ miles})$
$= -3 \text{ miles}.$

(vii) When he goes 8 miles to the east, 5 miles to the west, then 7 miles to the west, and is thus 4 miles to the west.

Symbolic representation: $(+8 \text{ miles}) + (-5 \text{ miles})$
$+ (-7 \text{ miles}) = -4 \text{ miles}.$

(viii) When he goes 8 miles to the west, 5 miles to the east, then 7 miles to the east, and is thus 4 miles to the east.

Symbolic representation: $(-8 \text{ miles}) + (+5 \text{ miles})$
$+ (+7 \text{ miles}) = +4 \text{ miles}.$

Money Illustrations. Let a stand for *one* rupee.

Arithmetical examples	Algebraic representation
(i) Rs. 8 gain + Rs. 3 loss = Rs. 5 gain.	$(+8a)+(-3a)=+5a$.
(ii) Rs. 8 loss + Rs. 3 gain = Rs. 5 loss.	$(-8a)+(+3a)=-5a$.
(iii) Rs. 8 gain + Rs. 3 loss + Rs. 4 gain = Rs. 9 gain.	$(+8a)+(-3a)+(+4a)=+9a$.
(iv) Rs. 8 loss + Rs. 3 gain + Rs. 4 loss = Rs. 9 loss.	$(-8a)+(+3a)+(-4a)=-9a$.

From these and similar examples it follows:

Rule 2. **The sum of a number of like terms, not all of the same sign, is a single term. To obtain its co-efficient add together the co-efficients of the positive terms and the co-efficients of the negative terms, take the difference of these two results, and prefix the sign of the greater.**

Example 3. Find the sum of

$$+6ab^2c-4ab^2c+8ab^2c-12ab^2c.$$

Here all the terms are alike, but have different signs; hence the letters in the sum are ab^2c, the co-efficient is 2, *i.e.* $(\overline{6+8}-\overline{4+12})$, and the sign to be prefixed is $-$.

$$\text{sum}=-2ab^2c.$$

EXERCISE 9.

1. Prove *graphically* that:

(i) $(+7)+(+12)=+19$, (ii) $(-7)+(-12)=-19$,
(iii) $(+7)+(-12)=-5$, (iv) $(-7)+(+12)=+5$,
(v) $+5-3=+2$, (vi) $-5+3=-2$,
(vii) $+8-8=0$, (viii) $-12+12=0$,
(ix) $+5-7+6=+4$, (x) $-5+7-6=-4$.

Add (*orally*):

2. $$\begin{array}{r} -7x \\ \underline{+4x} \end{array}$$

3. $$\begin{array}{r} 5ax \\ \underline{-7ax} \end{array}$$

4. $$\begin{array}{r} +19axy \\ \underline{-\ \ 9axy} \end{array}$$

5. $$\begin{array}{r} +8ab \\ -7ab \\ \underline{+5ab} \end{array}$$

6. $$\begin{array}{r} -\ \ 4xyz \\ +\ \ 7xyz \\ \underline{-13xyz} \end{array}$$

7. $$\begin{array}{r} +\ \ 9x^2y \\ -\ \ 5x^2y \\ \underline{+12x^2y} \end{array}$$

8. $$\begin{array}{r} -5y^2z \\ -8y^2z \\ \underline{9y^2z} \end{array}$$

9. $$\begin{array}{r} -\ \ 2m^2n \\ +17m^2n \\ \underline{-\ \ 5m^2n} \end{array}$$

10. $$\begin{array}{r} +\ \ 8p^2q^2 \\ -12p^2q^2 \\ \underline{+\ \ 4p^2q^2} \end{array}$$

Simplify:

11. $4ab+3ab-2ab-7ab.$

12. $13xyz-4xyz+5xyz.$

13. $-5abx^2+8abx^2-9abx^2+4abx^2-2abx^2.$

14. $-7pq^2r-12pq^2r+7pq^2r-3pq^2r+5pq^2r-8pq^2r.$

Simplify by collecting like terms:

15. $a-2b+3c-4a+6b-5c+2a.$

16. $x-4y+5z-3y+4z-5x+2y-10z+5y+x.$

17. $3p^2-3pq+4q^2-p^2+7pq-6p^2-2pq+5q^2.$

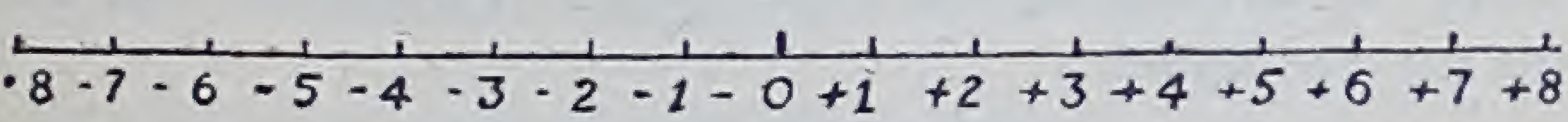
18. $5ab-4bc+ca-2ab+6ca+7bc-3ca+bc.$

19. $6x^2yz+xy^2z-4x^2yz-5xyz^2+3xy^2z+7xyz^2.$

20. $4ax^2y-5a^2xy+8ax^2y-7axy^2+2a^2xy-4axy^2.$

* *Def.* In a scale of directed numbers, the number midway between any two of the numbers is called their **Arithmetic mean.**

Thus in this scale



(i) the Arithmetic mean of $+1$ and $+7$ is $+4$,
(ii) the Arithmetic mean of $+2$ and -4 is -1,
(iii) the Arithmetic mean of -2 and -6 is -4.

But (i) half the sum of $+1$ and $+7$ is $+4$,
(ii) half the sum of $+2$ and -4 is -1,
(iii) half the sum of -2 and -6 is -4.

therefore, the Arithmetic mean of two directed numbers may be obtained by dividing their algebraic sum by 2.

Similarly, the Arithmetic mean or the average of, say 8 directed numbers $= \dfrac{\textit{their algebraic sum}}{8}$.

*** 21.** Find the Arithmetic mean of the following directed numbers :

(i) $+16$ and $+24$, (ii) -8 and $+14$,
(iii) -8 and -20, (iv) $+8$ and -22,
(v) -7, $+4$, -5, $+6$, (vi) -9, $+5$, $+3$, -6, $+7$.

*** 22.** Draw a straight line XOX' on squared paper and represent on it 5 points whose heights are $+8$, $+4$, -2, $+3$, -3. Find the average height of these points; draw a straight line LM at the average height above XOX'. Express the height of these points relative to the straight line LM.

*** 23.** Represent on squared paper 9 points whose heights are $+10$, -8, $+2$, -7, $+3$, $+5$, -4, $+6$, $+11$, with reference to a st. line AB.

Draw another st. line parallel to AB and at the average height of these points. Express the height of each of these points relative to this st. line.

Example 4. Add together $5a^3-4a^2b+6ab^2-2b^3$, $a^3-2ab^2-3a^2b$, and $b^3-3a^3+ab^2$.

Here, by arranging the like terms in columns and adding them up, we have

$$\begin{array}{rrrr}
5a^3 & -4a^2b & +6ab^2 & -2b^3 \\
a^3 & -3a^2b & -2ab^2 & \\
-3a^3 & & +ab^2 & +b^3 \\
\hline
3a^3 & -7a^2b & +5ab^2 & -b^3
\end{array}$$

Add together:

24. $x^2+3xy, -2xy+y^2$. **25.** $x^2+xy, -xy+y^2$.

26. $a^2+ab, -ab-b^2$. **27.** $p^2-3pq+q^2, -5p^2+6pq+2q^2$.

28. $a^2+a+1,\ 3a^2-a+2,\ 5a^2-4a-7$.

29. $-5x^2+3xy-y^2,\ -8x^2+xy-y^2,\ 7x^2-3y^2$.

30. $2x+3y-4z,\ -5x-y-z,\ -3x-4y$.

31. $b^2-b^2c+c^2b,\ 2b^2c-3c^2b,\ -4b^2-2c^2b$.

32. $3a^2x^2-4ax-10,\ 2a^2x^2+ax-6,\ -4a^2x^2+ax+9$.

33. $a^2-5ab-4b^2,\ -3a^2+2ab-6b^2,\ 5a^2+3ab+2b^2$.

34. $5p^2-2pq+7q^2,\ -9p^2+3pq-6q^2,\ +2p^2-5pq+10q^2$.

35. $-1-3x-2x^3,\ 5x-3+4x^3,\ x^2+2x^3-2x+2$.

36. $3a-4b+5c-d+6e,\ -2a+4b-2c-7d+3f,$
$-6a+4b-3c-7d+9f,\ -5a-2b+5c-4d-3e,$
$9a-2b-c+2e-6f$.

37. $3x^2+2y^2+6x-9,\ 7x^2-3x-4,\ -9x^2-7y^2+4,$
$-4z^2+yz-3xy,\ 6y^2+z^2-3xy-3yz,\ 3z^2+4yz-2xy$.

38. $\frac{1}{2}a^2-\frac{1}{3}ab+b^2,\ 2ab-\frac{1}{2}b^2+\frac{3}{4}a^2,\ \frac{1}{3}a^2+\frac{2}{3}ab-\frac{1}{3}b^2$.

39. If $A=x^2-5x+9$, $B=4x-3$, $C=6-x^2$, find the value of $A+B+C$.

40. If $x=4a^2-3a+5$, $y=-2a+7a^2+4$ and $z=-9+4a-11a^2$, find the value of $x+y+z$.

SUBTRACTION

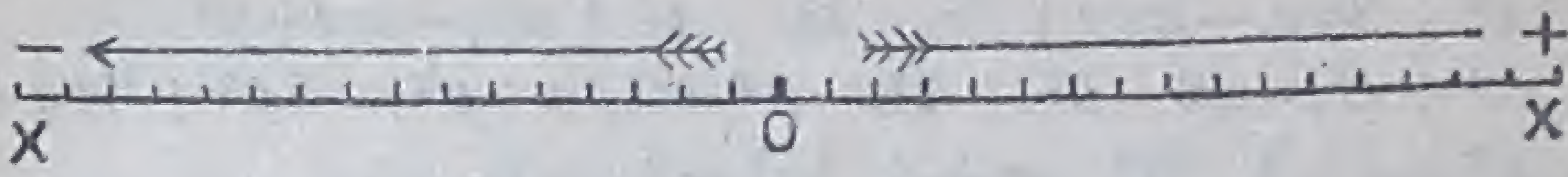
5. *Principle.* In Arithmetic $8-5$ means that we have to find the number which must be added to 5 to make 8; or, *generally*, to find the number which must be added to the subtrahend to make the minuend.

In Algebra to subtract subtrahend from minuend means to find out:

(i) **How far the minuend is from the subtrahend, and**

(ii) **the direction in which it lies.**

Graphic Illustrations.



(i) To subtract $+5$ from $+8$ means marking the positions of $+5$ and $+8$ and finding out the *direction* and the *number* of steps to be taken from $+5$ to reach $+8$.

Evidently, in this case, we take three steps in the positive direction ; hence the answer is $+3$.

(ii) To subtract -5 from $+8$ means marking the positions of -5 and $+8$ and finding out the *direction* and the *number* of steps to be taken from -5 to reach $+8$.

Evidently, in this case, we take thirteen steps in the positive direction ; hence the answer is $+13$.

(iii) To subtract $+5$ from -8 means marking the positions of $+5$ and -8 and finding out the *direction* and the *number* of steps to be taken from $+5$ to reach -8.

Evidently, in this case, we take thirteen steps in the negative direction; hence the answer is -13.

(iv) To subtract -5 from -8 means marking the positions of -5 and -8 and finding out the *direction* and the *number* of steps to be taken from -5 to reach -8.

Evidently, in this case, we take three steps in the negative direction ; hence the answer is -3.

Let us now compare algebraic addition with algebraic subtraction :

<table>
<tr><th colspan="2">Algebraic addition</th><th colspan="2">Algebraic subtraction</th></tr>
<tr><td>(i) $\begin{array}{r} +8 \\ -5 \\ \hline +3 \end{array}$</td><td>(ii) $\begin{array}{r} +8 \\ +5 \\ \hline +13 \end{array}$</td><td>(i) $\begin{array}{r} +8 \\ +5 \\ \hline +3 \end{array}$</td><td>(ii) $\begin{array}{r} +8 \\ -5 \\ \hline +13 \end{array}$</td></tr>
<tr><td>(iii) $\begin{array}{r} -8 \\ -5 \\ \hline -13 \end{array}$</td><td>(iv) $\begin{array}{r} -8 \\ +5 \\ \hline -3 \end{array}$</td><td>(iii) $\begin{array}{r} -8 \\ +5 \\ \hline -13 \end{array}$</td><td>(iv) $\begin{array}{r} -8 \\ -5 \\ \hline -3 \end{array}$</td></tr>
</table>

On comparison we find that the answer to

(i) in addition is the same as to (i) in subtraction,
(ii) ,, ,, ,, (ii) ,, ,,
(iii) ,, ,, ,, (iii) ,, ,,
(iv) ,, ,, ,, (iv) ,, ,,

but in each of the above corresponding cases the second terms (*i.e.*, the subtrahend and the corresponding term in addition) have opposite signs.

Rule. **Change the sign of the subtrahend and then add.**

When one compound expression is to be subtracted from another, we take the following steps:

(i) **Arrange the like terms in columns.**
(ii) **Change the signs of all the terms in the subtrahend.**
(iii) **Add the terms in columns.**

Example 1. Subtract $3a-5b-c$ from $5c-8b+4a$.

It means: To arrange the like terms in columns, change the signs of all the terms in the subtrahend, and then add.

Thus we have

$$\begin{array}{rl} +4a-8b+5c & \\ -3a+5b+\ c & \text{[Signs changed.]} \\ \hline +\ a-3b+6c & \text{[Addition.]} \end{array}$$

Example 2. Subtract

$12xy-5x^2+2c^2-1$ from $32x^2+20xy-3y^2$.

Arranging the terms properly, changing the signs in the subtrahend, and adding the columns, we have

$$\begin{array}{r} 32x^2+20xy-3y^2 \\ +5x^2-12xy-2c^2+1 \\ \hline 37x^2+\ 8xy-3y^2-2c^2+1 \end{array}$$

EXERCISE 10.

Subtract with the help of a graduated scale:

1. $+\ 7$ from $+12$. 2. $-\ 7$ from $+12$.
3. $+\ 7$ from $-\ 12$. $-\ 7$ from -12.

5. $+12$ from $+7$.
6. -12 from $+7$.
7. $+12$ from -7.
8. -12 from -7.

Subtract (*orally*):

9. $+7a$ from $+12a$.
10. $-7a$ from $+12a$.
11. $+7a$ from $-12a$.
12. $-7a$ from $-12a$.
13. $+12m$ from $+7m$.
14. $-12p$ from $+7p$.
15. $+12m$ from $-7m$.
16. $-12x$ from $-7x$.
17. $+5x$ from $-5x$.
18. $-7m$ from $+7m$.
19. $+9n$ from $+9n$.
20. $-13q$ from $-13q$.
21. $-m$ from 0.
22. $+m$ from 0.
23. $x-y$ from $x+y$.
24. $x-y$ from $x-y$.
25. $+b+c$ from a.
26. $-b+c$ from a.
27. $-3x+4y$ from $7x+5y$.
28. $+2m-6n$ from $3m-4n$.

Subtract by arranging like terms in columns:

29. $3x+2y+z$ from $5x+2y+3z$.
30. $2ab-3bc+7ca$ from $-ab+4bc+6ca$.
31. $-x^2+4y^2-xy$ from $-x^2-2y^2+3xy$.
32. $11-y^2+3x^2$ from $2x^2-3y^2-7$.
33. $1-m^2+m^3$ from $3-2m^2-m^4$.
34. $x^2+2y^2+z^2-6$ from $4x^2-7+z^2-2y^2$.
35. $a^2-2ab+3b^2$ from $-6a^2+ab-9b^2$.
36. $-8x^2y^2-3y^2z^2-z^2x^2$ from $-12x^2y^2-4y^2z^2+7z^2x^2$.
37. $3p^3+4p^2q-pq^2-2q^3+3$ from $4p^3-p^2q-pq^2+2q^3-3$.
38. $-x^2+\frac{1}{3}xy-4y^2+yz-z^2$ from $x^2+\frac{2}{3}xy-2y^2+yz-4z^2$.
39. Subtract a^2+b^2 from the sum of $a^2+2ab+b^2$ and $a^2-2ab+b^2$.
40. Subtract the sum of x^2+xy and $xy+y^2$ from the sum of x^2+xy+y^2 and x^2-xy+y^2.

If $x=3a^2-1$, $y=2b^2+4$, $z=a^2-b^2+7$, find:

41. $x-y$.
42. $y-z$.
43. $z-x$.
44. $x+y-z$.
45. $x-y+z$
46. $y+z-x$.
47. Which is greater and by how much:
(i) $-12, -7$?
(ii) $0, -4$?

6. Removal and Insertion of Brackets.

To add a and $+b+c$ is the same as $a+(+b+c)$, but we know that when a and $+b+c$ are added together, the sum is $a+b+c$.

$\therefore \quad a+(+b+c)=a+b+c.$

Similarly, to add a and $-b+c$ is the same as $a+(-b+c)$, but we know that when a and $-b+c$ are added together, the sum is $a-b+c$.

$\therefore \quad a+(-b+c)=a-b+c.$

Hence we conclude that **the brackets preceded by $+$ sign may be removed without making any change in the signs of the enclosed terms.**

To subtract $+b+c$ from a means the same as $a-(+b+c)$, but we know that when $+b+c$ is subtracted from a, the answer is $a-b-c$.

$\therefore \quad a-(+b+c)=a-b-c.$

Similarly, to subtract $-b+c$ from a means the same as $a-(-b+c)$, but we know that when $-b+c$ is subtracted from a, the answer is $a+b-c$.

$\therefore \quad a-(-b+c)=a+b-c.$

Hence we conclude that **the brackets preceded by — sign may be removed on changing the signs of the enclosed terms, i.e., from $+$ to $-$ and from $-$ to $+$.**

Example 1. Simplify $7x+\{-3x+(5-2x)\}$.

The expression

$$\begin{aligned} &=7x+\{-3x+5-2x\} \\ &=7x+\{-5x+5\} \\ &=7x-5x+5 \\ &=2x+5. \end{aligned}$$

Example 2. Simplify $7x-\{-3x-(5-2x)\}$.

The expression

$$\begin{aligned} &=7x-\{-3x-5+2x\} \\ &=7x-\{-x-5\} \\ &=7x+x+5 \\ &=8x+5. \end{aligned}$$

Example 3. Simplify by removing the brackets

$$(a+b+c+d)-(a-b+c-d)+(a-b+c-d).$$

The expression $=a+b+c+d-a+b-c+d+a-b+c-d$

$=a+b+c+d.$

Example 4. Shew that

$$(1+3x)-(2-5x)+(2-3x)-(1+5x)=0.$$

The expression $=1+3x-2+5x+2-3x-1-5x$

$=1-2+2-1+3x+5x-3x-5x$

$=1+2-2-1+3x+5x-3x-5x$

$=3-3+8x-8x=0.$

EXERCISE 11.

Write down the value of the following without any process:

1. (i) $+(+5)$, (ii) $-(+3)$, (iii) $+(-7)$, (iv) $-(-6)$.
2. (i) $-(-3m)$, (ii) $+(-5q)$, (iii) $-2-(2-2)$, (iv) $-(2-4)-2$.
3. (i) $-x^2-(-a^2)+(-a^2)+(+x^2)$, (ii) $(+p-p)-(p-p)$.
4. (i) $+(-1)+(-1)+(-1)$, (ii) $-(+3)-(+3)-(+3)$, (iii) $-(-3)-(-3)-(-3)$.

Simplify :

5. $4p+(-3q-5p)$.
6. $(4a-5b)+(-2a+4b)$.
7. $(6p-2q)+(3q-4p)$.
8. $4p-(-3q+5p)$.
9. $7k-(-4k-2l)$.
10. $(3a-4b)-(-2a+5b)$.
11. $(l-5m+2n)-(-3n+4m-5l)$.
12. $-3a-\overline{3b+c}+\overline{3a+3b}-(+c)$.
13. $p+\{2q+\overline{3-5p}-(-q+7)+5\}$.
14. $2p-(-3+\overline{p-2q}-5)-7$.
15. $2a+(+5b-\overline{7b-2a})$.
16. $2-\{a-3+($ [illegible] $-5)\}$
17. $5y-[3x+\{2y-(3y-z)+5z\}]$.
18. $-(-(-(-(-(-1)))))$.
19. $-3+(-(+(-(+(-1)))))$.

20. $1-[1-\{1-(1-\overline{1-1})\}]$.

21. If $a=-1$, $b=-2$, $c=-3$, find the value of
$$1-b+a-[c-\{a+(b-c-\overline{a-b-c})\}].$$

Find the value of :

22. $(a+b+c)-(a-b-c)+(-a-b-c)+(a+b-c)$.

23. $(x+y-z)+(y+z-x)-(z+x-y)-(x+y+z)$.

24. $(a-b+c)-(3a-3b+3c)+(2a-2b+2c)$.

25. $(a^3+b^3-c^3+abc)-(a^3-b^3+c^3-abc)+$
$(a^3-b^3+c^3-abc)-(a^3+b^3-c^3+abc)$.

26. $[a-\{b+(c+d)\}]-[a+\{b-(c-d)\}]$
$+[a-\{b-(c+d)\}]-[a-\{b-(c-d)\}]$.

27. Find whether
$5-(3-9)$ or $9-(5-3)$ is greater.

From the above rules for the removal of brackets, we deduce the following rules for the **insertion of brackets** :

(i) **Any number of terms in an expression may be put within a pair of brackets and the sign + prefixed.**

(ii) **Any number of terms in an expression may be put within a pair of brackets and the sign − prefixed, provided the signs of all the terms placed within the brackets be changed, *i.e.*, from + to − and from − to +.**

For example, $a+b-c+d-e+f$ may be written in the form

$a+b-(c-d+e-f)$

or $a-(-b+c-d+e-f)$

or $+(a+b)-(c-d)-(e-f)$

or $-(-a-b+c)+(d-e+f)$

or $-(-a-b)+(-c+d)-(e-f)$.

28. Enclose all the terms after the first, within a pair of brackets preceded by (*a*) positive sign, (*b*) negative sign :

(i) $-a^2+m^2+b^2-n^2+b^2+c^2$,

(ii) $x^2-a^2+bc-b^2-ca+c^2-ab$.

* **29.** Bracket like powers of x, so that the sign before the brackets may be (*a*) positive, (*b*) negative :

(i) $2x^4+3x^2-4x^3+x^2-5x^4+x^3$

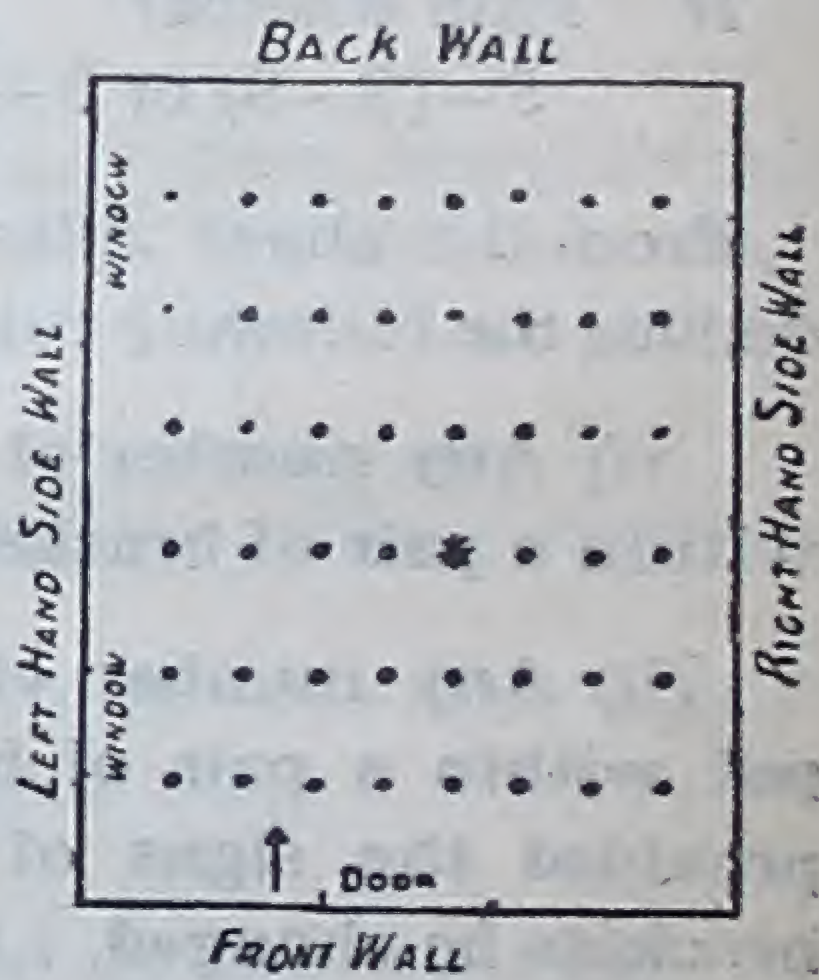
(ii) $ax^3-bx^2+cx-dx^3-ex^2-fx.$

* **30.** Fill in the blanks :

(i) $-1=+(-(+(-(+(\qquad)))))$,

(ii) $+a=-(-(-(+(-(+(\qquad))))))$.

7. **Co-ordinates of a point.** The annexed figure is the plan of an examination hall. The position of each candidate is indicated by means of a dot.



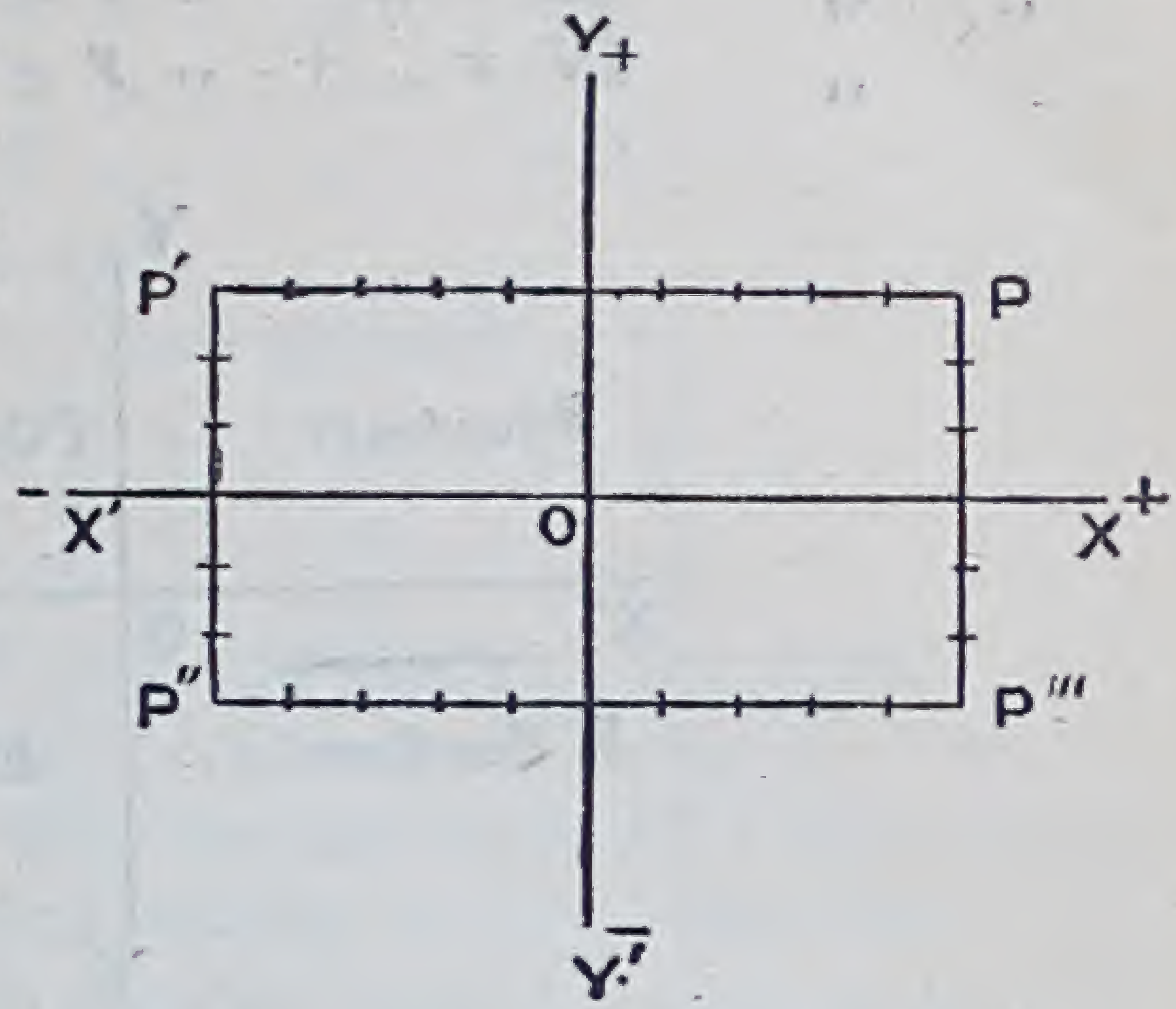
Suppose the position * of a particular candidate is to be pointed out to the superintendent from some distance.

If we merely say that he is in the 3rd row, it is not enough, for the 3rd row may be from the front wall or back wall. Again, if we say that he is in the 3rd row from the front wall, even that is not enough, for there are in all 8 candidates in that row and he may be *any* out of these 8.

Even if we add that he is in the 5th column, it is not enough, for the 5th column may be from the left-hand side wall or from the right-hand side wall. If, however, we say that he is in the 5th column from the left-hand side wall, his position is definitely fixed.

Hence to locate definitely the position of a candidate in an examination hall, we have to refer to his position with respect to two adjacent walls.

If we have to fix the position of a point P in a plane, with reference to two straight lines (say XOX' and YOY') at right angles to one another, it is not enough to say that the point P is $5'$ from YOY' and $3'$ from XOX', for there could be 4 points P, P', P'', P''' (see fig.) each satisfying the given condition; but if we say that it is at the distance of $+5'$ from YOY' and $+3'$ from XOX', the point is definitely fixed in position.



In order to fix the position of P', we shall have to say that it is $-5'$ from YOY' and $+3'$ from XOX'.

Similarly, the point P'' is $-5'$ from YOY' and $-3'$ from XOX', and the point P''' is $+5'$ from YOY' and $-3'$ from XOX'.

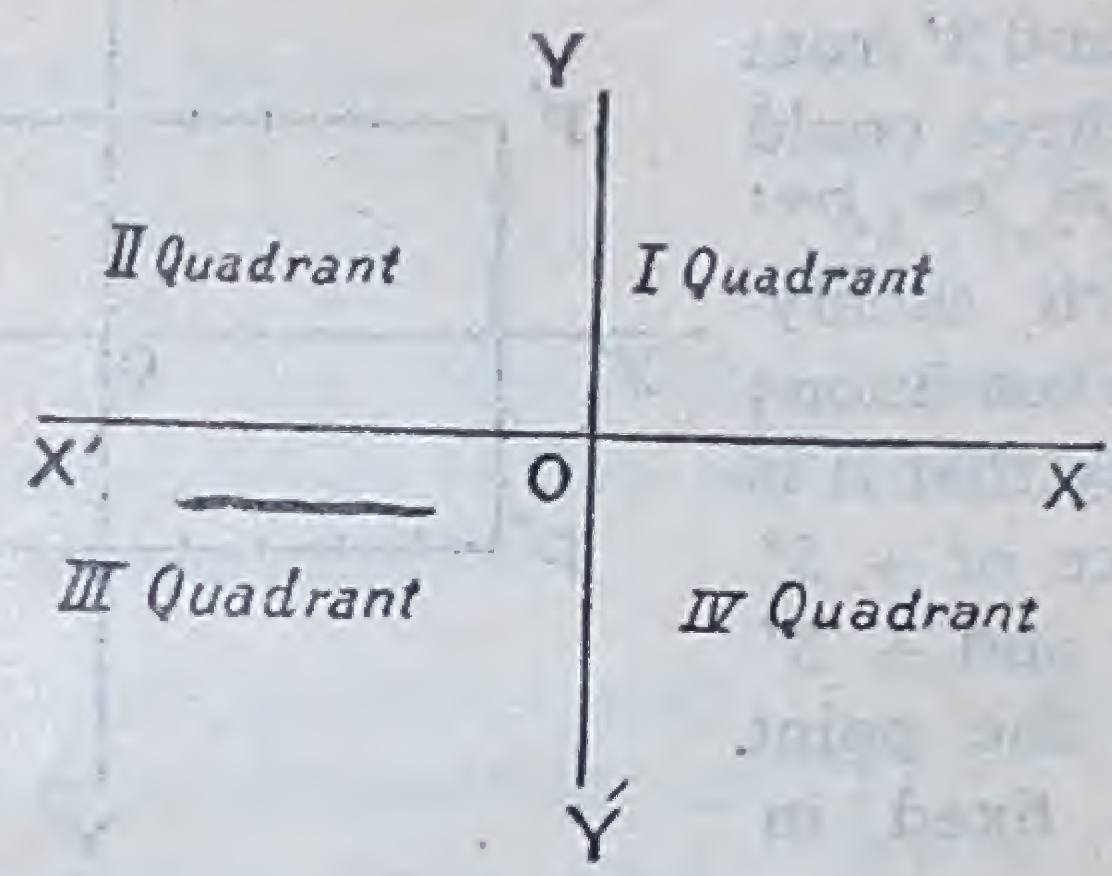
These distances of a point from the lines of reference are called **co-ordinates** of the point.

Thus, the co-ordinates of P are $(+5, +3)$,
" " P' " $(-5, +3)$,
" " P'' " $(-5, -3)$,
" " P''' " $(+5, -3)$.

The distance of a point from YOY' is called the **abscissa** and is usually represented by x, whereas its distance from XOX' is called the **ordinate** and is usually represented by y. The lines XOX' and YOY' are called the *x-axis* and *y-axis* respectively. O, the point of intersection of the axes from which all measurements are made, is called the **origin**.

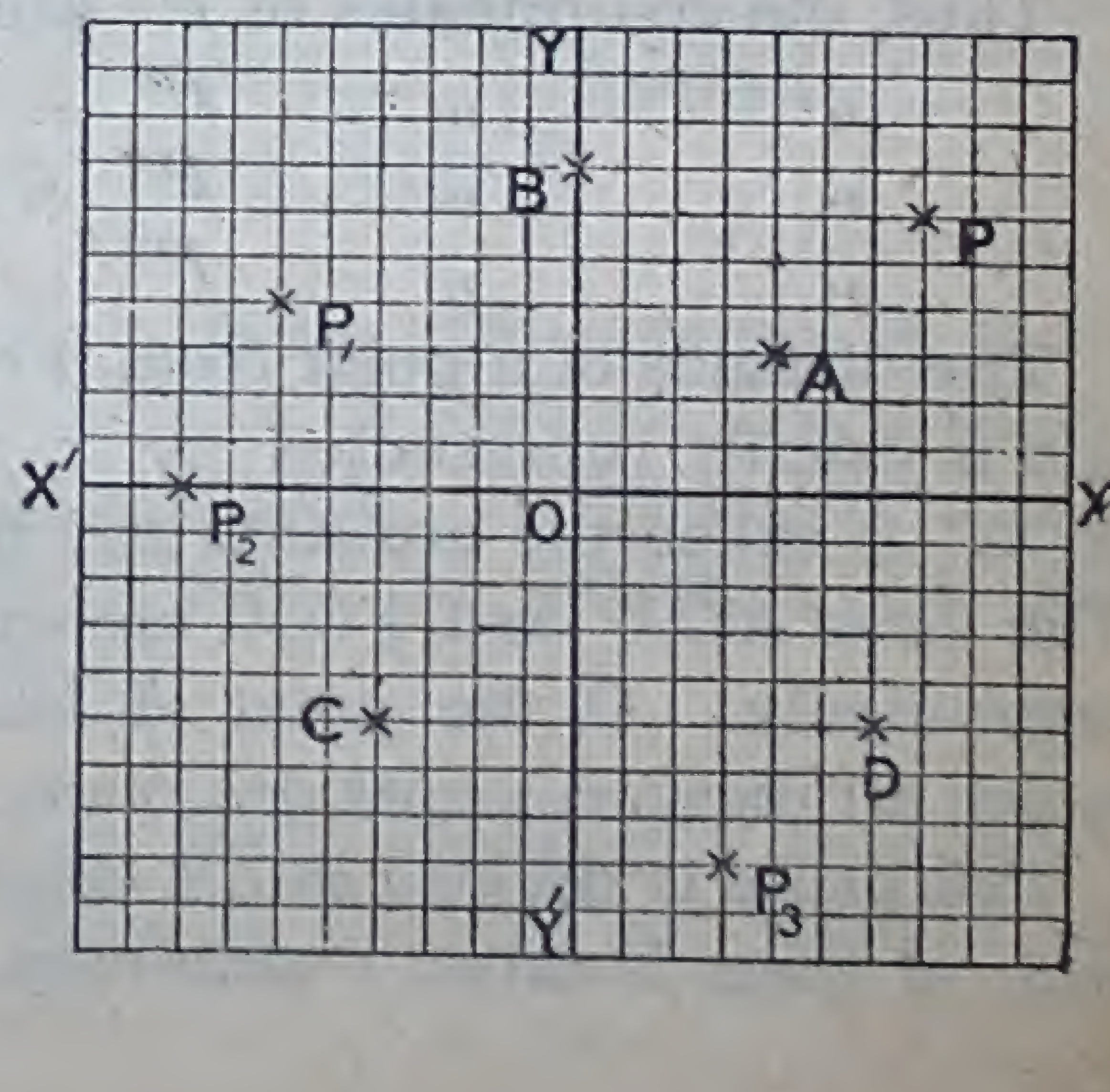
The plane is divided into 4 compartments by the axes, and each of them is called a **quadrant**. (See figure on page 48.)

In quadrant	I	x is $+$ and y is $+$
,, ,,	II	x ,, $-$,, y ,, $+$
,, ,,	III	x ,, $-$,, y ,, $-$
,, ,,	IV	x ,, $+$,, y ,, $-$.



The process of determining the position of a point in a plane, with reference to the two axes, is called *plotting the point.*

For the graphic representation of points and lines we use ordinary squared paper, in which the side of a small square is ·1″ and the side of a big square is 1″.



Example 1. (i) Write down the co-ordinates of the points A, B, C, D taking ·1″ as a unit.

(ii) Write down the co-ordinates of the points P, P_1, P_2, P_3, taking 1″ as a unit. (See fig.)

(i) If we take $\cdot 1''$ as a unit,

the abscissa of A is $+4$ and ordinate $+3$,
„ „ „ B „ 0 „ „ $+7$,
„ „ „ C „ -4 „ „ -5.
„ „ „ D „ $+6$ „ „ -5.

(ii) If we take $1''$ as a unit,

the abscissa of P is $+\cdot 7$ and ordinate $+\cdot 6$,
„ „ „ P_1 „ $-\cdot 6$ „ „ $+\cdot 4$,
„ „ „ P_2 „ $-\cdot 8$ „ „ 0,
„ „ „ P_3 „ $+\cdot 3$ „ „ $-\cdot 8$.

Example 2. Take $\cdot 1''$ as a unit and plot the points

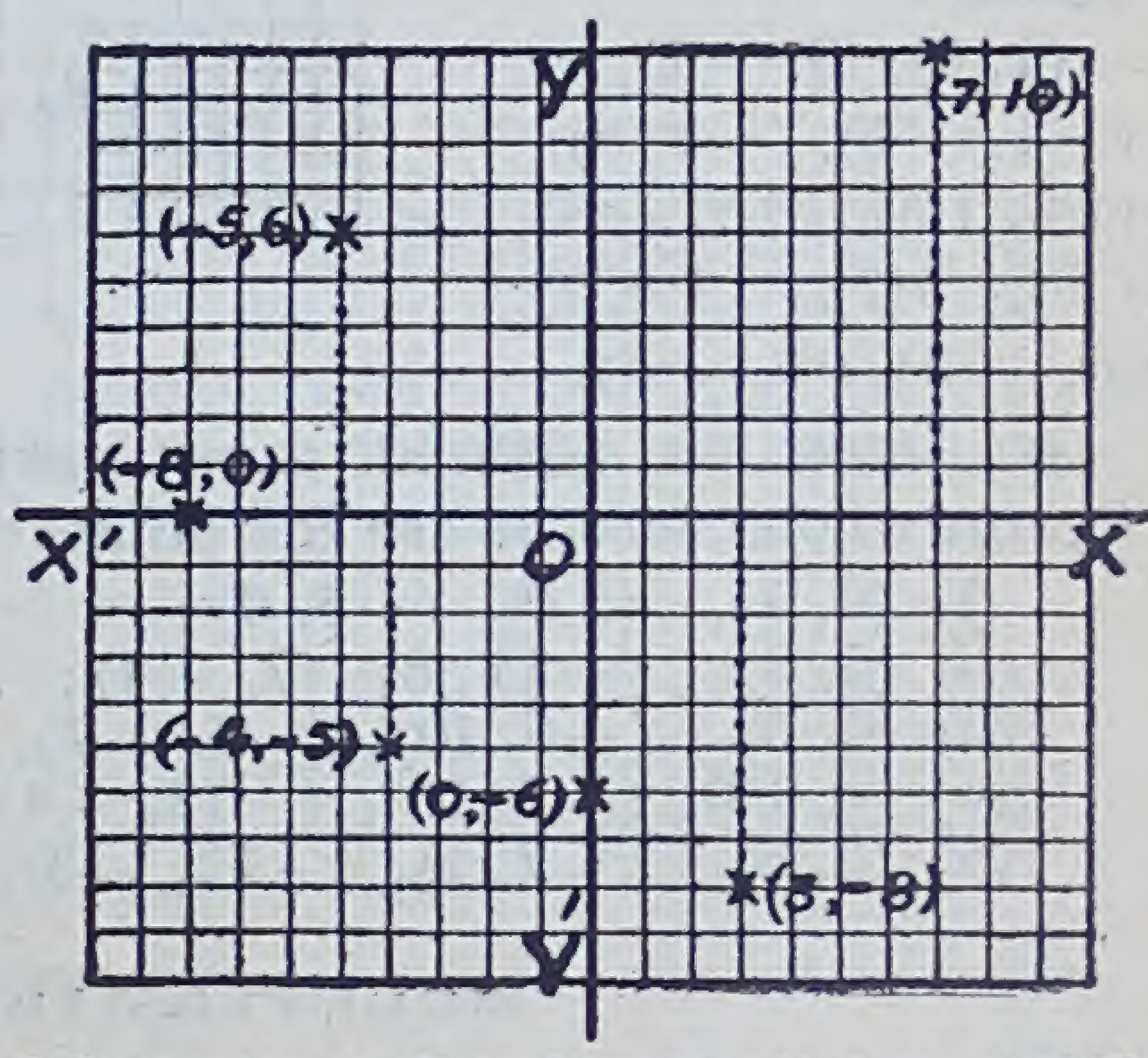
$(7, 10)$, $(-5, 6)$ $(-8, 0)$, $(-4, -5)$, $(0, -6)$, $(3, -8)$.

To plot the point $(7, 10)$, we take from O, 7 units along OX and from there 10 units parallel to OY. (See fig.)

To plot the point $(-5, 6)$ we take from O, 5 units along OX' and from there 6 units parallel to OY. (See fig.)

To plot the point $(-8, 0)$, we take from O, 8 units along OX' and from there no unit above or below it. (See fig.)

To plot the point $(-4, -5)$, we take from O, 4 units along OX' and from there 5 units parallel to OY'. (See fig.)



To plot the point $(0, -6)$, we take no units along OX or OX' and 6 units parallel to OY' from O. (See fig.)

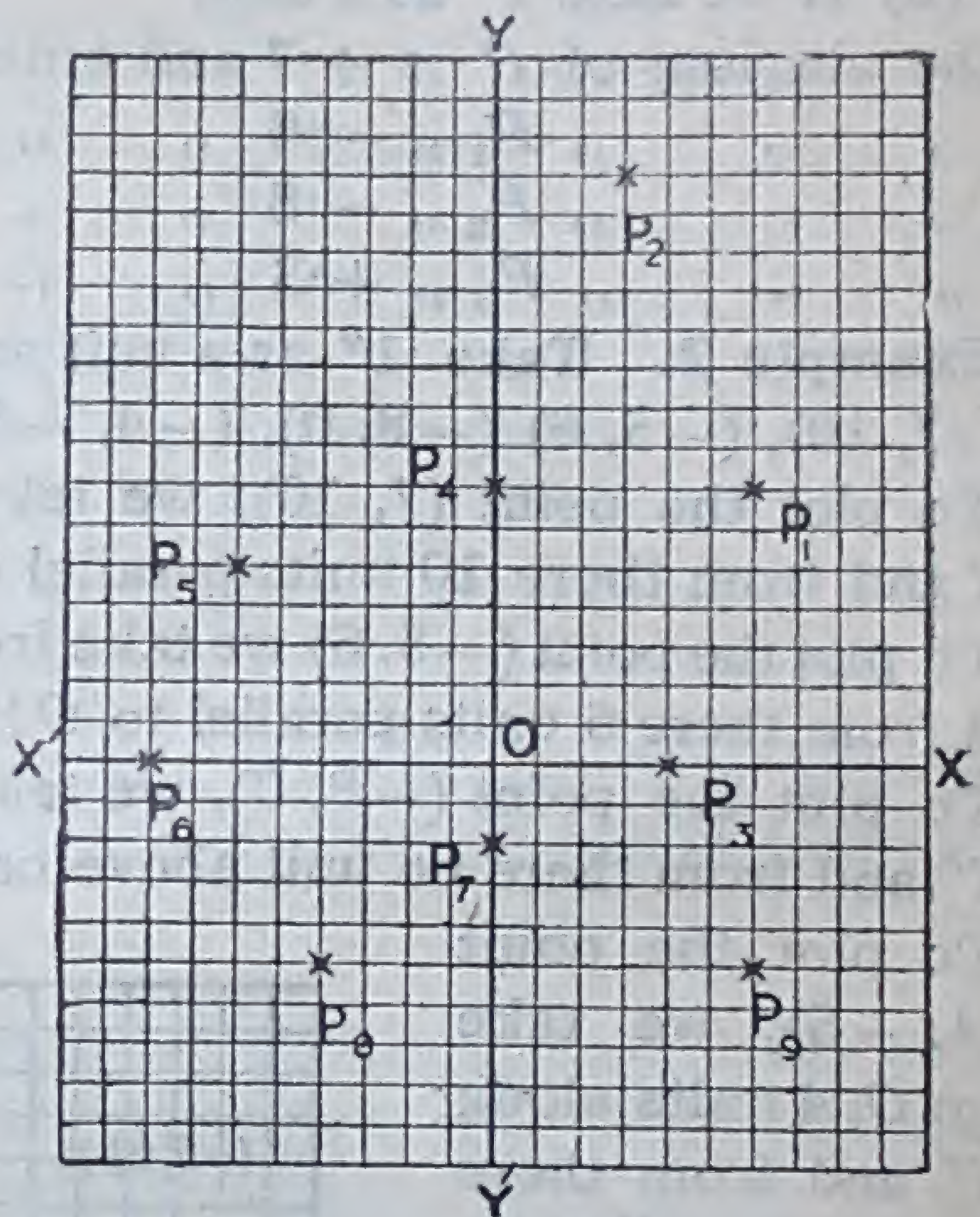
To plot the point $(3, -8)$, we take from O, 3 units along OX and from there 8 units parallel to OY'. (See fig.)

EXERCISE 12

1. *Write down* the co-ordinates of the points P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8, P_9 in this figure :

(i) taking ·1″ as a unit.
(ii) taking 1″ as a unit.

2. Take ·1″ as a unit and plot the points:
(9, 7), (0, 6), (– 5, 8), (6, 0), (– 7, – 4). (0, 7), (– 8, – 5), (– 7, + 8), (9, – 6). (17, – 1), (– 20, 4), (0, 0), (13, 0), (0, – 13), (9, – 9), (– 9, – 9), (0, 11).



3. Take 1″ as a unit and plot the points :
(1, 0), (0, – 1), (– 1, – 1), (– 1, 0), (1·3, 1·5), (3·2, – ·6), (3·6, – 1·2), (– 1·4, 0), (0, – ·5), (·6, – ·5).

4. Draw on squared paper taking ·1″ as a unit, the figures whose vertices are the points :

(i) (0, 7), (5,0), (– 5, 0) ;
(ii) (5, 9), (– 11, 6), (7, – 4) ;
(iii) (6, 8), (– 6, 10), (– 5, – 7), (12, – 9) ;
(iv) (6, 10), (– 7, 10), (– 12, – 4), (14, – 4) ;
(v) (3, 6), (– 7, 7), (– 12, – 3), (– 4, – 6), (9, – 8).

MULTIPLICATION

8. Law of signs. (*a*) Let us consider the changes in the annual saving of a municipality by the arrival and

departure of taxpayers and orphans, the former paying and the latter receiving Rs. 8 each annually.

(i) If five new taxpayers come in, the municipal saving is increased by Rs. 40.

(ii) If five taxpayers go away, the saving is decreased by Rs. 40.

(iii) If five new orphans come in, the saving is decreased by Rs. 40.

(iv) If five orphans go away, the saving is increased by Rs. 40.

If a tax of Rs. 8 be represented by $+8$, an increase of Rs. 40 in the saving would be represented by $+40$; a payment of Rs. 8 would be represented by -8, and a decrease of Rs. 40 in the saving would be represented by -40. If the coming of five men be represented by $+5$, the going away of five men would be represented by -5.

With such notation, the above four processes stand thus:

$$(+8)\times(+5)=+40 \quad . \quad . \quad . \quad \text{(i)}$$
$$(+8)\times(-5)=-40 \quad . \quad . \quad . \quad \text{(ii)}$$
$$(-8)\times(+5)=-40 \quad . \quad . \quad . \quad \text{(iii)}$$
$$(-8)\times(-5)=+40 \quad . \quad . \quad . \quad \text{(iv)}$$

(*b*) Let us consider the position of a ship, taking the equator as the starting place, the degrees north of the equator positive, the degrees south of the equator negative, the days to come positive, and the days passed away negative.

(i) Suppose a ship sailing *north* at the rate of 3° per day is at the equator, 5 days hence it will be 15° *north* of the equator, and 5 days back it was 15° *south* of the equator:

or,
$$(+3)\times(+5)=+15 \quad . \quad . \quad . \quad \text{(i)}$$
$$(+3)\times(-5)=-15 \quad . \quad . \quad . \quad \text{(ii)}$$

(ii) Suppose a ship sailing *south* at the rate of 3° per day is at the equator, 5 days hence it will be 15° *south* of the equator and 5 days back it was 15° *north* of the equator:

or,
$$(-3)\times(+5)=-15 \quad . \quad . \quad . \quad \text{(iii)}$$
$$(-3)\times(-5)=+15 \quad . \quad . \quad . \quad \text{(iv)}$$

From these and similar examples, we generalise the process as given on the next page:

$$(+a)\times(+b) = +ab \quad . \quad . \quad . \quad \text{(i)}$$
$$(-a)\times(+b) = -ab \quad . \quad . \quad . \quad \text{(ii)}$$
$$(+a)\times(-b) = -ab \quad . \quad . \quad . \quad \text{(iii)}$$
$$(-a)\times(-b) = +ab \quad . \quad . \quad . \quad \text{(iv)}$$

***Law.* The product of two quantities with like signs is positive.**

The product of two quantities with unlike signs is negative.

NOTE. (i) $(+a)\times 0 = +(a\times 0) = +0 = 0$,

(ii) $0\times(+a) = +(0\times a) = +0 = 0$,

(iii) $(-a)\times 0 = -(a\times 0) = -0 = 0$,

(iv) $0\times(-a) = -(0\times a) = -0 = 0$.

Hence, **if one factor is zero the product is zero.**

Graphic Illustrations

Convention. An area is usually considered positive, if it is on our left-hand side, when we go round it. (See fig. 1.)

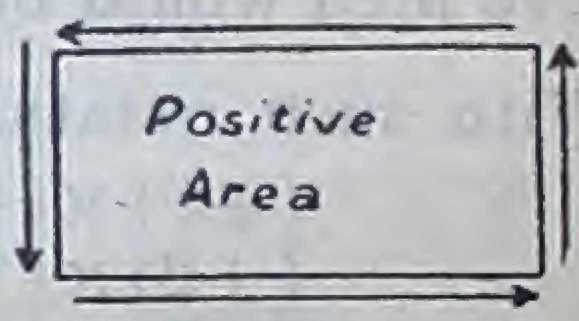


Fig. 1.

An area is usually considered negative, if it is on our right-hand side when we go round it. (See fig. 2.)

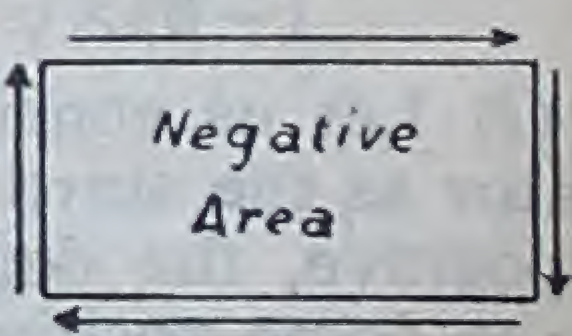
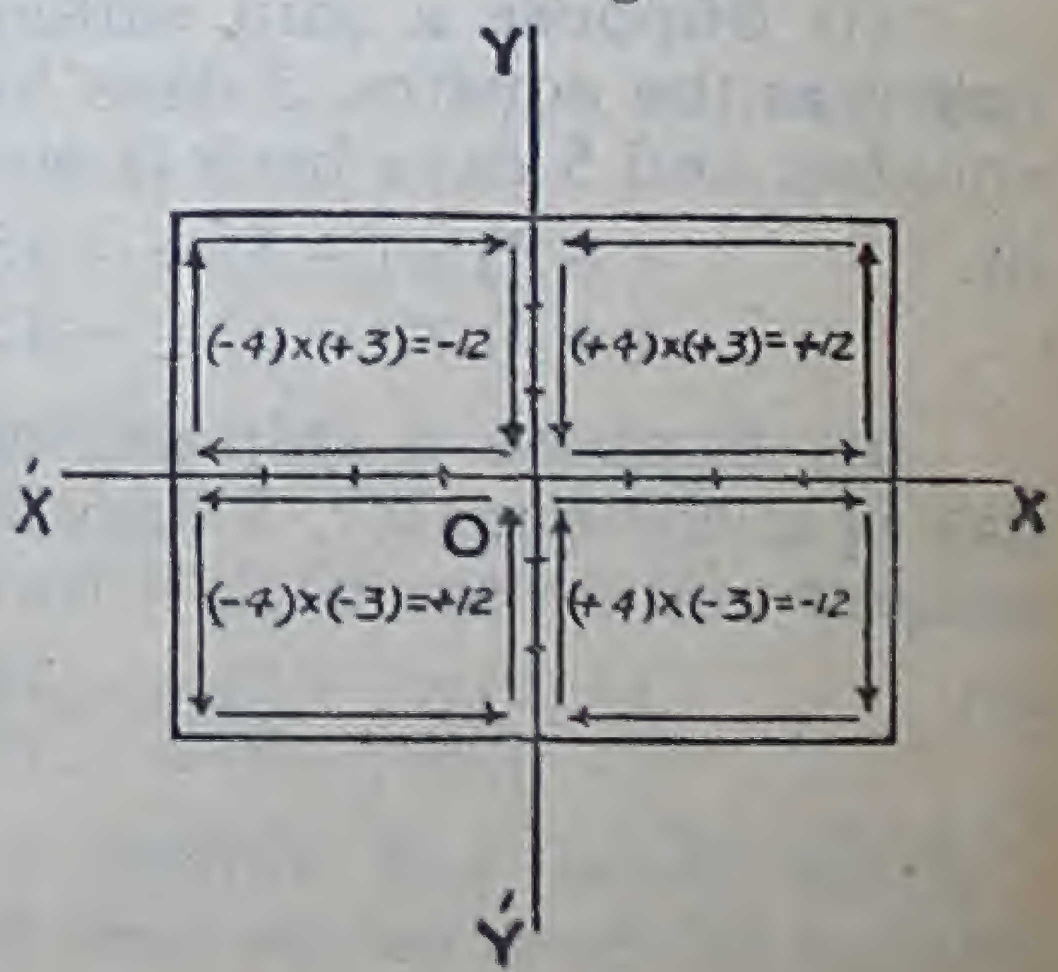


Fig. 2.

With this convention, we can easily illustrate the law of sign *graphically*, as discussed below:



XX' and YY' are the axes and O is the origin. In each quadrant a rectangle is constructed whose sides are 4 units and 3 units. In each case

we start from the origin, move along the x-axis and then complete the circuit. It is obvious that

(i) the sides of the rectangle in the first quadrant are $(+4)$ and $(+3)$, and its area is $(+12)$;

(ii) the sides of the rectangle in the second quadrant are (-4) and $(+3)$, and its area is (-12);

(iii) the sides of the rectangle in the third quadrant are (-4) and (-3), and its area is $(+12)$;

(iv) the sides of the rectangle in the fourth quadrant are $(+4)$ and (-3), and its area is (-12).

EXERCISE 13.

1. Multiply (*orally*):

(i) $(+5)$ by $(+7)$, (ii) $(+7)$ by (-2),
(iii) (-9) by $(+5)$, (iv) (-8) by (-4),
(v) 12 by -5, (vi) -15 by 8,
(vii) $+5a$ by $+8$, (viii) $+7m$ by -4,
(ix) $-8m$ by $+4$, (x) $-7m$ by -8,
(xi) $6a$ by -12, (xii) 12 by $-2m$,
(xiii) -8 by 0, (xiv) 0 by -6 (xv) $+3a$ by 0.

2. Multiply (*orally*):

(i) $4\times0\times(-3)$, (ii) $(+4m)\times0\times(-3m)$,
(iii) $(+2a)\times(-3b)\times0$, (iv) $0\times(-7p)\times(-2q)$,
(v) $(+5)\times(+5)$, (vi) $(-5)\times(-5)$,
(vii) $(-a)\times(-a)$, (viii) $(-b)\times(-b)\times(-b)$,
(ix) $(-c)\times(-c)\times(-c)\times(-c)$,
(x) $(-p)\times(-p)\times(-p)\times(-p)\times(-p)$,
(xi) $(-1)\times(-1)$, (xii) $(-1)\times(-1)\times(-1)$,
(xiii) $(-1)\times(-1)\times(-1)\times(-1)$,
(xiv) $(-1)\times(-1)\times(-1)\times(-1)\times(-1)$.

3. Show that:

(i) $(+a)^2=+a^2$ and $(-a)^2=+a^2$,
(ii) $(+x)^3=+x^3$ and $(-x)^3=-x^3$,
(iii) $(+p)^4=+p^4$ and $(-p)^4=+p^4$,

(iv) $(+m)^5=+m^5$ and $(-m)^5=-m^5$,

(v) $(+1)^2=+1$ and $(-1)^2=+1$,

(vi) $(+1)^3=+1$ and $(-1)^3=-1$,

(vii) $(+1)^4=+1$ and $(-1)^4=+1$,

(viii) $(+1)^5=+1$ and $(-1)^5=-1$.

4. What rule do you deduce from the above eight examples? Apply that rule to find out the values of the following:

(i) $(-1)^{57}$, (ii) $(-1)^{64}$, (iii) $(-a)^{83}$, (iv) $(-a)^{94}$, (v) $(+p)^{43}$, (vi) $(+m)^{54}$, (vii) $(+n)^{71}$.

5. $8x=0$, what is the value of x?

6. $4(x-3)=0$, what is the value of $x-3$?

7. Find the value of $3x^3-2x^2y-5xy^2+y^3$ by substituting for x and y the values given in the table:

x	-3	-2	$+1$	-1	-1	0
y	$+2$	-3	-2	-1	0	-1

8. Find the value of:

(i) $5x^3-3x^2+4x+1$,

(ii) $-3x^4+x^3+2x^2-x+4$,

(iii) $-4x^3-2x^2+6x-9$,

when x has values -2, $+3$, -1, 0.

9. Find the product of:

(i) $(-2y)^3$, $(3y)^4$, $(-y)^5$,

(ii) $(-a)^6$, $(-2ab^2)^2$, $(-a^2b)^3$.

10. Simplify $4x^2(x^2-3xy+2y^2)-2xy(3x^2-2xy+y^2)-y^2(2x^2-xy-y^2)$.

9. Just as $(a+b)\times 2=(a+b)+(a+b)$

$=a+b+a+b=2a+2b,$

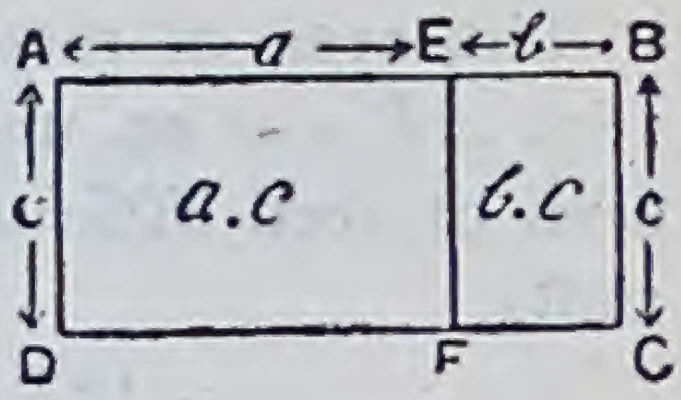
and $(a-b)\times 2=(a-b)+(a-b)$

$=a-b+a-b=2a-2b,$

so $(a+b)\times c = ac+bc,$

and $(a-b)\times c = ac-bc.$

Graphic Illustrations. (i) Take a rectangle $ABCD$.



$AB=(a+b)$ units, $AD=c$ units, $AE=a$ units, and $EB=b$ units.

Draw EF parallel to AD.

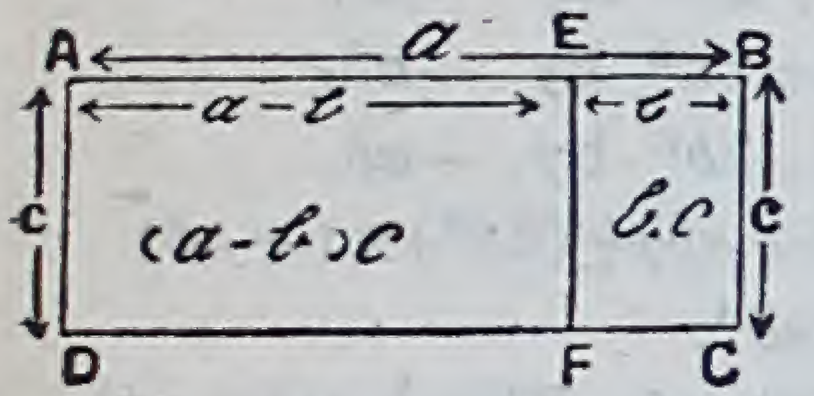
The whole rectangle AC

$=$ rect. $AF+$ rect. EC.

$\because$ The area of rectangle $AC=(a+b)c$ sq. units, the area of rectangle $AF=ac$ sq. units and that of rectangle $EC=bc$ sq. units,

$\therefore$ $(a+b)c=ac+bc.$

(ii) Take a rectangle $ABCD$.



$AB=a$ units, $EB=b$ units, and $AD=c$ units.

$\therefore$ $AE=(a-b)$ units,

The rect. $AF=$ rect. $AC-$ rect. EC.

$\because$ The area of rect. $AF=(a-b)c$ sq. units, the area of rect. $AC=ac$ sq. units, and that of rect. $EC=bc$ sq. units,

$\therefore$ $(a-b)c=ac-bc.$

Rule. **The product of a compound expression and a single factor is found by multiplying in succession each term of the expression by that factor.**

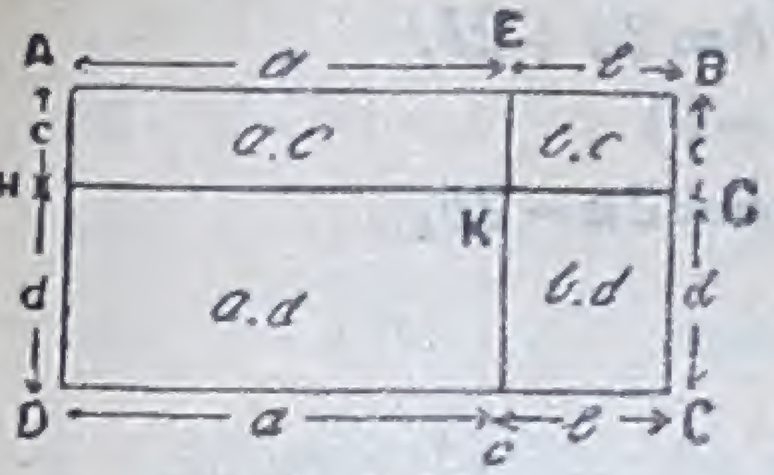
Since $(a+b)\ m=am+bm,$

if we substitute $c+d$ for m, we have

$$(a+b)\ (c+d)=a(c+d)+b(c+d)$$

$$=ac+ad+bc+bd.$$

Graphic Illustration. Take a rectangle $ABCD$ in which



$AE=a$ units, $EB=b$ units, $AH=c$ units, and $HD=d$ units.

Through E and H draw straight lines parallel to the opposite sides, intersecting at K.

Rect. $AC=$ rect. $AK+$ rect. $HF+$ rect. $EG+$ rect. KC.

The area of rect. $AC=(a+b)(c+d)$ sq. units, the area of rect. $AK=ac$ sq. units, the area of rect. $HF=ad$ sq. units, the area of rect. $EG=bc$ sq. units, and that of rect. $KC=bd$ sq. units.

$$\therefore \quad (a+b)(c+d)=ac+ad+bc+bd.$$

***Rule.* Multiply each term of the first expression by each term of the second and add all the partial products thus obtained.**

Example 1. Multiply (i) $3a^2+2b^2$ by $+ab$,
(ii) $3a^2-2b^2$ by $+ab$,
(iii) $3a^2+2b^2$ by $-ab$,
(iv) $3a^2-2b^2$ by $-ab$.

(i) $(3a^2+2b^2)\times(+ab)=(3a^2)\times(+ab)+(2b^2)\times(+ab)$
$=3a^3b+2ab^3.$

(ii) $(3a^2-2b^2)\times(+ab)=\{(+3a^2)+(-2b^2)\}\times(+ab)$
$=(+3a^2)\times(+ab)+(-2b^2)\times(+ab)$
$=3a^3b-2ab^3.$

(iii) $(3a^2+2b^2)\times(-ab)=(3a^2)\times(-ab)+(+2b^2)\times(-ab)$
$=-3a^3b-2ab^3.$

(iv) $(3a^2-2b^2)\times(-ab)=\{(+3a^2)+(-2b^2)\}\times(-ab)$
$=(+3a^2)(-ab)+(-2b^2)(-ab)$
$=-3a^3b+2ab^3$

Example 2. Multiply (i) $(3x+2y)$ by $(3x+4y)$,
(ii) $(3x+2y)$ by $(3x-4y)$,
(iii) $(3x-2y)$ by $(3x-4y)$.

(i) $(3x+2y)\times(3x+4y)=(3x+2y)(+3x)+(3x+2y)(+4y)$

$$=9x^2+6xy+12xy+8y^2$$
$$=9x^2+18xy+8y^2.$$

The process is usually arranged thus :

$$\begin{array}{lr} & 3x+2y \\ & 3x+4y \\ \hline (3x+2y)\times(+3x)= & 9x^2+6xy \\ (3x+2y)\times(+4y)= & +12xy+8y^2 \\ \hline \text{Product} = & 9x^2+18xy+8y^2 \end{array}$$

After some practice, each partial product is written with like terms in columns and the left-hand side of the process is omitted, as illustrated below :

(ii)
$$\begin{array}{r} 3x+2y \\ 3x-4y \\ \hline 9x^2+6xy \\ -12xy-8y^2 \\ \hline 9x^2-\ 6xy-8y^2 \end{array}$$

(iii)
$$\begin{array}{r} 3x-2y \\ 3x-4y \\ \hline 9x^2-6xy \\ -12xy+8y^2 \\ \hline 9x^2-18xy+8y^2 \end{array}$$

EXERCISE 14.

1. Multiply (*orally*):

(i) $(4a+5b)\times(+3)$, (ii) $(5x+7y)\times(-3)$,
(iii) $(2m-3n)\times(+7)$, (iv) $(3m-5n)\times(-7)$,
(v) $(3p-2q)\times(+4p)$, (vi) $(3p-2q)\times(-5q)$,
(vii) $(2a+b-3c)\times(+4c)$, (viii) $(2a-b-2c)\times(-5c)$.

Multiply :

2. $(x+2)\times(x+3)$. **3.** $(x+5)\times(x+2)$.

4. $(a+b)\times(b+c)$. **5.** $(a+b)\times(a+b)$.

6. $(2a+b)\times(3a+2b)$. **7.** $(2x+3y)\times(x+y)$.

8. $(a^2-ab)\times(ab-b^2)$. **9.** $(x^3-10)\times(-x^3+10)$.

10. $(-x^2+2xy)\times(3xy-y^2)$. **11.** $(x^2+xy)\times(-x-y)$.

12. $(2x^2+5ab)\times(2x^2-5ab)$. 13. $(3m^2-2mn)\times(n^2-3mn)$.
14. $(-\frac{1}{2}+5x)\times(\frac{1}{4}x-3)$. 15. $(-2a^2-\frac{1}{4})\times(5a-\frac{1}{5})$.

Example 3. Multiply x^2+xy-y^2 by x^2-xy+y^2.

$$\begin{array}{lllll}
x^2 & +xy & -y^2 & & \\
x^2 & -xy & +y^2 & & \\
\hline
x^4 & +x^3y & -x^2y^2 & & \\
 & -x^3y & -x^2y^2 & +xy^3 & \\
 & & +x^2y^2 & +xy^3 & -y^4 \\
\hline
x^4 & & -x^2y^2 & +2xy^3 & -y^4
\end{array}$$

Before the actual process of multiplication, it is generally useful to arrange the multiplicand and the multiplier according to descending or ascending powers of some common letter.

Example 4. Multiply $1+a^2+a$ by $-a+1+a^2$.

(i) Arranging according to decending powers of a,

$$\begin{array}{lllll}
a^2 & +a & +1 & & \\
a^2 & -a & +1 & & \\
\hline
a^4 & +a^3 & +a^2 & & \\
 & -a^3 & -a^2 & -a & \\
 & & +a^2 & +a & +1 \\
\hline
a^4 & & +a^2 & & +1
\end{array}$$

(ii) Arranging according to ascending powers of a,

$$\begin{array}{lllll}
1 & +a & +a^2 & & \\
1 & -a & +a^2 & & \\
\hline
1 & +a & +a^2 & & \\
 & -a & -a^2 & -a^3 & \\
 & & +a^2 & +a^3 & +a^4 \\
\hline
1 & & +a^2 & & +a^4
\end{array}$$

Example 5. Multiply $3a^2b^2-ab^3+2a^4-a^3b$ by $-2ab+b^2-3a^2$.

Arranging according to descending powers of a,

$$\begin{array}{lllllll}
2a^4 & -a^3b & +3a^2b^2 & -ab^3 & & & \\
-3a^2 & -2ab & +b^2 & & & & \\
\hline
-6a^6 & +3a^5b & -9a^4b^2 & +3a^3b^3 & & & \\
 & -4a^5b & +2a^4b^2 & -6a^3b^3 & +2a^2b^4 & & \\
 & & +2a^4b^2 & -a^3b^3 & +3a^2b^4 & -ab^5 & \\
\hline
-6a^6 & -a^5b & -5a^4b^2 & -4a^3b^3 & +5a^2b^4 & -ab^5 &
\end{array}$$

Multiply:

16. $1+a+a^2$ by $1-a$. 17. $1-a+a^2$ by $1+a$.
18. a^2+ab+b^2 by $a+b$. 19. a^2-ab+b^2 by $a-b$.
20. $a^4+a^2b^2+b^4$ by a^2-b^2. 21. $a^4-a^2b^2+b^4$ by a^2+b^2.

22. $7a^2-12ab+9b^2$ by $3a^2-4b^2$.
23. $\frac{3}{2}x^2+xy-\frac{2}{3}y^2$ by $\frac{1}{2}x-\frac{1}{3}y$.
24. $6x^2-3xy+4y^2$ by $2x^2+5xy-3y^2$.
25. $7x^3-2x^2y+xy^2-4y^3$ by $3x^2-4xy+y^2$.
26. x^2+xy+y^2 by x^2-xy+y^2.
27. $x^3+y^3+z^3-3xyz$ by $x+y+z$.
28. $x^3+y^3-z^3+3xyz$ by $x+y-z$.
29. $x^4-x^3y+x^2y^2-xy^3+y^4$ by $2x-3y$.
30. $x^2-2xy+y^2+z^2$ by $x^2-2xy+y^2-z^2$.

Find the product of :

31. a^3-3a^2+3a-1 and $1-2a+a^2$.
Check the result by putting $a=2$.
32. $3a+2a^2+1$ and $1+2a^2-3a$.
Check the result by putting $a=-3$.
33. $1+a+a^3+a^2+a^4$ and $1-a$.
34. $a^2+b^2+c^2-ab-ac-bc$ and $b+a+c$.
35. $y^2+z^2+xy+x^2-yz+xz$ and $x-z-y$.
36. $1+x^2+y^2-xy+x+y$ and $y+x-1$.
Check the result by putting $x=2$ and $y=-1$.
37. $xy+yz-zx-y^2$ and $x^2+xy-yz-z^2$.
38. $1+4x^2+6x+2x^3+x^4$ and $1+4x^2-6x-2x^3+x^4$.
39. $\frac{1}{2}ab+\frac{1}{3}b^2-\frac{1}{4}a^2$ and $\frac{1}{2}ab+\frac{1}{3}a^2-\frac{1}{4}b^2$.
40. $6a^2x^2-4ax^3-4a^3x+x^4+a^4$ and a^2+x^2-2ax.
41. $1+3x^6-2x^5-x+x^3$ and $3x-x^3-1+4x^2$.
42. $3a^2+a^6+1-5a-a^3$ and $2a^5-1+a^2-a^3$.

Example 6. Find the continued product of
$a-b$, $a+b$, a^2+b^2 and a^4+b^4.

Here we multiply $a-b$ by $a+b$, the product by a^2+b^2, and that product by a^4+b^4.

(i)
$$\begin{array}{r} a-b \\ a+b \\ \hline a^2-ab \\ +ab-b^2 \\ \hline a^2 \quad\quad -b^2 \end{array}$$

(ii)
$$\begin{array}{r} a^2-b^2 \\ a^2+b^2 \\ \hline a^4-a^2b^2 \\ +a^2b^2-b^4 \\ \hline a^4 \quad\quad -b^4 \end{array}$$

(iii)
$$\begin{array}{r} a^4-b^4 \\ a^4+b^4 \\ \hline a^8-a^4b^4 \\ +a^4b^4-b^8 \\ \hline a^8 \quad\quad -b^8 \end{array}$$

Find the continued product of :

43. $1+a$, $1-a$, $1+a^2$.
44. $a+1$, $a+2$, $a+3$.
45. $a-2$, $a+3$, $a+5$.
46. $x+a$, $x+b$, $x+c$.
47. $(x+y)$, $(x+2y)$, $(x+3y)$.
48. $(a+b)$, $(a+b)$, $(a+b)$.
49. $(a-b)$, $(a-b)$, $(a-b)$.
50. $(x+y)$, $(x-y)$, $(x^4+x^2y^2+y^4)$.
51. (a^2+ab+b^2), (a^2-ab+b^2), $(a^4-a^2b^2+b^4)$.
52. $(a-b)$, (a^2+ab+b^2), (a^3+b^3), (a^6+b^6).
53. $(a+b+c)$, $(a-b+c)$, $(a+b-c)$, $(-a+b+c)$.

Find the square of :

54. $a+b+c$.
55. $a-b+c$.
56. $2a-3b+4c$.

Find the cube of :

57. $a+b+c$.
58. $a-b+c$.

Example 7. Find by inspection, the co-efficient of x^2 in the product of $(3x^2-5x+1)$, $(2x-5)$.

Here we have to collect the terms containing x^2 from the partial products.

$2x \times 3x^2$ gives us a term of the 3rd degree, hence neglected.

$2x \times (-5x) = -10x^2$.

$2x \times 1$ gives us a term of the first degree, hence neglected.

$-5 \times 3x^2 = -15x^2$.

$-5 \times (-5x)$ gives us a term of the first degree, hence neglected.

-5×1 is a numerical term, hence neglected.

Thus the term containing x^2 is $\{(-10x^2)+(-15x^2)\}$ or $-25x^2$ and its co-efficient is -25.

NOTE.—Generally the above process is done mentally and only the terms containing the required power of x are taken down.

Example 8. If $A=x+y$ and $B=x-y$ find the value of

$$(3A+B)(2A-B).$$

$$3A+B=3(x+y)+(x-y)$$
$$=3x+3y+x-y=4x+2y.$$

$$2A-B=2(x+y)-(x-y)$$
$$=2x+2y-x+y=x+3y.$$
$$\therefore (3A+B)(2A-B)=(4x+2y)(x+3y)$$
$$=4x^2+2xy+12xy+6y^2$$
$$=4x^2+14xy+6y^2.$$

Find by inspection the co-efficient of :

59. x^2 in the product of (x^2-5x+2), $(x+4)$.
60. x^2 ,, ,, $(3x^2+2x-5)$, $(7x-4)$.
61. x ,, ,, $(2x^2+x-1)$, $(x-3)$.
62. x ,, ,, $(6x^2+xy-y^2)$, $(2x-y)$.
63. a^2 ,, ,, $(4a^2-7a+3)$, $(3a+5)$.
64. a ,, ,, $(a-4)$, $(2a^2-9a+3)$.
65. x^2 ,, ,, $(1-3x)$, $(4-5x^2+6x)$.

Find the term containing :

66. x^2 in the continued product of $(x-1)$, $(x+2)$, $(x-3)$.
67. x ,, ,, ,, $(2x+5)$, $(3x-4)$, $(2x-1)$.
68. x ,, ,, ,, $(4x-5)$, $(x+1)$, $(x-2)$.
69. y ,, ,, ,, $(1-3y+3y^2)$, $(2+3y)$.
70. If $A=x+y$ and $B=x-y$, find the value of $3A^2+4AB+5B^2$.
71. Find the value of $(A+B)(A-B)$ when $A=(2x-y)$ and $B=(x-2y)$.

*72. Evaluate $P^2+2PQ+Q^2$ when $P=(x+a)^2$, $Q=(x-a)^2$.

73. Simplify $(2x+3)(3-4x)-(4+3x)(2x-3)$.
74. Simplify $\dfrac{3x+6}{4}\cdot\dfrac{2x+15}{3}-\dfrac{5x-12}{6}\cdot\dfrac{x-2}{2}$.

*75. Find the value of
$$(2x+1)^3-3(x+2)(x+1)^2+4(x+1)(x-2)^2.$$

76. Simplify $(1+x)(1+x+x^2)-(1-x)(1-x+x^2)$.
77. Simplify
$$2(3x^2+4x-1)(7x^2-5)-3(4x^2+x+1)(x^2-3x-1).$$

*78. Multiply $x^{m+n}+2y^{m-n}$ by $x^{m+n}-y^{m-n}$.

*79. Multiply $2x^2+3x+1$ by $3x^2+x+2$, and hence find the product of 231 and 312.

*80. Find the co-efficient of x^3 in the product of
ax^3+bx^2+cx+d and px^2-qx+r.

81. If $x=a^2-bc$, $y=b^2-ca$, $z=c^2-ab$, find the values of (i) $ax+by+cz$, (ii) $bx+cy+az$, (iii) $cx+ay+bz$.

DIVISION

10. Just as in Arithmetic $5\times 3=15$ and $15\div 5=3$, similarly, in Algebra if $a\times b=c$, then $c\div a=b$. When division is exact, as $15\div 5=3$ or $c\div a=b$, then obviously

Dividend = Divisor × Quotient.

Def. When the product of two quantities is unity, each is called the **reciprocal** of the other; for example, as $a\times\frac{1}{a}=1$, a is the reciprocal of $\frac{1}{a}$ and $\frac{1}{a}$ is the reciprocal of a.

From the definition, we have

$$1=\frac{1}{n}\times n \qquad \ldots \qquad \ldots \qquad \ldots \qquad \text{(i)}$$

Multiplying both sides of (i) by m, we have

$$m\times 1=m\times\frac{1}{n}\times n,$$

or $$m=m\times\frac{1}{n}\times n. \qquad \ldots \qquad \ldots \qquad \ldots \qquad \text{(ii)}$$

Dividing both sides of (ii) by n, we have

$$m\div n=m\times\frac{1}{n}\times n\div n.$$

But $n\div n=1$

$$\therefore \quad m\div n=m\times\frac{1}{n}\times 1$$

$$=m\times\frac{1}{n}.$$

Or, to divide by a quantity is the same thing as to multiply by its reciprocal.

Thus division is the *inverse* of multiplication.

Cor. $1 \div a = 1 \times \frac{1}{a} = \frac{1}{a}$, and $\frac{1}{a}$ means $1 \div a$.

Just as in Arithmetic $105 \times 7 \div 5 = 105 \div 5 \times 7$
and $105 \div 7 \div 5 = 105 \div 5 \div 7$,
so, in Algebra $a \times b \div c = a \div c \times b$
and $a \div b \div c = a \div c \div b$.

***Law.* The order of operations in a chain of multiplication and division is immaterial.**

Just as in Arithmetic $105 \times (5 \div 7) = 105 \times 5 \div 7$
and $105 \div (5 \div 7) = 105 \div 5 \times 7$,
so, in Algebra $a \times (b \div c) = a \times b \div c$
and $a \div (b \div c) = a \div b \times c$.

***Law.* When a bracket contains the operations of multiplication and division only, it may be removed, every sign remaining unchanged if $\times$ precedes the bracket and every sign reversed if $\div$ precedes the bracket.**

Just as in Arithmetic $(15 + 12 + 9) \div 3 = (15 \div 3) + (12 \div 3) + (9 \div 3)$
so, in Algebra $(a + b + c) \div d = (a \div d) + (b \div d) + (c \div d)$.

Thus, when a multinomial expression is to be divided by a monomial:

divide each term of the dividend by the divisor and take the sum of the partial quotients.

Since $+ab = (+a) \times (+b)$

$$\therefore \quad \frac{+ab}{+a} = \frac{(+a) \times (+b)}{+a} = +b \quad . \quad . \quad \text{(i)}$$

Since $-ab = (-a) \times (+b)$

$$\therefore \quad \frac{-ab}{-a} = \frac{(-a) \times (+b)}{-a} = +b \quad . \quad . \quad \text{(ii)}$$

In (i) and (ii) the dividend and the divisor have *like signs* and the quotient is *positive*.

Again, $\qquad +ab = (-a) \times (-b)$

$$\therefore \qquad \frac{+ab}{-a} = \frac{(-a) \times (-b)}{-a} = -b \quad . \quad . \qquad \text{(iii)}$$

Also, $\qquad -ab = (+a) \times (-b)$

$$\therefore \qquad \frac{-ab}{+a} = \frac{(+a) \times (-b)}{+a} = -b \quad . \quad . \qquad \text{(iv)}$$

In (iii) and (iv) the dividend and the divisor have *unlike signs* and the quotient is *negative*.

Law. **Like signs produce + ; unlike signs produce −.**

Example 1. Divide $-15a^6b^4$ by $5a^3b^2$.

$$\begin{aligned} &(-15a^6b^4) \div (5a^3b^2) \\ &\qquad = (-15 \div 5) \times (a^6 \div a^3) \times (b^4 \div b^2) \\ &\qquad = -3a^{6-3}b^{4-2} = -3a^3b^2. \end{aligned}$$

After some practice this process should be performed *mentally*.

Example 2. Divide $x^6 - 5ax^5 + 3a^2x^4$ by $-2x^4$.

$$\begin{aligned} \text{The quotient } &= \frac{x^6 - 5ax^5 + 3a^2x^4}{-2x^4} \\ &= \frac{x^6}{-2x^4} - \frac{5ax^5}{-2x^4} + \frac{3a^2x^4}{-2x^4} \\ &= -\tfrac{1}{2}x^2 + \tfrac{5}{2}ax - \tfrac{3}{2}a^2. \end{aligned}$$

EXERCISE 15.

Divide (*mentally*):

1. $9x^5$ by $3x^3$. $\qquad$ 2. $18a^2b^4$ by $-6ab^2$.
3. $-12a^4b^5$ by $-4a^3b^3$. $\qquad$ 4. $-15a^3b^2$ by $10a^2b^2$.
5. $-18a^3x^3b^3y^4$ by $-6a^3x^3b^2y^3$.
6. $-35a^3b^3c^4x^4y^3z^3$ by $-5a^3b^3x^3z^3$.
7. $-m^4n^3p^4q^3$ by $m^2n^2p^2q^2$.
8. $-14a^3b^5c^4$ by $+2a^2b^2c^2$.
9. $a^3m^2np^5$ by $-6a^2mp^3$. $\qquad$ 10. $-\frac{3}{5}p^3q^4$ by $-\frac{9}{20}p^2q^3$.
11. $\frac{4}{7}a^3b^4c^5m^4$ by $-\frac{8}{21}a^3m^3b^2c^2$.
12. $-\frac{5}{9}a^3b^2c^2x^4$ by $\frac{25}{3}a^2b^3x^2$.

Divide:

13. $-abcd+axcm$ by $-ac$.
14. $6abxy-4acxz+2abcxyz$ by $12ax$.
15. $-a^2b^2c-ab^2c^2-a^2bc^2$ by $-abc$.
16. $25a^4x^5-20a^3x^4-35a^2x^3$ by $5a^2x^2$.
17. $18a^4b^5x^3-12a^2b^3x^4+6a^3b^4x^5$ by $-6a^2b^3x^3$.
18. $12a^4b^2c-18a^2b^4c-36ab^2c^4$ by $-6ab^2c$.
19. $-\frac{2}{3}abc^3+\frac{3}{4}ab^3c-\frac{2}{5}a^3bc$ by $-\frac{1}{60}abc$.
20. $-\frac{1}{2}a^2b^3c^4+\frac{2}{3}b^2c^3a^4-\frac{3}{4}c^2a^3b^4$ by $-\frac{1}{12}a^2b^2c^2$.

Division of one compound expression by another is analogous to 'Long Division' in Arithmetic, as illustrated below in the division of 483 by 23:

Compact process.

$$\begin{array}{r|l} 23 & 483\,(21 \\ & \underline{46} \\ & 23 \\ & \underline{23} \end{array}$$

Expanded process.

$$\begin{array}{r|l} 2.10+3 & 4.10^2+8.10+3\,(2.10+1 \\ & \underline{4.10^2+6.10} \\ & \qquad\qquad 2.10+3 \\ & \qquad\qquad \underline{2.10+3} \end{array}$$

In dividing $4x^2+8x+3$ by $2x+3$ we can proceed in exactly the same way.

Thus
$$\begin{array}{r|l} 2x+3 & 4x^2+8x+3\,(2x+1 \\ & \underline{4x^2+6x} \\ & \qquad\quad 2x+3 \\ & \qquad\quad \underline{2x+3} \end{array}$$

Explanation. The first term of the dividend is divided by the first term of the divisor. Thus $4x^2\div 2x=2x$. This gives us the first term of the quotient. The whole divisor is multiplied by $2x$ and the product is subtracted from the dividend. Thus we have the remainder $2x+3$. We treat it as the new dividend and divide its first term by the first term of the divisor. Thus $2x\div 2x=1$. This gives us the second term of the quotient. The whole divisor is multiplied by 1 and the product is subtracted from $2x+3$. There being no remainder, the process is complete, and the quotient is $2x+1$.

Example 1. Divide $4x^3+7x^2-3x-15$ by $4x-5$.

$$
\begin{array}{r|l|l}
4x-5 & 4x^3+7x^2-3x-15 & x^2+3x+3 \\
 & \underline{4x^3-5x^2} & \\
 & \quad 12x^2-3x & \\
 & \quad \underline{12x^2-15x} & \\
 & \qquad 12x-15 & \\
 & \qquad \underline{12x-15} &
\end{array}
$$

The required quotient is x^2+3x+3.

In this example the dividend and the divisor are already arranged in descending powers of x; however, if they are not arranged in descending or ascending powers of a common letter, we have to arrange them so, before doing the actual process.

Example 2. Divide $2-3x^2+x-x^3+x^4$ by x^2+2-3x.

Arrange the expressions according to descending powers of x.

$$
\begin{array}{r|l|l}
x^2-3x+2 & x^4-x^3-3x^2+x+2 & x^2+2x+1 \\
 & \underline{x^4-3x^3+2x^2} & \\
 & \quad 2x^3-5x^2+x & \\
 & \quad \underline{2x^3-6x^2+4x} & \\
 & \qquad x^2-3x+2 & \\
 & \qquad \underline{x^2-3x+2} &
\end{array}
$$

The required quotient $=x^2+2x+1$.

Thus the process of Algebraic long division can be stated as follows :

Rule. **(i) Arrange the dividend and the divisor in descending or ascending powers of a common letter.**

(ii) Divide, the term on the left of the dividend by the term on the left of the divisor. The result is the first term of the quotient.

(iii) Multiply the whole divisor by this quotient and subtract the product from the dividend.

(iv) Bring down to the remainder as many terms of the dividend as may be necessary to form a new dividend, and go on repeating the above steps till all the terms of the dividend have been used.

Divide:

21. $x^2+8x+15$ by $x+5$.
22. $x^2+2x-15$ by $x-3$.
23. $x^2-8x+15$ by $x-5$.
24. $x^2-14x+49$ by $x-7$.
25. $2x^2-3x-2$ by $2x+1$.
26. $10x^2-14x-12$ by $2x-4$.
27. $3m^2+m-2$ by $m+1$.
28. $3a^2+10a+3$ by $3a+1$.
29. $6a^2-7a-3$ by $3a+1$.
30. $4p^2-4p-3$ by $2p-3$.
31. $5m^2-17m+6$ by $5m-2$.
32. $-21a^2+a+10$ by $-7a+5$.
33. $1-x+x^2-x^3$ by $1+x^2$.
34. $3a-3a^2+a^3-1$ by $a-$
35. $3x^3-2y^3+9xy^2-11x^2y$ by $3x-2y$.
36. $3a^3-a^2b-2b^3-5ab^2$ by $2b+3a$.
37. $6a^4-a^3+5a-6+4a^2$ by $3a^2-2+a$.
38. $a^4-4a^3-18a^2-11a+2$ by a^2-7a+1.
39. $12a^4+a^3-8a^2+7a-2$ by $3a^2-2a+1$.
40. $2a^4+a^2-3a^3-9+9a$ by $2a^2-3+a$.
41. $4x-1-8x^2-6x^4+11x^3$ by $1-x+3x^2$.
42. $3-a^2-14a-4a^3+a^4$ by $a+3+a^2$.
43. $1+a+a^2+a^3+a^4+a^5$ by $1+a^2+a^4$.
44. $p^4-3p^3q+pq^3+3p^2q^2-6q^4$ by $p^2-pq-2q$
45. $3x^5+3x^4+2x^3+1$ by $3x^3-x+1$.

Example 3. Divide $1-a^4$ by $1-a$.

$$
\begin{array}{r|l}
1-a & 1-a^4 \quad (1+a+a^2+a^3 \\
 & \underline{1-a} \\
 & \quad a-a^4 \\
 & \quad \underline{a-a^2} \\
 & \qquad a^2-a^4 \\
 & \qquad \underline{a^2-a^3} \\
 & \qquad\quad a^3-a^4 \\
 & \qquad\quad \underline{a^3-a^4}
\end{array}
$$

The required quotient $=1+a+a^2+a^3$.

Example 4. Divide $a^4+a^2b^2+b^4$ by a^2-ab+b^2.

$$
\begin{array}{r|l}
a^2-ab+b^2 & a^4+a^2b^2+b^4 \quad (a^2+ab+b^2 \\
 & \underline{a^4-a^3b+a^2b^2} \\
 & \quad a^3b+b^4 \\
 & \quad \underline{a^3b-a^2b^2+ab^3} \\
 & \qquad a^2b^2-ab^3+b^4 \\
 & \qquad \underline{a^2b^2-ab^3+b^4}
\end{array}
$$

The required quotient $=a^2+ab+b^2$.

Divide:

46. x^3+y^3 by $x+y$.
47. x^5+y^5 by $x+y$.
48. x^5-y^5 by $x-y$.
49. x^8-y^8 by $x+y$.
50. x^9-1 by x^2+x+1.
51. $a^3+b^3+c^3-3abc$ by $a+b+c$.
52. x^6-1 by x^3+2x^2+2x+1.
53. x^6-2x^3+1 by x^2-2x+1.
54. x^9+y^9 by x^3+y^3.
55. $x^3+3xy+y^3-1$ by $x+y-1$.

    Check the result by putting $x=2$, $y=1$.
56. x^6-y^6 by $x^3+2x^2y+2xy^2+y^3$.
57. $x^{12}-2x^6y^6-3y^{12}$ by $x^4-x^2y^2+y^4$.
58. $a^4-\frac{1}{81}b^4$ by $a-\frac{1}{3}b$.

    Check the result by putting $a=-1$ and $b=-2$.
59. $x^4-\frac{5}{4}x^3+\frac{11}{8}x^2-\frac{1}{2}x$ by $x^2-\frac{1}{2}x$.
60. $\frac{1}{4}x^2-\frac{1}{9}y^2-\frac{1}{36}z^2+\frac{1}{9}yz$ by $\frac{1}{2}x-\frac{1}{3}y+\frac{1}{6}z$.
61. $x^4+y^4-z^4+2x^2y^2-2z^2-1$ by $x^2+y^2-z^2-1$.
62. $a^3-b^3-c^3-3abc$ by $a^2+b^2+c^2+ab+ac-bc$.
63. $x^3+8y^3-27z^3+18xyz$ by $x-3z+2y$.
64. $a^3+8b^3+27c^3-18abc$ by $a^2+4b^2+9c^2-2ab-6bc-3ca$.
65. $x^8+x^6y^2+x^4y^4+x^2y^6+y^8$ by $x^4-x^3y+x^2y^2-xy^3+y^4$.
66. $x^{3n}-y^{3n}$ by x^n-y^n.

INEXACT DIVISION

Example 5. Divide $x^2+7x+12$ by $x+2$.

$$
\begin{array}{r|l|l}
x+2 & x^2+7x+12 & x+5 \\
 & x^2+2x & \\ \hline
 & 5x+12 & \\
 & 5x+10 & \\ \hline
 & 2 &
\end{array}
$$

It is an instance of inexact division, where 2 is left as the remainder and $x^2+7x+12=(x+2)(x+5)+2$. In all similar cases of inexact division, if we denote the dividend by D, the divisor by d, the quotient by Q and the remainder by R, we have

$$D=dQ+R,$$

and the complete quotient $=Q+\dfrac{R}{d}$.

Example 6. Divide $x^4-3x^3-13x^2+12x+4$ by x^2-x+2.

First method. Arrange the divisor and the dividend according to *descending powers of x.*

$$
\begin{array}{r|l}
x^2-x+2 & x^4-3x^3-13x^2+12x+4\;(x^2-2x-17 \\
 & \underline{x^4-x^3+2x^2} \\
 & \quad -2x^3-15x^2+12x \\
 & \quad \underline{-2x^3+2x^2-4x} \\
 & \qquad\quad -17x^2+16x+4 \\
 & \qquad\quad \underline{-17x^2+17x-34} \\
 & \qquad\qquad\quad -x+38
\end{array}
$$

The quotient $=x^2-2x-17$ and the remainder $=-x+38$.

The complete quotient $=x^2-2x-17+\dfrac{-x+38}{x^2-x+2}$.

Second method. Arrange the divisor and the dividend according to *ascending powers of x.*

$$
\begin{array}{r|l}
2-x+x^2 & 4+12x-13x^2-3x^3+x^4\;(2+7x-4x^2-7x^3 \\
 & \underline{4-2x+2x^2} \\
 & \quad 14x-15x^2-3x^3 \\
 & \quad \underline{14x-7x^2+7x^3} \\
 & \qquad -8x^2-10x^3+x^4 \\
 & \qquad \underline{-8x^2+4x^3-4x^4} \\
 & \qquad\qquad -14x^3+5x^4 \\
 & \qquad\qquad \underline{-14x^3+7x^4-7x^5} \\
 & \qquad\qquad\qquad -2x^4+7x^5
\end{array}
$$

The quotient $=2+7x-4x^2-7x^3$ and the remainder $=-2x^4+7x^5$.

The complete quotient $=2+7x-4x^2-7x^3+\dfrac{-2x^4+7x^5}{2-x+x^2}$.

NOTE. It may be observed that the results in *inexact division* by the above two methods are different.

Divide and give the complete quotient in each case :

67. $a^2+b^2+2ab+3$ by $1+a+b$.
68. $a^3+7a^2+21a+17$ by a^2+5a+6.
69. $a^4+7a^3+14a^2+9a+4$ by a^2+3a+1.
70. $2a^4+3ab^3-4a^2b^2+a^3b-5b^4$ by a^2-2b^2+ab.

Divide up to 4 terms; give the remainder and the complete quotient in each case :

71. 1 by $1+x$. **72.** $1-2a$ by $1+3a$. **73.** $1-a$ by $1-2a+2a^2$.

Prove that :

74. $\dfrac{a^2+11a+35}{a+5}=a+6+\dfrac{5}{a+5}$.

75. $\dfrac{x^2-9x+25}{x-4}=x-5+\dfrac{5}{x-4}$.

76. $\dfrac{a^2-13a+45}{a-7}=a-6+\dfrac{3}{a-7}$.

77. $\dfrac{x^2-16x+60}{x-9}=x-7-\dfrac{3}{x-9}$.

78. $x^6-6x^5+15x^4-20x^3+15x^2-6x+1$ is the product of two expressions of which one is x^2-2x+1. Find the other.

79. Find the divisor when the dividend is $x^3-6x^2+12x-5$, the quotient $x-3$ and the remainder $x+1$.

80. What must be subtracted from $12a^2+8a-9$, so that it may be exactly divisible by $2a-1$?

81. What must be added to $3a^3-8a^2+7a$, so that it may be exactly divisible by $3a-2$?

82. What must be added to $3a^3+7a^2-6a-5$, so that it may be exactly divisible by a^2+2a-1?

***83.** Find the condition that x^4-3x^2+m may be exactly divisible by x^2+1?

***84.** For what value of k is $9a^3-6a^2+3a-k$ exactly divisible by a^2-2a+3?

***85.** For what value of x will $2x^3+5x^2-mx+4$ be exactly divisible by x^2+2x-1?

***86.** Find a and b in order that $x^4-2x^3+8x^2+ax+b$ may be exactly divisible by x^2+x+5.

CHAPTER III

SIMPLE EQUATIONS AND PROBLEMS

1. If we *simplify* $(3x+4)+2(x-1)$, the result is $5x+2$,

or $$(3x+4)+2(x-1)=5x+2.$$

Here, if we put $x=1$,

the left-hand side $=(3.1+4)+2(1-1)=7$

and the right-hand side $=5.1+2=7$.

Thus the left-hand side $=$ the right-hand side.

Again, if we put $x=2$,

the left-hand side $=(3.2+4)+2(2-1)=12$

and the right-hand side $=5.2+2=12$.

Thus the left-hand side $=$ the right-hand side.

If in the above we substitute 3, 4, 5, 6,...*any number* for x, the left-hand side is always $=$ the right-hand side.

Def. When one expression is equal to another expression for all values of the letter or letters involved (or all values of the quantities used), it is said to form an **Identical Equation** or simply an **Identity.**

Now if we substitute in $(3x+4)+2(x-1)=4x+7$, different values of x, say 1, 2, 3, 4, 5, 6,...etc., we find that the left-hand side $=$ the right-hand side *only when* $x=5$ and for no other values of x. In this case the equality is obviously *not universal* but *conditional* and holds good only when $x=5$. Such an equality is called **Conditional Equation** or simply an **Equation.**

Def. When one expression is equal to another expression for particular value or values of the quantity or quantities used, an **Equation** is formed.

The sign $=$ is used for an equation and $\equiv$ for an identity.

It is useful to note the effect of ordinary processes of addition, subtraction, multiplication or division, as illustrated below in the four examples:

(i) When we add x^2+xy+y^2 and x^2-xy+y^2, the result is $2x^2+2y^2$ for all values of x and y,
or $(x^2+xy+y^2)+(x^2-xy+y^2)\equiv 2x^2+2y^2$.

(ii) When we subtract x^2-xy+y^2 from x^2+xy+y^2, the result is $2xy$ for all values of x and y,
or $(x^2+xy+y^2)-(x^2-xy+y^2)\equiv 2xy$.

(iii) When we multiply x^2+xy+y^2 and x^2-xy+y^2, the result is $x^4+x^2y^2+y^4$ for all values of x and y
or $(x^2+xy+y^2)(x^2-xy+y^2)\equiv x^4+x^2y^2+y^4$,

(iv) When we divide $x^4+x^2y^2+y^4$ by x^2+xy+y^2 the result is x^2-xy+y^2 for all values of x and y,
or $(x^4+x^2y^2+y^4)\div(x^2+xy+y^2)\equiv x^2-xy+y^2$.

Example 1. Prove the identity
$(x+y)^2-4xy\equiv(x-y)^2$ and verify it when
(i) $x=3$ and $y=1$, (ii) $x=2$ and $y=1$, (iii) $x=4$ and $y=2$.

The left-hand side $=x^2+2xy+y^2-4xy=x^2-2xy+y^2$ and the right-hand side $=x^2-2xy+y^2$.

Since by *expanding*, both sides can be transformed into the same expression, $\therefore$ the identity is true.

(i) When $x=3$ and $y=1$,
the left-hand side $=(3+1)^2-4.3.1=16-12=4$
and the right-hand side $=(3-1)^2=2^2=4$.

(ii) When $x=2$ and $y=1$,
the left-hand side $=(2+1)^2-4.2.1=9-8=1$
and the right-hand side $=(2-1)^2=1^2=1$.

(iii) When $x=4$ and $y=2$,
the left-hand side $=(4+2)^2-4.4.2=36-32=4$
and the right-hand side $=(4-2)^2=2^2=4$.

Thus in each case, the left-hand side = the right-hand side.

EXERCISE 16.

Prove the following identities and verify each when
(i) $a=4$ and $b=2$, (ii) $a=3$ and $b=1$, (iii) $a=5$ and $b=3$:

1. $(a+b)^2\equiv a^2+2ab+b^2$. 2. $(a-b)^2\equiv a^2-2ab+b^2$.

3. $(a+b)(a-b)\equiv a^2-b^2$. 4. $(a+b)^2-(a-b)^2\equiv 4ab$.

5. $(a+b)^2+(a-b)^2\equiv 2(a^2+b^2)$.

6. $(a+b)(a^2-ab+b^2)\equiv a^3+b^3$.

7. $(a-b)(a^2+ab+b^2)\equiv a^3-b^3$.

8. $(a+3b)^2+(a-3b)^2\equiv 2(a+3b)(a-3b)+36b^2$.

In a conditional equation like $5x+2=4x+7$, the letter x *stands for a number* whose value is not known, and this x is generally called the **unknown.** Any particular value of the unknown which makes the two sides of an equation equal is said to **satisfy** the equation and is called the **root** or the **solution** of the equation. To *solve* an equation is to find its root.

When an equation, reduced to its simplest form, contains no power of the unknown higher than the first, it is called a **Simple Equation.**

Find by *inspection* the values of x which satisfy the following equations :

9. $6x=12$. 10. $8x=32$. 11. $4x=1$. 12. $5x=0$.

13. $x+5=16$. 14. $x+7=18$. 15. $\frac{x}{2}=4$. 16. $\frac{x}{3}=6$.

17. $x+3=7$. 18. $x-3=6$. 19. $x-5=8$. 20. $x-5=0$.

21. $x-7=0$. 22. $x+4=0$. 23. $x+6=0$. 24. $x+8=0$.

25. $\frac{2x}{3}=6$, find $2x$ and then x.

26. $\frac{3x}{5}=12$, find $3x$ and then x.

27. $\frac{x+1}{2}=4$, find $x+1$ and then x.

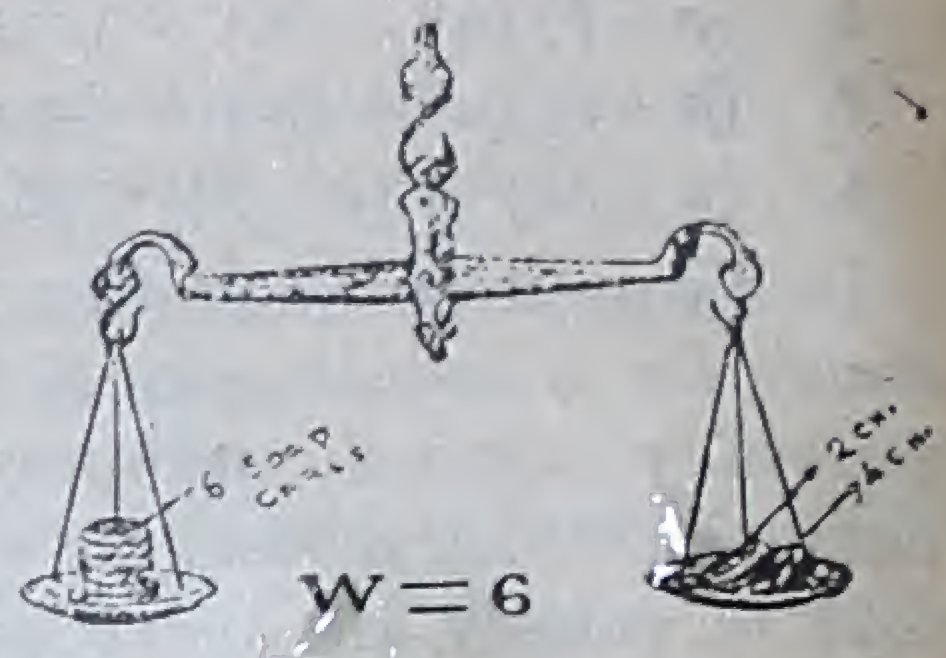
28. $\frac{x-1}{3}=6$, find $x-1$ and then x.

29. $ax=c$, find x. 30. $\frac{x}{a}=b$, find x.

31. $x+a=b$, find x. 32. $x-a=b$, find x.

2. Methods of solving simple equations.

An *equation* may be compared to a *balance*. An equation has two sides connected by the sign of equality ; a balance has two scale-pans connected by a beam. The two sides of an equation are equal, and the two scale-pans, with equal weights, keep the beam horizontal.



Suppose there are a few soap cakes in the left-hand scale-pan, and they are balanced by 6 chattak weights placed in the right-hand scale-pan.

Representing it symbolically, we have

$$w=6.$$

Now, if a certain amount of soap (say 2 ch.) be added to or removed from the left-hand scale-pan, weights equal to that amount must be added to or removed from the right-hand scale-pan to keep the beam horizontal.

Representing it symbolically, we have

$$w+2=6+2 \quad \ldots \quad \text{(i)}$$

and

$$w-2=6-2 \quad \ldots \quad \text{(ii)}$$

If the amount of soap be doubled or trebled, the weights will have to be doubled or trebled to keep the beam horizontal.

Representing it symbolically, we have

$$\left.\begin{aligned} w\times 2&=6\times 2 \\ \text{and} \quad w\times 3&=6\times 3 \end{aligned}\right\} \quad \ldots \quad \text{(iii)}$$

Again, if the amount of soap be halved or reduced to one-third, the weights will have to be halved or reduced to one-third to keep the beam horizontal.

Representing it symbolically, we have

and
$$\left.\begin{aligned}\frac{w}{2}&=\frac{6}{2}\\ \frac{w}{3}&=\frac{6}{3}\end{aligned}\right\} \quad . \quad . \quad . \quad \text{(iv)}$$

From these illustrations and from the process of oral exercises from 9 to 32 done above it is clear that the following axioms are very useful in the solution of equations:

If equals be added to equals, the sums are equal.

[Axiom I]

If equals be subtracted from equals, the remainders are equal.

[Axiom II]

If equals be multiplied by equals, the products are equal.

[Axiom III]

If equals be divided by equals, the quotients are equal.

[Axiom IV]

Example 1. Solve the equation $x-7=10$.

Adding 7 to both sides, we have

$$x-7+7=10+7 \quad . \quad . \quad \text{[Axiom I]}$$

$$\therefore \quad x=17.$$

Example 2. Solve the equation $x+9=13$.

Subtracting 9 from both sides, we have

$$x+9-9=13-9 \quad . \quad . \quad \text{[Axiom II]}$$

$$\therefore \quad x=4.$$

Example 3. Solve the equation $\frac{x}{5}=4$.

Multiplying both sides by 5, we have

$$\frac{x}{5}\times 5=4\times 5 \quad . \quad . \quad \text{[Axiom III]}$$

$$\therefore \quad x=20.$$

Example 4. Solve the equation $3x=12$.

Dividing both sides by 3, we have

$$\frac{3x}{3}=\frac{12}{3} \quad . \quad . \quad \text{[Axiom IV]}$$

$$\therefore \quad x=4.$$

Example 5. Solve the equation $3y+2=17$.

Subtracting 2 from both sides, we have

$$3y+2-2=17-2 \quad . \quad . \qquad [\text{Axiom II}]$$

$$\therefore \quad 3y=15.$$

Dividing both sides by 3, we have

$$\frac{3y}{3}=\frac{15}{3} \quad . \quad . \qquad [\text{Axiom IV}]$$

$$\therefore \quad y=5.$$

Example 6. Solve the equation $\frac{z}{4}-6=10$.

Adding 6 to both sides, we have

$$\frac{z}{4}-6+6=10+6 \quad . \quad . \qquad [\text{Axiom I}]$$

$$\therefore \quad \frac{z}{4}=16.$$

Multiplying both sides by 4, we have

$$\frac{z}{4}\times 4=16\times 4 \quad . \quad . \qquad [\text{Axiom III}]$$

$$\therefore \quad z=64.$$

EXERCISE 17.

Solve the following equations and do *full process* :

1. $3x=18$.	2. $5x=35$.	3. $7x=21$.
4. $11x=44$.	5. $\frac{y}{4}=7$.	6. $\frac{y}{6}=3$.
7. $\frac{y}{8}=5$.	8. $\frac{y}{12}=4$.	9. $\frac{1}{3}z=6$.
10. $\frac{1}{5}z=2$.	11. $x-5=13$.	12. $x-3=12$.
13. $x-9=4$.	14. $x-14=3$.	15. $x-7=6$.
16. $x-2=9$.	17. $x+4=11$.	18. $x+6=15$.
19. $x+8=17$.	20. $x+10=21$.	21. $2y+3=15$.
22. $3z+4=22$.	23. $5x+2=27$.	24. $4x+5=21$.

25. $3y-1=8$.
26. $7y-6=15$.
27. $6z-3=33$.
28. $5z-7=38$.
29. $3x-5-x=13$.
30. $5x-9-2x=12$.
31. $2x+7+x=10$.
32. $4x+3+x=15$.
33. $6x-2=3x+7$.
34. $4z+6=2z+11$.
35. $5y-1=2y+9$.
36. $4y+7=y+17$.
37. $\frac{x}{2}-1=2$.
38. $\frac{z}{3}+2=4$.
39. $\frac{z}{6}+\frac{1}{6}=\frac{5}{6}$.
40. $\frac{z}{5}-\frac{1}{5}=\frac{2}{5}$.
41. $\frac{x}{12}+\frac{5}{4}=2$.
42. $\frac{5x}{6}-\frac{1}{4}=\frac{3}{8}$.
43. $\frac{3x}{8}+\frac{1}{6}=\frac{3}{4}$.
44. $\frac{5x}{12}-\frac{3}{8}=\frac{1}{2}$.
45. $5x-8+3x=12-4x$.
46. $17x-3+2x=7x+18+5x$.
47. $2x-(x+4)=9$.
48. $5y-(3y-2)=22$.
49. $17x+(8x-11)=39$.
50. $z-(6-4z)+5=14$.
51. $7y+(5y-9)-6=21$.
52. $4x-(x+5)+3=0$.
53. $9x+(12-3x)-26=0$.
54. $2y-(7-5y)-24=0$.

3. Transposition. Consider the equation

$$2x-7=5 \quad . \quad . \quad . \quad . \quad \text{(i)}$$

Adding 7 to both sides, we have

$$2x-7+7=5+7 \quad . \quad . \quad . \quad \text{[Axiom 1]}$$

$$\therefore \quad 2x=5+7 \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

Comparing (i) and (ii), we notice that -7 *disappears* from the left-hand side and *appears* on the right-hand side *with its sign changed.*

Again, let us consider the equation

$$5x=15+2x \quad . \quad . \quad . \quad \text{(iii)}$$

Subtracting $2x$ from both sides, we have

$$5x-2x=15+2x-2x \quad \text{[Axiom II]}$$

$$\therefore \quad 5x-2x=15 \quad . \quad . \quad . \quad . \quad \text{(iv)}$$

Comparing (iii) and (iv), we notice that $+2x$ *disappears* from the right-hand side and *appears* on the left-hand side *with its sign changed.*

By taking a few more similar equations and proceeding as above, we notice that the process is based on the application of addition and subtraction axioms but for all practical purposes it assumes the form of the following rule :—

We may transpose a term from one side of an equation to the other, if at the same time we change the sign of the term from + to – or from – to +, as the case may be.

For the sake of convenience this process is called the process of **transposition.**

Abbreviations. **R.H.S.** stands for the *right-hand side* and **L.H.S.** stands for the *left-hand side.*

Example 7. Solve $11x-5=4x+9$.

Transposing $4x$ and 5, we have

$$11x-4x=9+5$$

$\therefore$ $7x=14$ and $x=2$.

[**Verification.** Put 2 for x in the above equation.

$$\text{L.H.S.}=11\times2-5=17.$$
$$\text{R.H.S.}=4\times2+9=17.$$

Thus L.H.S. = R.H.S., and the solution is correct.]

Example 8. Solve $5x-18-2x=x+2-3x$.

Transposing all the terms containing x to the left-hand side and all other terms to the right-hand side, we have

$$5x-2x-x+3x=+2+18$$

$\therefore$ $5x+3x-3x=20$

$\therefore$ $5x=20$

or $x=4$.

[**Verification.** Put 4 for x in the above equation.

$$\text{L.H.S.}=5\times4-18-2\times4=-6.$$
$$\text{R.H.S.}=4+2-3\times4=-6.$$

Thus L.H.S. = R.H.S., and the solution is correct.]

NOTE. In practice it is *usual* to bring all the terms containing x to the left-hand side and all other terms to the right-hand side.

Example 9. Solve $4(2x+5)-2(x+1)=0$.

Removing the brackets, we have

$$8x+20-2x-2=0$$

or $$6x+18=0.$$

Transposing $+18$, we have

$$6x=-18$$

$\therefore$ $$x=-3.$$

[**Verification.** Put -3 for x in the above equation.

$$\text{L.H.S.}=4\{2\times(-3)+5\}-2(-3+1)$$
$$=4\times(-1)-2\times(-2)=-4+4=0.$$
$$\text{R.H.S.}=0.$$

Thus L.H.S. = R.H.S., and the solution is correct.]

Example 10. Solve $(x-3)(2x-1)=5+(2x+3)(x-2)$.

Simplifying both sides of the equation, we have

$$2x^2-x-6x+3=5+2x^2-4x+3x-6$$

$\therefore$ $2x^2-x-6x-2x^2+4x-3x=5-6-3$ ... [*Transposition.*]

or $$-6x=-4$$

$\therefore$ $$6x=4$$

or $$x=\tfrac{2}{3}.$$

[**Verification.** Put $\frac{2}{3}$ for x in this equation.

$$\text{L.H.S.}=(\tfrac{2}{3}-3)(2\times\tfrac{2}{3}-1)=(-\tfrac{7}{3})(\tfrac{1}{3})=-\tfrac{7}{9}.$$
$$\text{R.H.S.}=5+(2\times\tfrac{2}{3}+3)(\tfrac{2}{3}-2)$$
$$=5+\tfrac{13}{3}\times(-\tfrac{4}{3})=5-\tfrac{52}{9}=-\tfrac{7}{9}.$$

Thus L.H.S. = R.H.S., and the solution is correct.]

NOTE. It will be found useful to remember the following hints:

(i) We may change the signs of *all* the terms on both sides of an equation without altering the equality. Thus, in $-x+5=-8$, if we divide both sides by -1, we get $x-5=8$.

(ii) We may interchange the position of the two sides of an equation without changing the signs.

Thus, $11=x-3$ may be written as $x-3=11$.

(iii) We may cancel the same term from both sides of an equation. Thus $2x-5=6+x-5$ is the same thing as $2x=6+x$, and $17x-4+2x=5x+6+2x$ is the same thing as $17x-4=5x+6$.

In solving the following equations apply the *method of transposition*, and *verify* each solution :

55. $3x-5=2x-3$.
56. $4x-1=6x+5$.
57. $2x+3=x-8$.
58. $5x+7=2x+3$.
59. $11-2x=15-x$.
60. $7+3x=12-2x$.
61. $15+3x=8+2x$.
62. $9-5x=13+3x$.
63. $-4+2x=6-3x$.
64. $12-4x=-5+2x$.
65. $-3y-5=-7y+1$.
66. $-5+7x=-11-2x$.
67. $3(5x+6)=4(2-x)$.
68. $9x+20-4x=45$.
69. $3+3x-4=5x-1$.
70. $7x-6+x=16+3x$.
71. $-5x+2=8x-20-4x$.
72. $5x+23+11x=-8x+46+x$.
73. $5x-9+3x-7=0$.
74. $3x-6+5x-12=0$.
75. $9x-8+2x-4-8x+10=0$.
76. $4x+20-8(13-x)=0$.
77. $6x-2(4-3x)=7-3(17-x)$.
78. $3(x-5)-7(2x+3)=0$.
79. $4(2x-3)+5(x-2)=0$.
80. $2(3x-1)-6(2x-5)=0$.
81. $-7(12+3x)+5(x-2)=0$.
82. $-3(5-4x)+2(3x-1)=0$.
83. $6-\{2x-(3x-4)-1\}=0$.
84. $2x-(3-5x)+2=4x-7$.
85. $(2x-7)(3x+1)=(2x+5)(3x+1)$.
86. $(4x-3)(2x-1)=(2x+5)(4x-1)$.
87. $(4x+5)(3x-2)=(6x-1)(2x-5)$.
88. $(x-6)(12x+1)=(3x+2)(4x-1)$.
89. $(3x+2)(8x+5)=(6x-1)(4x+7)$.
90. $-(x-8)(x+12)=(x+1)(6-x)$.
91. $-(7-6x)(3-2x)=-(4x-3)(3x-2)$.
92. $(x+3)^2+(4-x)^2=2x^2+x$.
[**Hint.** $(x+3)^2=(x+3)(x+3)$ and $(4-x)^2=(4-x)(4-x)$].
93. $(x+5)^2-(2-x)^2=7$.

94. $(2x+1)^2-8=(2x-1)^2$.

95. $14-5(x-4)(x+8)=188-5(x+3)^2$.

96. $9(x-2)^2+3(x-4)^2=(3x-7)(4x-19)+42$.

97. Find the value of x which will make $5(x-4)+2$ equal to 52.

98. Find the value of x which will make $2(x-3)+7$ equai to zero.

99. For what value of m will $(2m-3)(m+4)+5-2m^2$ be equal to zero ?

100. For what values of x are the following statements true :

(i) $(12x+5)(x-3)=(3x+4)(4x+3)+29$?

(ii) $(2x-1)(6x+5)=4(3x-2)(x+1)+3$?

(iii) $(2x-1)(3x-4)=(6x-5)(x-1)$?

EQUATIONS INVOLVING NUMERICAL DENOMINATORS

4. To get rid of the numerical denominators in an equation, multiply both sides of it by the L. C. M. of the denominators.

Example 1. Solve $\dfrac{3x}{4}+\dfrac{5x}{6}=38$.

The L. C. M. of 4 and 6 is 12.

Multiplying both sides of the equation by 12, we have

$$12\times\frac{3x}{4}+12\times\frac{5x}{6}=12\times 38$$

or $$9x+10x=12\times 38$$

or $$19x=12\times 38$$

or $$x=\frac{12\times 38}{19}$$

or $$x=24.$$

[**Verification.** Since $\dfrac{3\times 24}{4}+\dfrac{5\times 24}{6}=18+20=38$, therefore the answer is correct.]

Example 2. Solve $\frac{3x}{5}+\frac{5}{6}=\frac{x}{2}+\frac{3}{4}$.

The L. C. M. of 5, 6, 2 and 4 is 60.

Multiplying both sides by 60, we have

$$36x+50=30x+45$$

$$\therefore \quad 36x-30x=45-50$$

$$\therefore \quad 6x=-5$$

$$\therefore \quad x=-\tfrac{5}{6}.$$

[Verification. L. H. S. $=\frac{3}{5}\times(-\frac{5}{6})+\frac{5}{6}=-\frac{1}{2}+\frac{5}{6}=\frac{1}{3}$.

R. H. S. $=\frac{1}{2}\times(-\frac{5}{6})+\frac{3}{4}=-\frac{5}{12}+\frac{3}{4}=\frac{1}{3}$.

Thus L. H. S. = R. H. S., and the solution is correct.]

Example 3. Solve $\frac{2x+7}{5}-6=\frac{3x+4}{10}+8$.

By transposition, we have

$$\frac{2x+7}{5}-\frac{3x+4}{10}=8-6=2.$$

Multiplying both sides by 10, the L. C. M. of 5 and 10, we have $2(2x+7)-(3x+4)=20.$

Removing the brackets, we have

$$4x+14-3x-4=40$$

or $$+10=20$$

or $$x=10.$$

[Verification. L. H. S. $=\frac{2\times 10+7}{5}+6=11\frac{2}{5}$.

R. H. S. $=\frac{3\times 10+4}{10}+8=11\frac{2}{5}$.

Thus L. H. S. = R. H. S., and the solution is correct.]

Example 4. Solve $\cdot 4x - \cdot 35 = \cdot 34x + \cdot 01$.

Expressing the decimals as vulgar fractions, we have

$$\frac{2x}{5} - \frac{7}{20} = \frac{17x}{50} + \frac{1}{100}.$$

The L.C.M. of 5, 20, 50, 100 is 100.

Multiplying both sides by 100, we have

$$40x - 35 = 34x + 1$$

$\therefore$ $\quad 40x - 34x = 35 + 1$

or $\quad 6x = 36$

and $\quad x = 6.$

Another method

$$\cdot 4x - \cdot 35 = \cdot 34x + \cdot 01.$$

By transposition, we have

$$\cdot 4x - \cdot 34x = \cdot 01 + \cdot 35$$

or $\quad \cdot 06x = \cdot 36$

or $\quad x = 6.$

EXERCISE 18.

Solve the following equations and *verify* each solution :

1. $\frac{x}{2} + \frac{x}{3} = 5.$ 2. $\frac{x}{4} - \frac{x}{6} = 3.$

3. $\frac{x}{2} + \frac{2x}{3} = \frac{7}{2}.$ 4. $\frac{5x}{6} - \frac{x}{4} = 14.$

5. $\frac{x}{2} - 5 = 7 - \frac{5x}{6}.$ 6. $\frac{2x}{5} - \frac{3}{4} = \frac{x}{2} - \frac{5}{6}.$

7. $\frac{3}{x} = 2.$ [*Hint.* Multiply both sides by x.]

8. $\frac{1}{x} - \frac{1}{6} = \frac{1}{8}.$ 9. $\frac{3x+1}{5} = \frac{2x-1}{11}.$

10. $\frac{y-3}{5} + \frac{y-5}{3} = 0.$ 11. $\frac{3y-2}{7} - \frac{5y-1}{14} = 0.$

12. $\frac{x}{2} - 3 = \frac{3x}{4} - \frac{5x}{6} + 7.$ 13. $\frac{6x}{5} + \frac{3x}{4} - 11 = \frac{3x}{20} - 18 + \frac{5x}{2}.$

14. $\frac{4x-1}{5}=\frac{3x+1}{4}-1.$ 15. $\frac{x-1}{14}+\frac{x-2}{21}=\frac{x-3}{7}.$

16. $\frac{2x+3}{11}+2=\frac{2x-1}{22}+\frac{x+3}{6}.$

17. $3+\frac{5x-8}{16}=\frac{x+4}{8}+6\frac{1}{2}.$ 18. $12+\frac{x+8}{6}=4+\frac{3x-16}{8}.$

19. $\frac{2x-1}{3}-\frac{1}{4}=x-\frac{x+1}{2}.$ 20. $\cdot 5x+1\cdot 2x=3\cdot 4.$

21. $1\cdot 32x+\cdot 02x=1\cdot 19+x.$ 22. $\cdot 2y+\cdot 05=\cdot 12+\cdot 15y.$

23. $\frac{\cdot 24-\cdot 5x}{6}=\frac{\cdot 3x+1}{4}.$ 24. $\frac{\cdot 3x-1}{5}=\frac{\cdot 6-\cdot 2x}{3}.$

25. $x-\frac{2x-\cdot 3}{\cdot 7}=\frac{\cdot 5-x}{\cdot 35}.$ 26. $1\cdot 2x-\frac{\cdot 16x-\cdot 05}{\cdot 5}=\cdot 4x.$

27. $\frac{x}{\cdot 5}-\frac{1}{\cdot 05}+\frac{x}{\cdot 005}-\frac{1}{\cdot 0005}=0.$

28. $\frac{x+\cdot\dot{3}}{1\cdot\dot{6}}+\frac{2x-\cdot\dot{6}}{1\cdot\dot{3}}=6x+2\cdot 5.$

29. $\frac{1}{2}(x-2)+\frac{1}{3}(x-3)=\frac{1}{4}(x-4).$

30. $\frac{7x-3}{2}+1=\frac{3x+5}{4}+\frac{7x-2}{3}.$

31. $\frac{2x-1}{2}+\frac{3x-4}{6}=\frac{6+x}{4}-\frac{x+1}{9}.$

32. $\frac{5+3x}{5}-\frac{3x-1}{2}=\frac{3+5x}{3}+\frac{2x+1}{4}.$

33. $\frac{x+2\frac{1}{2}}{15}+\frac{x+3\frac{1}{3}}{25}=\frac{x+4\frac{1}{6}}{55}.$

34. $\frac{1}{3}(x-4)-\frac{1}{6}(2x+3)=\frac{7(5x+6)}{9}$ correct up to 2 decimal places.

35. What value of x will make $\frac{x}{2}+\frac{x}{3}+\frac{x}{4}-2$ equal to x?

5. Change of subject in a formula.

Let us consider the formula

$$I=\frac{PRT}{100} \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(i)}$$

Here the subject of the formula is I and it can be determined if P, R and T be known.

If this formula be reduced to the form

$$100\times I=PRT,$$

we can deduce from it

$$P=\frac{100\times I}{RT} \quad . \quad . \quad . \quad . \qquad \text{(ii)}$$

$$R=\frac{100\times I}{PT} \quad . \quad . \quad . \quad . \quad . \qquad \text{(iii)}$$

$$T=\frac{100\times I}{PR} \quad . \quad . \quad . \quad . \quad . \qquad \text{(iv)}$$

In (ii) the subject of the formula is P and it can be determined if I, R and T be known.

In (iii) the subject of the formula is R and it can be determined if I, P and T be known.

In (iv) the subject of the formula is T and it can be determined if I, P and R be known.

The process illustrated above is called **changing the subject** of a formula.

Example 1. A rectangular room is l ft. long, b ft. broad and h ft. high. If the area of the four walls is A sq. ft., we know that $A=2h(l+b)$.

Here A is the subject of the formula ; change the formula so that l may become the subject.

$$A=2h\,(l+b)$$

$$\therefore \qquad 2h\,(l+b)=A.$$

Divide each side by $2h$,

$$\therefore \qquad l+b=\frac{A}{2h}.$$

Subtract b from both sides,

$$l=\frac{A}{2h}-b.$$

EXERCISE 19.

1. If the area of a triangle whose base is b in. and height h in. is A sq. in., $A=\frac{1}{2}bh$.

Make (i) b the subject, (ii) h the subject.

2. If c ft. is the circumference of a circle of radius r ft., $c=2\pi r$. Make r the subject.

3. In a trapezoid a ft. and b ft. are the parallel sides and h ft. is the height. If its area is A sq. ft., $A=\frac{1}{2}(a+b)h$.

Make (i) h the subject.
(ii) $(a+b)$ the subject.

4. The radius of the base of a cylinder is r ft. and its height is h ft. If the area of its curved surface is A sq. ft., $A=2r\pi h$. Make (i) r the subject, (ii) h the subject.

5. If the radius of a circle is r ft. and its area A sq. ft., $A=r^2\pi$. Make r the subject.

6. If e ft. is the edge of a cube and A sq. ft. the area of its six faces, $A=6\cdot e^2$. Make e the subject.

7. If in an isosceles triangle the vertical angle is x degrees and a base angle y degrees, $x+2y=180$.

Make (i) x the subject. (ii) y the subject.

8. If x be increased by $r\%$, the result $y=x\left(1+\frac{r}{100}\right)$

Make r the subject. Find r if $x=20$ and $y=21$.

9. If r ft. is the radius of the base of a cylinder, h ft. its height and V cubic ft. its volume, $V=r^2\pi h$. Make (i) r the subject, (ii) h the subject.

10. Given $v=u+ft$; find (i) f in terms of u, v, t, (ii) t in terms of u, v, f.

11. Given $S=16t^2$. Make t the subject.

12. If a ft. is one side of a right-angled triangle, b ft. the other side and c ft. its hypotenuse, $c^2=a^2+b^2$. Make a the subject.

13. If F degrees Fahrenheit is the same temperature as C degrees Centigrade, $F=32+\frac{9C}{5}$; make C the subject, and find C when $F=50$.

14. If the radius of the base of a circular cone is r ft., its height h ft. and volume V cubic ft., $V=\frac{1}{3}\pi r^2 h$.

Make (i) h the subject, (ii) r the subject.

15. If $s=ut+\frac{1}{2}ft^2$, make f the subject.

16. The temperature t, the pressure p and the volume v of a quantity of gas are connected by the formula $\frac{pv}{t+273}=10$. Make t the subject, and find t if $p=3$, $v=1050$.

Symbolical Expressions

6. Algebra is largely used for the solution of problems of various kinds, but before attempting them, the student must have sufficient practice in expressing the given statements symbolically.

EXERCISE 20. (Oral)

1. What is the sum of 7 and a?
2. What is the sum of m and n?
3. What is the difference between q and 15?
4. What is the difference between p and q?
5. One part of x is 7; what is the other?
6. One part of $2m$ is n; what is the other?

7. One part of $2m+n$ is $m+n$; what is the other?

8. By how much is m greater than 17?

9. By how much is m less than 20?

10. The sum of two numbers is m and one of them is n; what is the other?

11. The sum of two numbers is $2x+19$, one of them is $x+5$; what is the other?

12. The difference between two numbers is d, the greater is 25; what is the other?

13. What is the selling price if the cost price is Rs. 160 and the profit is Rs. x?

14. What is the excess of 53 over n?

15. What is the excess of m over 34?

16. What is the excess of p over q?

17. What is the defect of 19 from 23?

18. What is the defect of m from 40?

19. What is the defect of x from y?

20. What number is less than 60 by m?

21. What number is less than $2m$ by n?

22. By how much does 21 exceed n?

23. By how much does $2p$ exceed q?

24. If n is a factor of 28, what is the other factor?

25. If m is a factor of n, what is the other factor?

26. If m be divided into 5 equal parts, what is the value of each part?

27. If the product of two numbers is $2p^2$ and one of them is p, what is the other?

28. How many times is 8 contained in 48?

29. How many times is a contained in 48?

30. How many times is m contained in n?

31. There are three numbers, each equal to a; what is their product?

32. What is five times x increased by 15?

33. What is seven times n diminished by 12?

34. What is three times p increased by q?

35. What is six times m diminished by n?

36. x is multiplied by 4 and 13 is added to the product; what is the result?

37. y is doubled and then 5 is added to it; what is the result?

38. One-fifth of m is increased by 7; what is the result?

39. One-third of x is diminished by 9; what is the result?

40. Three-fourths of y is increased by 11; what is the result?

41. Five-sixths of m is diminished by 13; what is the result?

42. Write down three consecutive numbers of which 15 is the middle one; of which x is the middle one.

43. Write down three consecutive numbers of which 11 is the least one; of which k is the least one.

44. Write down three consecutive numbers of which 27 is the greatest one; of which p is the greatest one.

45. Write down three consecutive numbers of which $2m+3$ is the middle one.

46. Write down three consecutive even numbers of which $2n+4$ is the middle one.

47. Write down three consecutive even numbers of which $2x-6$ is the least one.

48. Write down three consecutive even numbers of which $2x+2$ is the greatest one.

49. Write down three consecutive odd numbers of which $2n-3$ is the least one.

50. Write down three consecutive odd numbers of which $2m+1$ is the middle one.

51. Write down three consecutive odd numbers of which $2n+5$ is the greatest one.

If the digits of a number beginning from the left-hand side be 3 and 7, the number is 37, and its value is $10 \times 3 + 7$; when the digits of 37 are reversed, it becomes 73, and its value is $10 \times 7 + 3$.

Similarly, if the digits of a number beginning from the left-hand side be x and y, the value of the number $= 10 \times x + y = 10x + y$; when these digits are reversed, the value of the number thus formed $= 10 \times y + x = 10y + x$.

52. Represent the numbers whose digit in the tens place is 8 and the digit in the units place is n.

53. Represent the number whose digit in the tens place is k and the digit in the units place is 4.

54. If the digits of a number beginning from the left be m and n, what is its value?

55. In the preceding question, if the digits be reversed, how would you represent that number?

56. Represent the number whose digit in the tens place is $3p$ and the digit in the units place is $2q$.

57. A man is m years old now; how old was he 16 years back? n years back?

58. A man is 35 years old now; how old will he be 12 years hence? x years hence?

59. How old will a man be k years hence, if he is now y years old?

60. How old is a man now who was 17 years of age 13 years ago? x years ago?

61. How old is a man now who was p years of age 18 years ago? n years ago?

62. A man will be 60 years old in k years; how old is he now?

63. A man will be n years old in m years; how old is he now?

64. The age of a boy is $(m-5)$ years; what was his age k years ago? What will be his age p years hence?

65. A father is 37 years older than his son. The age of the son is $(m-3)$ years; what is the age of the father?

66. A man walks at the rate of x miles an hour; (i) how far does he walk in 3 hours? (ii) in h hours?

67. A man walks at the rate of x miles an hour; (i) how far does he walk in 4 hours? (ii) in t hours?

68. What is the speed per hour of a man who travels (i) 40 miles in 8 hours? (ii) 40 miles in x hours? (iii) m miles in t hours?

69. A train goes at the rate of 28 miles per hour; (i) in what time will it go 280 miles? (ii) m miles?

70. A train goes at the rate of s miles per hour; (i) in what time will it go 250 miles? (ii) m miles?

71. The cost price of an article is $3m$ rupees, and the gain is Rs. 5: find its selling price.

72. The cost price of an article is Rs. $7x$, and the loss is Rs. 9; find its selling price.

73. The selling price of an article is Rs. $2n$, and the loss is Rs. 6; find its cost price.

74. The price of 7 cows is Rs. 350; what is the price of one cow? of x cows?

75. If one cow costs k rupees; how many can be bought for r rupees?

76. What is $12\frac{1}{2}$ per cent. of 40? of x?

77. What is m per cent. of Rs. 250? of Rs. k?

78. A man buys goods for Rs. x, and sells them at a gain of 7 per cent.; what is the selling price?

79. A man buys goods for Rs. $3m$, and sells them at a loss of 12 per cent.; what is the selling price?

80. A man buys goods for Rs. c, and sells them at a gain of p per cent.; what is the selling price?

Solution of Problems

7. The application of equations in the solution of problems is of the utmost importance.

It is usual to take the following steps in the solution of problems:

(i) Read the problem *very carefully* and try to understand the relation between the quantities stated therein. (This step is very important.)

(ii) Represent the unknown quantity by the symbol x.

(iii) State the *conditions* of the problem in *symbolic language*, so as to obtain two expressions which are equal. (This step is very important.)

(iv) Solve the equation thus formed.

(v) Verify the solution.

Example 1. The excess of twice a certain number over 9 is 15; find the number.

Let x be the required number

$\therefore$ twice that number $=2x$

$\therefore$ the excess of twice the number over $9=2x-9$.

As this excess is equal to 15,

$\therefore$ $2x-9=15$

$\therefore$ $2x=24$

and $x=12$.

[**Verification.** Since $2\times 12-9=15$, the answer is correct.]

Example 2. One number is greater than another by 11 and their sum is 49; find them.

Let x be the greater number,

then the smaller number $=x-11$.

As their sum is equal to 49,

$\therefore$ $x+(x-11)=49$.

or $\quad 2x-11=49$

or $\quad 2x=60$

$\therefore \quad x=30$, the greater number

and $\quad x-11=30-11=19$, the smaller number.

[**Verification.** Since $30-19=11$ and $30+19=49$, the answer is correct.]

Example 3. Divide Rs. 64 between A and B, so that three times A's share may be greater than four times B's share by Rs. 10.

Let $\quad x$ be the number of rupees in A's share,

then $\quad (64-x)=$ the number of rupees in B's share.

$\therefore \quad$ three times A's share $=3x$

and $\quad$ four times B's share $=4\,(64-x)$.

According to the condition, $3x$ is greater than $4(64-x)$ by 10.

$\therefore \quad 3x=4(64-x)+10$

or $\quad 3x=256-4x+10$

or $\quad 3x=266-4x$

$\therefore \quad 7x=266$

and $\quad x=38.$

$\therefore \quad 64-x=64-38=26.$

$\therefore \quad A$'s share $=$ Rs. 38 and B's share $=$ Rs. 26.

[**Verification.** Since 3×38 is greater than 4×26 by 10 and $38+26=64$, the answer is correct.]

Example 4. I thought of a number, doubled it, then added 7, the result was 31; what was the number thought of ?

Let x be the number thought of, then the successive steps would be :

$$x,\ 2x,\ 2x+7,$$

$\therefore \quad 2x+7=31$

$\therefore \quad 2x=24$

$\therefore \quad x=12.$

[**Verification** $\because \quad 12\times 2+7=31$, $\therefore$ the answer is correct.]

EXERCISE 21.

Solve the following problems and *verify* the answer in each case :

1. The defect of an angle from 90° is 15° ; find the angle.

2. The excess of an angle over a right angle is 12° ; find the angle.

3. In an isosceles triangle, the vertical angle is one-half of each of the base angles. Find each angle.

4. One angle is one-fifth of another, and their difference is 20° ; find them.

5. A is twice as old as B, and the sum of their ages is 72 years ; find their ages.

6. A is three times as old as B, and the difference of their ages is 30 years ; find their ages.

7. One angle is greater than another by 32°, and their sum is 106° ; find them.

8. The sum of two numbers is 83, and one of them exceeds the other by 17 ; find them.

9. If 28 be added to five times a number, the result is 78 ; find the number.

10. If 7 be taken away from four times a number, the result is 69 ; find the number.

11. If 9 be added to one-third of a number, the result is 19 ; find the number.

12. If 11 be taken away from one-fifth of a number, the result is 23 ; find the number.

13. Think of a number, double it, and add 15. If the result is 71, what was the number thought of ?

14. Think of a number, divide it by 4, and add 9 to it ; the result is 15. What was the number thought of ?

15. Think of a number, take away 5 from it, multiply the result by 4, and add 10 to it ; it becomes equal to 90. Find the original number.

16. Divide Rs. 96 between A and B, so that four times A's share may be greater than three times B's share by Rs. 13.

17. Divide Rs. 150 between A and B, so that five times A's share may be less than six times B's share by 9.

18. A has x apples, B has $x+5$, and C has $x-4$, and they together have 49 apples ; find what each has got.

19. Divide Rs. 81 among A, B and C, so that B may get Rs. 7 more than A, and C 6 less than twice A's share.

20. Divide Rs. 340 among A, B and C, so that B may get Rs. 30 more than A and C Rs. 50 less than twice A's share.

21. Divide Rs. 500 among A, B and C, so that B may get Rs. 60 more than one-half of A's share and C may get Rs. 90 more than one-fourth of A's share.

22. The sum of the interior angles of a figure of n sides is always $(2n-4)$ right angles. If the sum of the interior angles of a figure is 16 right angles, how many sides has it ?

Example 5. Divide 180 into two parts, so that one-third of one part may be greater than one-fourth of the other by 18.

Let x be one of the parts ; then $180-x$ would be the other part.

One-third of the first part $= \frac{x}{3}$.

One-fourth of the second part $= \frac{1}{4}(180-x)$.

According to the condition given in the problem, we have

$$\frac{x}{3}=\frac{1}{4}(180-x)+18.$$

Multiplying both sides by 12, we have

$$4x=3(180-x)+216$$

or $$4x=540-3x+216$$

or $$7x=756$$

$\therefore$ $x=108$, the first part,

and $(180-x)=180-108=72$, the second part.

[Verification. Since $\frac{1}{3}\times 108 = 36$, and $\frac{1}{4}$ of $72 = 18$, also 36 is greater than 18 by 18, therefore the solution is correct.]

23. What is the number whose one-half, one-third, and one-fourth added together give us 65?

24. What is the number whose sixth part exceeds its seventh part by 10?

25. What is the number whose fourth part falls short of its third part by 5?

26. Divide 81 into two parts, so that five-sixths of the smaller may exceed seven-fifteenths of the larger by 9.

27. A post is one-sixth of its length in mud, one-fourth of its length in water, and 12 ft. above the water; find its length.

28. A man goes one-half of his journey by a railway train, one-fourth of it by a motor-car, one-sixth of it on a horse, and the remaining 10 miles on foot; find the total length of the journey.

29. Out of a cask of wine one-seventh had leaked away, 6 gallons were sold, and then the cask was three-fourths full; find the capacity of the cask.

30. A man leaves one-fourth of his property to his wife, one-sixth of it to each of his two sons, one-eighth of it to each of his two daughters, and the rest, amounting to Rs. 1,200 for a school; find his total property.

31. Find a number whose one-half added to 12 exceeds the sum of its third and fourth parts by 5.

32. One-sixth of a number is taken away from one-half of it; the result is less than the sum of its fourth and eighth parts by 15. Find the number.

33. To a certain number I add one-third of it; the result is as much above 100 as the number is below 103. Find the number.

34. Divide Rs. 860 between A and B so that five-sevenths of A's share may be less than three-fourths of B's share by Rs. 30.

35. The difference between two numbers is 16, and one-sixth of one is less than one-fourth of the other by 5; find the numbers.

36. The sum of two numbers is 678. When the greater is divided by the less, the quotient is 5 and the remainder is 6. Find the numbers.

[*Hint.* Dividend $=$ quotient $\times$ divisor $+$ remainder.]

37. The sum of two numbers is 1064. When the greater is divided by the less, the quotient is 4 and the remainder is 14. Find the numbers.

38. The difference between two numbers is 431. When the greater is divided by the less, the quotient is 3 and the remainder is also 3. Find the numbers.

Example 6. A is 24 years older than B; 10 years back A's age was five times the age of B; find their ages.

Let x years be the age of A,

then the age of B is $(x-24)$ years.

10 years back, the age of A was $(x-10)$ years and that of B was $\{(x-24)-10\}$ years.

According to the condition given in the problem we have

$$x-10=5\{(x-24)-10\}$$

$$\therefore \quad x-10=5(x-34)$$

$$\therefore \quad x-10=5x-170$$

$$\therefore \quad 4x=160$$

$$\therefore \quad x=40, \text{ the age of } A,$$

and $x-24=40-24=16$, the age of B.

[**Verification.** Since $40-16=24$ and $(40-10)$ or 30 is five times $(16-10)$ or 6, therefore the answer is correct.]

39. A's age is three-eighths of B's age; the difference of their ages is 15; find their ages.

40. A father is 20 years older than his son; 12 years back the age of the father was six times that of his son; find their ages.

41. A father is 24 years older than his son; 15 years hence the father will be $\frac{5}{3}$ times as old as his son; find their ages.

42. What is A's present age if he is now three times as old as B, and was 5 times as old as B 8 years back?

43. A father is twice as old as his son; 14 years ago he was three times as old as his son; find their ages.

44. A is 10 years older than B; 12 years ago five-sevenths of A's age exceeded five-eighths of B's age by 10; find their ages.

45. A father is 26 years older than his son; 16 years hence three-fifths of the father's age will exceed two-thirds of his son's age by 14; find their ages.

Example 7. Find three consecutive even numbers whose sum is 96.

Let $2n$ be the middle one of the consecutive even numbers,

then $2n-2$ = the least one of the consecutive even numbers,

and $2n+2$ = the greatest one of the consecutive even numbers.

Since their sum is given equal to 96,

$$\therefore \quad (2n-2)+2n+(2n+2)=96$$

$$\text{or} \quad 6n=96$$

$$\therefore \quad n=16.$$

$\therefore$ $2n-2=30$, the least number.

$2n=32$, the middle number,

and $2n+2=34$, the greatest number.

Example 8. Find two consecutive odd numbers so that one-third of the greater may exceed one-fifth of the less by 8.

Let $2n+1$ be the greater of the consecutive odd numbers,
then $2n-1=$,, lesser ,, ,, ,, ,,

According to the condition given in the problem, we have

$$\frac{1}{3}(2n+1)=\frac{1}{5}(2n-1)+8$$

or $$5(2n+1)=3(2n-1)+120$$

or $$10n+5=6n-3+120$$

or $$4n=112$$

or $$n=28.$$

$\therefore$ $2n+1=57$, the greater consecutive odd number,
and $2n-1=55$, ,, lesser ,, ,, ,,

46. Find three consecutive numbers whose sum is 87.

47. Find four consecutive numbers whose sum is 74.

48. Find three consecutive even numbers whose sum is 48.

49. Find three consecutive odd numbers whose sum is 81.

50. Find two consecutive numbers such that one-third of the greater exceeds one-fifth of the less by 7.

51. Find two consecutive even numbers such that one-eighth of the greater may exceed one-tenth of the less by 3.

52. Find two consecutive odd numbers such that one seventh of the greater may fall short of one-fifth of the less by 4.

53. Find two consecutive numbers such that seven-eighths of the greater may exceed five-sixths of the less by 6.

54. Find two consecutive odd numbers such that two thirds of the greater may exceed three-fifths of the less by 9.

Example 9 There are two digits in a number, the digit in the units place being 8. If 9 be added to the number, the order of the digits is reversed; find the number.

Let $\quad x=$ the digit in the tens place.

$\because \quad 8=$ the digit in the units place,

$\therefore$ the value of the number $= 10 \times x + 8$.

When the order of the digits is reversed, the value of the new number $= 10 \times 8 + x$.

According to the condition given in the problem, we have

$$(10x+8)+9=10\times 8+x$$

or $$10x+17=80+x$$

or $$9x=63$$

or $$x=7.$$

Thus the required number $= 78$.

[**Verification.** Since $78+9=87$, and we get 87 by reversing the digits of 78, therefore the answer is correct.]

55. A number consists of two digits, the sum of the digits is 12. If 36 be added to the number, the digits are reversed. Find the number.

56. A number consists of two digits, the digit in the units place is 5. If 27 be added to it, the digits are reversed. Find the number.

57. A number consists of two digits whose sum is 8. If 54 be subtracted from it, the digits are reversed. Find the number.

58. A number consists of two digits, the digit in the tens place being 5. If 18 be subtracted from it, the digits are reversed. Find the number.

59. A number consists of two digits, the digit in the units place is one-third of the digit in the tens place. If the digits be reversed, the new number falls short of the original by 36. Find the number.

60. A number consists of two digits, the digit in the units place is one-half of the digit in the tens place. If 9 be subtracted from the number, the digits are reversed. Find the number.

61. Reverse the digits of a number, it will be three-

eighths of what it was before; also the difference between the digits is 5. Find the number.

62. There is a number of two digits whose difference is 2. If it be increased by three times the sum of its digits, the digits are reversed. Find the number.

63. A number consists of two digits; the digit in the units place is three times the digit in the tens place. If the number be divided by the sum of the digits, the quotient is 3 and the remainder is 3. Find the number.

Example 10. Two trains with speeds 32 miles and 24 miles per hour respectively, starting at the same time, run in opposite directions between two stations P and Q, 336 miles apart one going from P to Q and the other from Q to P; find when and where they will meet.

Suppose that the two trains meet at O after x hours, then

the number of miles travelled by the first train in x hours $=$ $32x$ and the number of miles travelled by the second train in x hours $= 24x$.

The sum of these distances is equal to PQ, or 336 miles.

$$\therefore \quad 32x + 24x = 336$$

$$\text{or} \quad 56x = 336$$

$$\text{or} \quad x = 6.$$

Hence the two trains meet after 6 hours and at a distance of 6×32, or 192 miles from P.

Example 11. A and B are travelling on the same road in the same direction, A with the speed of 5 miles an hour and B in a motor-car with the speed of 60 miles an hour. A is 20 miles ahead of B. Find when and where B will overtake A.

Suppose B overtakes A after x hours; then the number of miles travelled by A in x hours $= 5x$ and the number of miles travelled by B in x hours $= 60x$.

According to the condition given in the problem, we have

$$60x-20=5x$$
$$55x=20$$
or
$$x=\tfrac{4}{11}.$$

Hence B overtakes A after $\frac{4}{11}$ hour or $21\frac{9}{11}$ minutes, and at a distance of $\frac{4}{11}\times 60$, or $21\frac{9}{11}$ miles, from his own starting-point.

64. A train running at the rate of 24 miles an hour leaves Amritsar for Jullundur at 7 p. m., and another train leaves Jullundur for Amritsar at 8 p. m. at the rate of 32 miles an hour. Find the time when the two trains meet, the distance between Amritsar and Jullundur being 50 miles.

65. There are two ports P and Q, the distance between them being 5625 miles. A steamer sails from P to Q with the speed of 240 miles a day; two days after, another steamer starts from Q and sails towards P with the speed of 250 miles a day Find when and where they meet.

66. A sets out for a walk at the rate of $3\frac{1}{2}$ miles an hour; 3 hours afterwards B cycles after him at the rate of 9 miles an hour. When will B overtake A?

67. A and B travel in opposite directions from two towns, 135 miles apart, and meet in 15 hours; if A goes twice as fast as B, what will be the speed of B?

68. A train with the speed of 32 miles an hour starts from a station $\frac{1}{2}$ hour after a goods train, and overtakes it in 50 minutes. Find the speed of the goods train.

69. A man walked to the top of a hill at the rate of $2\frac{1}{2}$ miles an hour and down the hill at the rate of $3\frac{1}{2}$ miles an hour. The journey took him 4 hours in all; find the length of his journey.

70. A man has $7\frac{1}{3}$ hours at his disposal; how far can he go in a motor-car, running with the speed of 36 miles

an hour so as to return home in time in a tonga running with the speed of 8 miles an hour?

71. A and B start at the same time from two towns, 39 miles apart, and meet in 4 hours and 20 minutes. If A walks 1 mile an hour faster than B, find where he meets B.

72. A train takes 8 hours to travel from P to Q. Had it gone 2 miles an hour faster, it would have taken $\frac{1}{2}$ hour less to reach Q. Find its speed and the distance between P and Q.

73. A and B travel along the same road, A on foot with the speed of 5 miles an hour, and B in a motor-car with the speed of 60 miles an hour. A is 10 miles ahead of B; find when and where B will overtake A.

74. In the last question, find when B will be ahead of A by (i) 22 miles, (ii) 15 miles, (iii) 30 miles.

[*Hint.* In the next three questions proceed as in the last two questions.]

75. At what time will the minute-hand overtake the hour-hand (i) between 2 and 3? (ii) between 4 and 5?

76. At what time will the minute-hand be ahead of the hour-hand by 30 minute-divisions (or opposite) (i) between 4 and 5? (ii) between 2 and 3?

77. At what time will the minute-hand be ahead of the hour-hand by 15 minute-divisions (or at right angles) (i) between 5 and 6? (ii) between 6 and 7?

Example 12. A person invested Rs. 440 in buying sheep and goats at the rate of Rs. 6 and Rs. 8 each respectively. If the total number of sheep and goats be 65, find the number of sheep and the number of goats.

Let the number of sheep purchased be x,
then the number of goats purchased $= 65-x$,
$\therefore$ the price of the sheep $=$ Rs. $6x$,
and the price of the goats $=$ Rs. $8(65-x)$.

According to the condition given in the problem, we have $6x+8(65-x)=440$

or $$6x+520-8x=440$$
or $$2x=80$$
or $$x=40 \text{ (the number of sheep)},$$
and $$65-x=65-40=25 \text{ (the number of goats)}.$$

[**Verification.** The price of the sheep $=40\times6$ rupees $=$ Rs. 240; the price of the goats $=25\times8$ rupees $=$ Rs. 200. The sum of the prices $=$ Rs. 240 $+$ Rs. 200 $=$ Rs. 440, therefore the answer is correct.]

Example 13. The population of a town is 15,000; the annual death-rate of males is $4\frac{1}{2}$% and that of females is $5\frac{1}{2}$%. The total number of deaths during the year is 745; find the number of males and females in the town.

Let the number of males in the town $=x$,
then the number of females in the town $=15000-x$.

Annual number of male deaths

$$=x\times\frac{4\frac{1}{2}}{100}=\frac{9x}{200}.$$

Annual number of female deaths

$$=(15000-x)\times\frac{5\frac{1}{2}}{100}=\frac{11(15000-x)}{200}.$$

According to the condition given in the problem, we have

$$\frac{9x}{200}+\frac{11(15000-x)}{200}=745.$$

Multiplying both sides by 200, we have

$$9x+11(15000-x)=149000$$
or $$9x+165000-11x=149000$$
or $$2x=16000$$
or $$x=8000, \text{ (the number of males)},$$
$$15000-x=15000-8000=7000, \text{ (the number of females)}.$$

The students should verify the answer.

78. Two pieces of cloth, measuring 80 yds., cost together Rs. 57. One piece was bought at $9a.$ per yd. and the other at $13a.$ per yd. Find the length of each.

79. If 8 pounds of tea and 12 pounds of coffee together cost Rs. 14 8 as. and a pound of tea cost 4 as. more than a pound of coffee, find the cost of each per pound.

80. A person bought a number of mangoes for Rs. 2 10 as. and found that 18 of them cost as much under 8 annas as 27 of them cost over 7 annas. Find the number of mangoes bought.

81. A person bought sugar of two different kinds, and paid in all Rs. 13 12 as., the better one at the rate of 5 as. per seer and the worse at 4 as. per seer; the total amount of both kinds of sugar was 50 seers. Find the number of seers of each kind.

82. A person had Rs. 9,900, a part of which he lent out at 8% and the rest at 6%; the interest amounted to Rs. 702. How much was lent out at 8%?

83. A sum of Rs. 3,200 was lent out at simple interest, partly at 5% and partly at $6\frac{1}{2}$% per annum. The total annual interest amounted to Rs. 190. How much was lent out at each rate?

84. A person had a capital of Rs. 4,600, a part of which he invested in a business yielding 10% annual profit, and the rest he lent out at 8% per annum; the whole annual income amounted to Rs. 416. Find how much he invested in the business.

85. The population of a town is 6,600; the annual death-rate of males is $3\frac{1}{2}$% and that of females is $3\frac{3}{4}$%; the total number of deaths during the year is 239. Find the number of males and females in the town.

***86.** A tradesman bought an equal number of goats of two kinds, one at £1 4*s*. each and the other at £1 10*s*. each. If he had spent his money equally on the two kinds, he would have had one goat more. How many did he buy of each kind?

Example 14. An army of 8,000 soldiers is formed into a solid square, and 79 men are left over. Find the number of men in the front row.

Let x be the number of men in the front row of the solid square,

then x^2 is the number of men in the solid square.

According to the condition given in the problem, we have

$$x^2+79=8000.$$

Let the student solve it further and verify the answer.

Example 15. The length of a room exceeds its breadth by 7 ft. If the length be increased by 5 ft. and the breadth be diminished by 3 ft., the area remains unaltered. Find the dimensions of the room.

Let x be the number of ft. in the breadth,

then $(x+7)$ is the number of ft. in the length,

and $x(x+7)$ is the number of sq. ft. in the area in the first case.

The number of ft. in the length in the second case $=(x+7)+5$,

The number of ft. in the breadth in the second case $=(x-3)$

and the number of sq. ft. in the area in the second case $=(x-3)\{(x+7)+5\}$.

According to the condition given in the problem, we have

$$x(x+7)=(x-3)\{(x+7)+5\}.$$

Let the student solve it further and verify the answer.

Example 16. Divide 36 into 4 parts such that if the first be increased by 2, the second be diminished by 2, the third be multiplied by 2, and the fourth divided by 2, the result in each case is the same.

Let x be the result we get in each case.

First part $+2=x$, $\therefore$ first part $=x-2$.

Second part $-2=x$, $\quad\therefore$ second part $=x+2$.

Third part $\times 2=x$, $\quad\therefore$ third part $=\frac{x}{2}$.

Fourth part $\div 2=x$, $\quad\therefore$ fourth part $=2x$.

Since the sum of the parts is equal to the whole,

$$\therefore \qquad (x-2)+(x+2)+\frac{x}{2}+2x=36$$

or $$\frac{9x}{2}=36$$

or $$x=8.$$

Hence the first part $=x-2=8-2=6$.

the second part $=x+2=8+2=10$.

the third part $=\frac{x}{2} \quad =\frac{8}{2} \quad =4$.

and the fourth part $=2x \quad =2\times 8=16$.

Example 17. A hare, 70 of her leaps before a greyhound, takes 5 leaps for every 4 leaps of the greyhound, but 2 leaps of the greyhound cover as much ground as 3 leaps of the hare. Find the number of leaps the greyhound must take to catch the hare.

Let $4x$ be the number of leaps the greyhound takes to catch the hare.

then the number of leaps the hare takes in the same time $=5x$.

Suppose the greyhound is at G and the hare at H, and the greyhound catches the hare at C.

Evidently $GC=4x$ leaps of the greyhound

$=\frac{3}{2}\times 4x$ leaps of the hare

$=6x$ leaps of the hare.

Again, $GC=GH+HC$

$=(70+5x)$ leaps of the hare.

$\therefore \qquad 70+5x=6x$

or $\qquad x=70.$

Hence the number of leaps the greyhound takes to catch the hare $=4x$, or 280.

87. An army of 2,000 soldiers is formed into a solid square, and 64 men are left over; find the number of men in the front row.

88. Find the length of the side of a square whose area is exactly equal to the area of a rectangle whose sides are 225 ft. and 289 ft.

89. The length of a rectangular room exceeds its breadth by 9 ft. If the length be increased by 7 ft. and the breadth diminished by 5 ft., the area remains the same. Find the dimensions of the room.

90. The length of a rectangular room exceeds its breadth by 6 ft. If the length be increased by 9 ft. and the breadth diminished by 3 ft. the area is increased by 15 sq. ft. Find its dimensions.

91. The length of a room exceeds its breadth by 12 ft. If the length be diminished by 6 ft. and the breadth increased by 2 ft., the area is diminished by 48 sq. ft. Find its dimensions.

92. A bag contains Rs. 325 in rupees and 8-anna bits. If the amount of the latter be less than that of the former by Rs. 15, how many coins are there of each kind?

93. A purse contains three times as many 4-anna bits as 2-anna bits. The value of the coins is Rs. 17 8*a*. How many coins are there of each kind?

94. Divide 180 into four parts such that if the first part be increased by 5, the second diminished by 5, the third multiplied by 5 and the fourth divided by 5, the result in each case is the same.

95. Divide 155 into four parts such that if the first part be diminished by 4, the second be increased by 5, the third be divided by 2, and the fourth be multiplied by 3, the result is the same.

96. Divide 81 into four parts such that if the first part be diminished by 5, the second be increased by 1, the third be multiplied by 2, and the fourth be divided by 3, the result is the same.

***97.** A certain sum of money is to be distributed amongst a number of boys; on calculation it is found that if Rs. 4 be given to each, Rs. 13 would be left over, and if Rs. 5 be given to each, Rs. 15 would be wanting. Find the number of boys and the sum.

***98.** A person has h hours at his disposal; he intends to walk a distance of d miles during this period. He finds that if he walk 3 miles an hour, he would require 40 minutes more to complete his journey, but if he walk 4 miles an hour, he would save 45 minutes. Find the values of h and d.

***99.** A hare, 30 of her leaps before a greyhound, takes 4 leaps for every 3 leaps of the greyhound, but 2 leaps of the greyhound cover as much ground as 3 leaps of the hare. Find the number of leaps the greyhound must take to catch the hare.

***100.** A hare, 22 of her leaps before a greyhound, takes 5 leaps for every 4 leaps of the greyhound, but 5 leaps of the greyhound cover as much ground as 9 leaps of the hare. Find the number of leaps the greyhound must take to catch the hare.

SECTIONAL REVISION I

PAPER 1

1. Use squared paper to show that
   (i) $8-3-12=-7$. (ii) $5a-8a+a=-2a$.
2. Find the value of $2x^3-3x^2+x-5$ when $x=-2, -1, 0$.
3. Divide $x^4-x^2y^2-x^2z^2+y^2z^2$ by $x^2-xy+xz-yz$.
4. Prove by different methods that $6a-2(8a-3a)\equiv-4a$.
5. Solve the equation $(3x+20)-(x-3x+2)=21$.
6. A number consists of two digits whose difference is 4. If it be increased by three times the sum of its digits, the digits are reversed; find the number.

PAPER 2

1. Subtract $b\{a-(b+c)\}$ from the sum of $a\{a-(c-b)\}$ and $c\{a-(b-c)\}$.
2. What must be added to $8x^3-12x^2-16x+21$ to make it exactly divisible by $2x-3$?
3. Simplify $\frac{a}{6}-\left[\frac{3b-4a}{6}-\left\{b-\frac{a-3b}{6}-\left(3b-\frac{4a+b}{3}\right)\right\}\right]$.
4. Find the continued product of
   $(a-b), (a^2+ab+b^2), (a^3+b^3)$.
5. Solve the equation $\frac{5}{3}(x-4)-\frac{7}{3}(2x-6)=20$.
6. A father is 21 years older than his son; 12 years hence three-fourths of the father's age will exceed two-thirds of the son's age by 18 years; find their ages.

PAPER 3

1. A person walks $3a-2b+c$ miles east, he then walks $2a-b-3c$ miles west, again walks $5a-b-4c$ miles east. How far is he then from the starting-point?

2. Use squared paper to illustrate the following :
(i) $5a+4a-7a=2a$, (ii) $3a-9a+4a=-2a$.

3. Divide $1+x$ by $1-2x+2x^2$ up to four terms.

4. If $x=2a-3b$ and $y=3a-2b$, find the value of $(2x-3y)(x-2y)$.

5. Solve $\frac{5}{6}(x-2)-\frac{3}{8}(2x+1)+2=0$.

6. A and B are travelling on the same road in the same direction, A with the speed of 4 miles an hour and B with the speed of $4\frac{3}{4}$ miles an hour. B gives A a start of 40 minutes; find when and where B will overtake A.

PAPER 4

1. Simplify $3a-[a-b-2\{a-b+c-(a-b-\overline{c-a})\}+a]$.

2. The product of two expressions is $x^4+4x^3+3x^2+2x-1$, and one of them is x^2+x+1; find the other.

3. Multiply $x^{2n}+x^n+1$ by x^n-1.

4. Find the value of $\frac{2a-b}{a+b}+\frac{2b-c}{b+c}+\frac{2c-a}{c+a}$ when $a=2$, $b=a+2$ and $c=a-2$.

5. Prove diagramatically :
(i) $(a+b)c=ac+bc$, (ii) $(a-b)c=ac-bc$,
(iii) $(a+b)(c+d)=ac+ad+bc+bd$.

6. Solve the equation $\frac{5(3-2x)}{4}=3(x+1)-\frac{3x+8}{6}$.

PAPER 5

1. Distinguish between (i) an index and a power,
(ii) an identical equation and a conditional equation.

2. Find the value of
(i) $(+1)^{2n}$, (ii) $(+1)^{2n-1}$, (iii) $(-1)^{2n}$, (iv) $(-1)^{2n+1}$.

3. Divide 1 by $1+a+a^2$ up to four terms.

4. Find the co-efficient of x^3 in the product of $3x^2-2x-1$ and $2x^2+3x+1$.

5. Find the continued product of (a^2+a+1), (a^2-a+1) and (a^4-a^2+1).

6. A cycles at the rate of 15 miles an hour from P to Q and returns at the rate of 12 miles an hour. B cycles both ways at the rate of 14 miles an hour, and takes 20 minutes less than A. Find the distance between P and Q

PAPER 6

1. Find the co-efficient of x in the product of $(x-a)$, $(x-2b)$, $(x-2c)$.

2. Multiply x^3-12x^2+5x-7 by $x^4-3x^3+6x^2-4x+8$, and arrange the product in ascending powers of x,

3. If $x=am^2+5bm+5c$ and $y=am^2-6bm-6c$, find the value of $6x+5y$.

4. Simplify $(x-a)(x-b)(x-c)$
$$-[bc(x-a)-\{(a+b+c)x-a(b+c)\}x].$$

5. Solve the equation $(x-3)(x-5)+1=(x+1)(x-2)+4$.

6. A travels from P to Q, a distance of 60 miles, and back again, at the rate of 12 miles an hour. On his way back he meets B who travels at the rate of 8 miles an hour and who started at the same time from P. Find the distance from P of the place where they meet.

PAPER 7

1. (i) From the formula $D=d\times Q+R$, find the value of R when $D=895$, $d=25$, $Q=35$.

(ii) One factor of $27a^4+11a-10$ is $3a^2-a+2$, find the other.

2. Add $a-3b+4c$, $3a+2b-5c$, $4a-7b+3c$, and multiply the result by the difference between $9a+6c$ and $10a-b-5c$.

3. Simplify $(5x+4y+1)(5x+4y-1)+(3x-5y)(3x+5y)$
$$-(4x-3y)(4x+3y)+1.$$

4. What must be added to $a^3-5a(a-3)-1$ to make it equal to $a^3+5a(a+3)+1$?

5. Solve the equation $(x+6)^2-(5-x)^2=18x+7$.

6. A person invested Rs. 5,800 at simple interest, partly at 4% and partly at 5% per annum. The total annual interest was Rs. 267; how much was lent at 4%?

PAPER 8

1. Prove by different methods that

$$x(x-1)(x-2)(x-3)+1\equiv(x^2-3x+1)^2.$$

2. If $x=m(p+q)$ and $y=n(p-q)$, find the value of

$$\frac{x}{m}+\frac{y}{n} \text{ and } \frac{x}{m}-\frac{y}{n}.$$

3. Find the dividend when the quotient is a^2+a+2, the divisor is a^2-3a+4, and the remainder is $7a+3$.

4. A man walks 6 miles North, then 10 miles South, then again 8 miles North. How far is he from the starting-point? Illustrate graphically.

5. Solve the equation $\dfrac{\cdot 3x-\cdot 2}{5}+1=\dfrac{\cdot 6-\cdot 5x}{3}-1.$

6. A farmer bought an equal number of sheep of two kinds, one at £1. 10*s*. each and the other at £2 each. If he had spent his money equally on the two kinds, he would have had one sheep more. How many of each kind did he buy?

PAPER 9

1. Simplify $3x+5-7x^2+4x-3x^2-4x^5-x^4+x^3+3x^4+6$, and arrange the result in ascending powers of x.

2. Find the value of $x(x+1)(x+2)(x+3)+1$ when $x=1$, 2, 3, 4...&c. Show that in each case the result is a perfect square.

3. In the formula $V=\frac{1}{3}\pi r^2 h$, V is the subject. Make (i) r, (ii) h the subject of the formula.

4. Find without actual multiplication the term containing x^2 in the product of $(2x^3-x^2+x+4)$, $(3x^2+2x+1)$.

5. Solve the equation $\dfrac{\cdot 5x}{2}-\dfrac{\cdot 2x}{3}=\dfrac{\cdot 02x-\cdot 9}{6}.$

6. The length of a room exceeds its breadth by 11 ft. If its length be increased by 5 ft. and the breadth diminished by 4 ft., the area is diminished by 44 sq. ft.; find the dimensions of the room.

PAPER 10

1. (i) What is the square of $\frac{3}{2} a^2b^3c^4d$?

(ii) What is the square root of $\frac{16}{25} a^4b^{10}c^2d^4$?

2. In India $m\%$ of the population are men and $n\%$ of them are married. What per cent of the population are the married men?

3. Find the value of (i) $\sqrt{s(s-a)(s-b)(s-c)}$.

(ii) $\sqrt{\dfrac{(s-a)(s-b)}{s(s-c)}}$

when $a=12$, $b=5$, $c=13$ and $s=\frac{1}{2}(a+b+c)$.

4. Subtract the sum of $4(2+x)-3(x^2-1)$ and $2(x^2-1)+4(x^2-1)$ from the sum of $x^3-(1-x+x^2)$ and $2x^2(x+1)$.

5. (i) Simplify $2x-[y+\{z-(x-\overline{y+z})\}]$.

(ii) For what value of k will $3a^4+5a^3-2a^2-a-1$ be equal to $3k^4-5k^3-2k^2+k-1$?

6. A hare, 24 of her leaps before a greyhound, takes 6 leaps for every 5 leaps of the greyhound but 7 leaps of the greyhound cover as much ground as 9 leaps of the hare. Find the number of leaps the greyhound must take to catch the hare.

CHAPTER IV

FORMULÆ

1. We add here a few important results, known as **multiplication formulae,** which should be thoroughly learnt by heart.

Formula I. $\boldsymbol{(a+b)^2=a^2+2ab+b^2}$.

Proof.

$$\begin{array}{l} a+b \\ a+b \\ \hline a^2+ab \\ \quad\;\; +ab+b^2 \\ \hline a^2+2ab+b^2. \end{array}$$

Statement of the formula in words:

The square of the sum of two quantities is equal to the sum of their squares plus twice their product.

Cor. $a^2+b^2=(a+b)^2-2ab.$

Example 1. Find square of $(3x+4y)$,

$(3x+4y)^2$ is in form similar to $(a+b)^2$ having $3x$ in place of a and $4y$ in place of b.

Thus $(3x+4y)^2=\{(3x)+(4y)\}^2=(3x)^2+2(3x)(4y)+(4y)^2$

$=9x^2+24xy+16y^2.$

Example 2. Find the missing term in the perfect square $49a^2+70ab+(\quad)$.

$49a^2=(7a)^2$ and

$70ab=2\times 7a\times 5b.$

$\therefore$ the 2nd quantity $=5b.$

$\therefore$ the missing term $=(5b)^2=25b^2.$

Example 3. Find algebraically the value of $(309)^2$.

$(309)^2=(300+9)^2=(300)^2+2(300)(9)+(9)^2$

$=90,000+5,400+81=95,481.$

EXERCISE 22.

Write down the square of :

1. $2x+5y$. 2. $3a+7b$. 3. $5x+3$. 4. $3+4p$.
5. $3ab+4b^2$. 6. $8a^2+3ab$. 7. $9ax+4y^2$. 8. m^2+n^2.
9. $px+qy$. 10. p^3+pq^2. 11. $2a^2b+3ab^2$. 12. $a+(-b)$.

Find the missing terms in the following perfect squares :

13. $16x^2+(\quad)+49y^2$. 14. $36m^2+60mn+(\quad)$.
15. $(\quad)+112pq+64p^2$. 16. $121a^2+(\quad)+81$.
17. $144m^2+72m+(\quad)$. 18. $(\quad)+104p+169$.

Find algebraically the value of :

19. $(302)^2$. 20. $(405)^2$. 21. $(703)^2$.
22. $(806)^2$. 23. $(611)^2$. 24. $(1001)^2$.

Example 4. Simplify $(2x+3y)^2-(x+2y)^2$.

The given expression $=(4x^2+12xy+9y^2)-(x^2+4xy+4y^2)$

$=3x^2+8xy+5y^2$.

Simplify :

25. $(a+2b)^2+(2a+b)^2$. 26. $(3x+2y)^2-(2x+3y)^2$.
27. $(px+qy)^2-(qx+py)^2$. 28. $(mx+n)^2-(mx+k)^2$.

Formula II. $\mathbf{(a-b)^2=a^2-2ab+b^2}$.

Proof.

$$\begin{array}{r} a-b \\ a-b \\ \hline a^2-ab \\ -ab+b^2 \\ \hline a^2-2ab+b^2. \end{array}$$

Statement of the formula in words:

The square of the difference of two quantities is equal to the sum of their squares minus twice their product.

Cor. $a^2+b^2=(a-b)^2+2ab$.

Example 5. Find the square of $(3x-4y)$.

Reducing the expression to the standard form, we have $(3x-4y)^2=\{(3x)-(4y)\}^2=(3x)^2-2(3x)(4y)+(4y)^2$

$=9x^2-24xy+16y^2$.

Example 6. Find the missing term in the perfect square $(\quad)-60mn+36n^2$.

$36n^2 = (6n)^2$

$-60mn = -2 \times 6n \times 5m.$

$\therefore$ the 2nd quantity $= 5m.$

$\therefore$ the missing term $= (5m)^2 = 25m^2.$

Example 7. Find the square of 395 by applying the formula $(a-b)^2 = a^2 - 2ab + b^2.$

$$(395)^2 = (400-5)^2 = (400)^2 - 2(400)(5) + (5)^2$$
$$= 160{,}000 - 4{,}000 + 25$$
$$= 156{,}025.$$

Write down the square of:

29. $3a-5b.$ **30.** $5m-3.$ **31.** $6p-5.$
32. $7x-1.$ **33.** $4-7p.$ **34.** $a-(-b).$
35. $4m^2-5n^2.$ **36.** $ax-by.$ **37.** $2a^2x-3b^2y.$
38. $7a^2-4b^2.$ **39.** $2a^3b-b^3.$ **40.** $7p^3q^2-8p^2q^3.$

Find the missing term in the following perfect squares:

41. $16a^2 - (\qquad) + 9b^2.$ **42.** $81x^2 - 72xy + (\qquad).$
43. $(\qquad) - 48pq + 36p^2.$ **44.** $49m^2 - (\qquad) + 121.$
45. $64p^2 - 96pq + (\qquad).$ **46.** $(\qquad) - 120xy + 144y^2.$

Find algebraically the value of:

47. $(198)^2.$ **48.** $(245)^2.$ **49.** $(396)^2.$
50. $(138)^2.$ **51.** $(497)^2.$ **52.** $(599)^2.$

Example 8. Simplify $(3m-5n)^2-(2m-3n)^2.$

The given expression

$$= (9m^2 - 30mn + 25n^2) - (4m^2 - 12mn + 9n^2)$$
$$= 5m^2 - 18mn + 16n^2.$$

Simplify:

53. $(a+b)^2-(a-b)^2.$ **54.** $(2a-3b)^2+(3a-2b)^2.$
55. $(px+qy)^2-(px-qy)^2.$ **56.** $(2mx-ny)^2-(mx+3ny)^2.$
57. $(3ab-2b^2)^2-4(a^2-b^2)^2.$

Example 9. Find the value of a^2+b^2 when $a+b=9$ and $ab=20.$

$$a^2+b^2 = (a+b)^2 - 2ab.$$

Substituting the values of $(a+b)$ and ab, we have the expression

$$= 9^2 - 2 \times 20$$
$$= 81 - 40 = 41.$$

Example 10. Find the value of a^2+b^2 when $a-b=7$ and $ab=60$.

$$a^2+b^2=(a-b)^2+2ab.$$

Substituting the values of $(a-b)$ and ab, we have the expression

$$=7^2+2\times 60$$
$$=49+120=169.$$

Find the value of a^2+b^2 when

58. $a+b=27$ and $ab=180$.
59. $a+b=19$ and $ab=88$.
60. $a-b=7$ and $ab=120$.
61. $a-b=9$ and $ab=112$.

Formula III. $\boldsymbol{(a+b)(a-b)=a^2-b^2}$.

Proof.
$$(a+b)(a-b)=a(a-b)+b(a-b)$$
$$=a^2-ab+ba-b^2$$
$$=a^2-b^2.$$

Statement of the formula in words :

The product of the sum and the difference of two quantities is equal to the difference of their squares.

Example 1. Multiply $4a+3b$ by $4a-3b$.

$$(4a+3b)(4a-3b)=(4a)^2-(3b)^2$$
$$=16a^2-9b^2.$$

Example 2. Multiply $a+b+c$ by $a-b-c$.

$$(a+b+c)(a-b-c)=\{a+(b+c)\}\{a-(b+c)\}$$
$$=a^2-(b+c)^2$$
$$=a^2-(b^2+2bc+c^2)$$
$$=a^2-b^2-2bc-c^2$$
$$=a^2-b^2-c^2-2bc.$$

Example 3. Simplify $(x-2y+3z)^2-(x+2y-3z)^2$.

The given expression $=\{(x-2y+3z)+(x+2y-3z)\}$
$$\times\{(x-2y+3z)-(x+2y-3z)\}$$
$$=2x\times(-4y+6z)$$
$$=-8xy+12xz.$$

Example 4. Find the value of $(236)^2-(232)^2$.

The given expression $=(236+232)(236-232)$
$$=468\times 4=1872.$$

EXERCISE 23.

Write down the product of :

1. $3x+5$ and $3x-5$.
2. $am+bn$ and $am-bn$.
3. $ax+b^2$ and $ax-b^2$.
4. $px+qy$ and $px-qy$.
5. $4a+5b$ and $4a-5b$.
6. $3xy-7xz$ and $3xy+7xz$.
7. $a+b$, $a-b$ and a^2+b^2.
8. $a+1$, $a-1$ and a^2+1.
9. a^2+b^2, a^2-b^2 and a^4+b^4.
10. $x+y-z$ and $x-y+z$.
11. $x-3y+4z$ and $x-3y-4z$.
12. x^2-xy+y^2 and x^2+xy+y^2.
13. x^2+x+1 and x^2-x+1.
14. x^4+x^2+1 and x^4-x^2+1.

Simplify :

15. $(2a-b+c)^2-(a+b-2c)^2$.
16. $(a^2+ab+b^2)^2-(a^2-ab+b^2)^2$.
17. $(a+b-c+d)^2-(a-b+c-d)^2$.
18. $(2a+3b-4c+5d)^2-(2a-3b+4c-5d)^2$.

Find the value of :

19. $(547)^2-(542)^2$.
20. $(784)^2-(779)^2$.
21. $328\times 328-318\times 318$.
22. $614\times 614-607\times 607$.

Formula IV. $(\boldsymbol{x}+\boldsymbol{a})(\boldsymbol{x}+\boldsymbol{b})=\boldsymbol{x}^2+(\boldsymbol{a}+\boldsymbol{b})\boldsymbol{x}+\boldsymbol{ab}$.

Proof. $(x+a)(x+b)=x(x+b)+a(x+b)$.

$$=x^2+xb+ax+ab.$$
$$=x^2+x(a+b)+ab.$$

From this formula, we can easily deduce the following results :

$$(x-a)(x-b)=x^2+(-a-b)x+(-a)(-b) \qquad \text{(i)}$$
$$(x+a)(x-b)=x^2+(a-b)x+a(-b) \qquad \text{(ii)}$$
$$(x-a)(x+b)=x^2+(-a+b)x+(-a)b. \qquad \text{(iii)}$$

Thus the above formula can be expressed in more general terms as follows :

$(x+a)(x+b)=x^2+$(*algebraic sum of 2nd terms*)$x+$ *product of 2nd terms.*

It is interesting to note that formulæ I and II can also be deduced from this formula, as follows :

$$(a+b)^2=(a+b)(a+b)$$
$$=a^2+(b+b)a+b.b$$
$$=a^2+2ab+b^2,$$

and
$$(a-b)^2=(a-b)(a-b)$$
$$=a^2+(-b-b)a+(-b)(-b)$$
$$=a^2-2ab+b^2.$$

Example 1. Write down the product of:

(i) $x+4$ and $x+5$. (ii) $x+6$ and $x-2$.
(iii) $x-3$ and $x+7$. (iv) $x-5$ and $x-6$.

(i) Since $4+5=9$ and $4\times 5=20$,
$\therefore \quad (x+4)(x+5)=x^2+9x+20.$

(ii) Since $+6-2=+4$ and $(+6)(-2)=-12$,
$\therefore \quad (x+6)(x-2)=x^2+4x-12.$

(iii) Since $-3+7=+4$ and $(-3)(+7)=-21$,
$\therefore \quad (x-3)(x+7)=x^2+4x-21.$

(iv) Since $(-5)+(-6)=-11$ and $(-5)(-6)=+30$,
$\therefore \quad (x-5)(x-6)=x^2-11x+30.$

EXERCISE 24.

Find out *mentally* the product of:

1. $x+3$ and $x+5$. **2.** $x+4$ and $x+6$.
3. $x+5$ and $x+7$. **4.** $x+7$ and $x+9$.
5. $m-8$ and $m+5$. **6.** $m-11$ and $m+7$.
7. $p-14$ and $p+9$. **8.** $a-20$ and $a+11$.
9. $x-7$ and $x+15$. **10.** $x-8$ and $x+12$.
11. $x-6$ and $x+17$. **12.** $x-12$ and $x+20$.
13. $x+10$ and $x-11$. **14.** $x+13$ and $x-8$.
15. $x+16$ and $x-9$. **16.** $x+12$ and $x-7$.
17. $x+7$ and $x-17$. **18.** $x+3$ and $x-15$.
19. $x-5$ and $x-18$. **20.** $x+7$ and $x-19$.
21. $x-7$ and $x-9$. **22.** $x-12$ and $x-15$.
23. $x-13$ and $x-10$. **24.** $x-9$ and $x-14$.

Example 2. Multiply together $x+y+z$ and $x+y-2z$.

Putting a for $(x+y)$ in both the expressions, we have

$$(x+y+z)(x+y-2z)=(a+z)(a-2z)$$
$$=a^2+(z-2z)a-2z^2$$
$$=a^2-az-2z^2.$$

Substituting the value of a, we have the product

$$=(x+y)^2-(x+y)z-2z^2$$
$$=x^2+2xy+y^2-xz-yz-2z^2$$
$$=x^2+y^2-2z^2+2xy-xz-yz.$$

Multiply :

25. $2x+3$ and $2x+5$.
26. $3m+5$ and $3m-2$.
27. $1+2x$ and $1-3x$.
28. $3+4x$ and $3-5x$.
29. $x-y+z$ and $x-y+2z$.
30. $2x+y+3z$ and $2x+y-z$.
31. $2x+3y+z$ and $2x+3y-5z$.

Formula V. $\boldsymbol{(x+a)(x+b)(x+c)}$

$$\boldsymbol{=x^3+(a+b+c)x^2+(ab+ac+bc)x+abc.}$$

The formula can be proved by actual multiplication.

From this formula, we can easily deduce the following results :

$$(x+a)(x+b)(x-c)$$
$$=x^3+(a+b-c)x^2+\{ab+a(-c)+b(-c)\}x+ab(-c). \quad \text{(i)}$$
$$(x+a)(x-b)(x-c)=x^3+(a-b-c)x^2$$
$$+\{a(-b)+a(-c)+(-b)(-c)\}x+a(-b)(-c). \quad \text{(ii)}$$
$$(x-a)(x-b)(x-c)=x^3+(-a-b-c)x^2$$
$$+\{(-a)(-b)+(-a)(-c)+(-b)(-c)\}x+(-a)(-b)(-c). \quad \text{(iii)}$$

Statement of the formula in more general terms :

$(x+a)(x+b)(x+c)=x^3+$(*algebraic sum of 2nd terms*)x^2 $+$(*sum of the products of 2nd terms, taken two at a time*)x $+$*product of 2nd terms.*

Example 1. Write down the product of $x+2$, $x+3$, and $x-6$.

$$(x+2)(x+3)(x-6)=x^3+(2+3-6)x^2$$
$$+\{2\times3+2(-6)+3(-6)\}x+2\times3\times(-6)$$
$$=x^3-x^2-24x-36.$$

Example 2. Write down the product of $x-3$, $x-4$ and $x-5$.

$$(x-3)(x-4)(x-5)=x^3+(-3-4-5)x^2$$
$$+\{(-3)(-4)+(-3)(-5)+(-4)(-5)\}x+(-3)(-4)(-5)$$
$$=x^3-12x^2+47x-60.$$

EXERCISE 25.

Write down the product of :

1. $(x+1)(x+2)(x+3)$.
2. $(x+2)(x+3)(x+4)$.
3. $(x+4)(x+5)(x+6)$.
4. $(x+3)(x+4)(x-5)$.
5. $(x+4)(x+5)(x-7)$.
6. $(x+5)(x+2)(x-4)$.
7. $(x+3)(x-4)(x-5)$.
8. $(x+4)(x-6)(x-3)$.
9. $(x-3)(x-4)(x-5)$.
10. $(x-4)(x-1)(x-7)$.
11. $(x-3)(x-5)(x-2)$.
12. $(x-4)(x-6)(x-8)$.
13. $(2x+1)(2x+3)(2x+5)$.
14. $(3x-1)(3x-2)(3x-4)$.
15. $(ax+2)(ax+3)(ax+6)$.
16. $(mx+1)(mx-4)(mx-5)$.
17. $(px-3)(px-4)(px-7)$.

Complete the following :

18. $(x+p)(\quad+q)(x+\quad) = x^3+(\quad)x^2+(\quad)x+pqr.$
19. $(\quad+a)(p+2b)(p-3c)=(p)^3+(\quad)p^2+(\quad)p+(\quad).$
20. $(m+3)(\quad-4)(m+\quad)=m^3+(\quad)m^2+(\quad)m-60.$

Formula VI. $\boldsymbol{(a+b)^3=a^3+3a^2b+3ab^2+b^3}$

or $\boldsymbol{=a^3+b^3+3ab(a+b)}$

The formula can be easily established by actual multiplication or can be deduced from formula V, as shown below :

$$(a+b)^3=(a+b)(a+b)(a+b)$$
$$=a^3+(b+b+b)a^2+(b\times b+b\times b+b\times b)a+b\times b\times b$$
$$=a^3+3ba^2+3b^2a+b^3$$
$$=a^3+3a^2b+3ab^2+b^3$$
$$=a^3+b^3+3ab(a+b).$$

Statement of the formula in words :

The cube of the sum of two quantities is equal to the sum of their cubes plus three times their product multiplied by their sum.

Cor. $a^3+b^3=(a+b)^3-3ab(a+b)$.

Example 1. **Find the cube of $2x+3y$.**

Reducing the expression to the standard form, we have

$$(2x+3y)^3 = \{(2x)+(3y)\}^3$$
$$=(2x)^3+(3y)^3+3(2x\times 3y)(2x+3y)$$
$$=8x^3+27y^3+36x^2y+54xy^2,$$

or

$$(2x+3y)^3=(2x)^3+3(2x)^2(3y)+3(2x)(3y)^2+(3y)^3$$
$$=8x^3+36x^2y+54xy^2+27y^3.$$

EXERCISE 26.

Find the cube of :

1. $2x+y$. 2. $x+3y$. 3. $2a+3b$. 4. $ax+by$.
5. $1+3x$. 6. $2+5x$. 7. $3x^2+2$. 8. a^2+b^2.
9. $3a^2+2b^2$.

Example 2. **Find the missing terms as well as the missing parts of terms in**

$$(\quad+\quad)^3=(2x)^3+3(2x)^2(\quad)+3(\quad)(5y)^2+(\quad)^3.$$

Since in the expansion the first term is $(2x)^3$ and the third term contains $(5y)^2$,

$\therefore$ the left-hand side $=(2x+5y)^3$

and the right-hand side $=(2x)^3+3(2x)^2(5y)+3(2x)(5y)^2+(5y)^3$.

Fill in the gaps in the following :

10. $(\quad+\quad)^3=(4m)^3+3(\quad)^2(\quad)+3(\quad)(\quad)^2+(2n)^3$.
11. $(2p+\quad)^3=(\quad)^3+3(\quad)(\quad)+3(2p)(\quad)+(3q)^3$.
12. $(\quad+\quad)^3=(5p)^3+3(\quad)(5p)^2+3(\quad)(\quad 2q)^2+(\quad)^3$.
13. $(2a+\quad)^3=(\quad)^3+(3b)^3+3(\quad)(2a+3b)$.

Example 3. Simplify :

$$(x+2y)^3+(2x-y)^3+3(3x+y)(x+2y)(2x-y).$$

Putting a for $x+2y$ and b for $2x-y$, we have the expression $=a^3+b^3+3(a+b)(a)(b)$, $[\because a+b=3x+y.]$
$=(a+b)^3$.

Substituting the values of a and b, we have the expression

$$=\{(x+2y)+(2x-y)\}^3$$
$$=(3x+y)^3$$

$$=(3x)^3+3(3x)^2y+3(3x)y^2+y^3$$
$$=27x^3+27x^2y+9xy^2+y^3.$$

Example 4. If $a+b=6$ and $ab=8$, find the value of a^3+b^3.

$$a^3+b^3=(a+b)^3-3ab(a+b).$$

Substituting the values of $(a+b)$ and ab, we have the expression

$$=6^3-3\times8\times6$$
$$=216-144=72.$$

Simplify :

14. $(3a+5b)^3+(4a-3b)^3+3(3a+5b)(4a-3b)(7a+2b)$.
15. $(2x-7y)^3+(3x+12y)^3+15(2x-7y)(3x+12y)(x+y)$.
16. $(a+2)^3+(a+1)^3$.
17. $(2a+3)^3-(2a+1)^3$.

Find the value of a^3+b^3 when

18. $a+b=5$ and $ab=6$.
19. $a+b=7$ and $ab=12$.
20. $a+b=9$ and $ab=18$.

Example 5. Find the value of $64+144x+108x^2+27x^3$ when $x=-3$.

$$64+144x+108x^2+27x^3=(4)^3+3(4)^2(3x)+3(4)(3x)^2+(3x)^3$$
$$=(4+3x)^3=(4-9)^3=(-5)^3=-125.$$

Find the value of:

21. $x^3+6x^2+12x+20$ when $x=5$.
22. $8x^3+36x^2+54x+35$ when $x=-3$.
23. $27x^3+27x^2+9x+21$ when $x=-2$.
24. If $a+b=7$, shew that $a^3+b^3+21ab=343$.
25. If $a+b=8$, shew that $a^3+b^3+24ab=512$.

Formula VII. $\boldsymbol{(a-b)^3=a^3-3a^2b+3ab^2-b^3}$.

or $\boldsymbol{=a^3-b^3-3ab(a-b)}$.

The formula can be proved by actual multiplication or by putting $(-b)$ for $(+b)$ in formula VI, as shown below:

$$(a-b)^3=\{a+(-b)\}^3=a^3+3a^2(-b)+3a(-b)^2+(-b)^3$$
$$=a^3-3a^2b+3ab^2-b^3$$
$$=a^3-b^3-3ab(a-b).$$

Statement of the formula in words :

The cube of the difference of two quantities is equal to the difference of their cubes minus three times their product multiplied by their difference.

Cor. $a^3 - b^3 = (a-b)^3 + 3ab(a-b)$.

Example 6. Find the cube of $3x - 2y$.

$$(3x-2y)^3 = (3x)^3 - (2y)^3 - 3(3x)(2y)(3x-2y)$$
$$= 27x^3 - 8y^3 - 54x^2y + 36xy^2$$
$$= 27x^3 - 54x^2y + 36xy^2 - 8y^3.$$

Example 7. If $a-b=4$ and $ab=12$, find the value of a^3-b^3.

$$a^3-b^3 = (a-b)^3 + 3ab(a-b)$$
$$= 4^3 + 3\times 12\times 4$$
$$= 64 + 144 = 208.$$

Example 8. Find the value of $8 - 9x + 27x^2 - 27x^3$ when $x=4$.

The given expression $= 1 - 3(3x) + 3(3x)^2 - (3x)^3 + 7$
$$= (1-3x)^3 + 7 = (1-12)^3 + 7$$
$$= (-11)^3 + 7 = -1331 + 7$$
$$= -1324.$$

Find the cube of :—

26. $2x-1$. **27.** $3-2x$. **28.** $3a-4b$. **29.** $4x-3y$.
30. $5x-4y$. **31.** $4m^2-3n$. **32.** $2ax-3by$.

Fill in the gaps in the following :

33. $(\quad - 3y)^3 = (2x)^3 - 3(\quad)^2(\quad) + 3(\quad)(\quad)^2 - (3y)^3$.
34. $(4m - \quad)^3 = (\quad)^3 - 3(\quad)^2(\quad) + 3(4m)(\quad)^2 - (5x)^3$.
35. $(\quad - 6b)^3 = (\quad)^3 - 3(4a)^2(\quad) + 3(\quad)(6b)^2 - (\quad)^3$.
36. $(2p - \quad)^3 = (\quad)^3 - (\quad)^3 - 3(\quad)(\quad)(2p-3q)$.

Simplify :

37. $(6m-7n)^3 - (3m+5n)^3 - 9(6m-7n)(3m+5n)(m-4n)$.
38. $(5p+3q)^3 - (4p-2q)^3 - 3(5p+3q)(4p-2q)(p+5q)$.
39. $(x+3)^3 - (x-1)^3$. **40.** $(2x+3)^3 - (2x-5)^3$.

Find the value of a^3-b^3 when

41. $a-b=3$ and $ab=18$. **42.** $a-b=5$ and $ab=14$.
43. $a-b=4$ and $ab=32$.

Find the value of :

44. $p^3-6p^2+12p-12$ when $p=3$.

45. $8p^3-36p^2+54p-35$ when $p=2$.

46. $27p^3-54p^2+36p-20$ when $p=4$.

Formula VIII. $(a+b)(a^2-ab+b^2)=a^3+b^3$.

Proof. $(a+b)(a^2-ab+b^2)=a(a^2-ab+b^2)+b(a^2-ab+b^2)$

$$=a^3-a^2b+ab^2+ba^2-ab^2+b^3$$

$$=a^3+b^3.$$

Conversely, $a^3+b^3=(a+b)(a^2-ab+b^2)$.

Example 1. Find the product of $3a+2b$ and $9a^2-6ab+4b^2$.

$$(3a+2b)(9a^2-6ab+4b^2)=\{(3a)+(2b)\}\{(3a)^2-(3a)(2b)+(2b)^2\}$$

$$=(3a^3+(2b)^3$$

$$=27a^3+8b^3.$$

Example 1. Simplify

$$(2x+3)(4x^2-6x+9)-(3x+2)(9x^2-6x+4).$$

Since $(2x+3)(4x^2-6x+9)=\{(2x)+(3)\}\{(2x)^2-(2x)(3)+(3)^2\}$

$$=(2x)^3+(3)^3=8x^3+27,$$

and $(3x+2)(9x^2-6x+4)=\{(3x)+(2)\}\{(3x)^2-(3x)(2)+(2)^2\}$

$$=(3x)^3+(2)^3$$

$$=27x^3+8,$$

$\therefore$ the given expression $=(8x^3+27)-(27x^3+8)$

$$=8x^3+27-27x^3-8$$

$$=19-19x^3.$$

EXERCISE 27.

Find the product of :

1. $x+3$ and x^2-3x+9.
2. $x+5$ and $x^2-5x+25$.
3. $x+7$ and $x^2-7x+49$.
4. $x+11$ and $x^2-11x+121$.
5. $25a^2-15a+9$ and $5a+3$.
6. $9m^2-15mn+25n^2$ and $3m+5n$.
7. $x^2y^2-4xyz+16z^2$ and $xy+4z$.
8. $4x+9y$ and $16x^2-36xy+81y^2$.
9. a^2+bc and $a^4-a^2bc+b^2c^2$.

10. $a^4-4a^2b^2+16b^4$ and a^2+4b^2.

11. x^2+1 and x^4-x^2+1.

Simplify :

12. $(x+5)(x^2-5x+25)-(x+3)(x^2-3x+9)$.

13. $(x+2y)(x^2-2xy+4y^2)+(2x+y)(4x^2-2xy+y^2)$.

14. $(4x+3y)(16x^2-12xy+9y^2)-(3x+4y)(9x^2-12xy+16y^2)$.

15. $(x^2+4)(x^4-4x^2+16)-(x^2+2)(x^4-2x^2+4)$.

Formula IX. $(a-b)(a^2+ab+b^2)=a^3-b^3$.

Proof.
$$\begin{aligned}(a-b)(a^2+ab+b^2)&=a(a^2+ab+b^2)-b(a^2+ab+b^2)\\&=a^3+a^2b+ab^2-ba^2-ab^2-b^3\\&=a^3-b^3.\end{aligned}$$

Conversely, $a^3-b^3=(a-b)(a^2+ab+b^2)$.

Example 3. Find the product of $3xy-z$ and $9x^2y^2-3xyz+z^2$.

$$\begin{aligned}(3xy-z)(9x^2y^2+3xyz+z^2)&=\{(3xy)-z\}\{(3xy)^2+(3xy)z+z^2\}\\&=(3xy)^3-z^3=27x^3y^3-z^3.\end{aligned}$$

Example 4. Simplify $(x-3)(x^2+3x+9)-(x-2)(x^2+2x+4)$.

$$\begin{aligned}(x-3)(x^2+3x+9)&=\{(x)-(3)\}\{(x)^2+(x)(3)+(3)^2\}\\&=x^3-3^3=x^3-27,\\(x-2)(x^2+2x+4)&=\{(x)-(2)\}\{(x)^2+(x)(2)+(2)^2\}\\&=x^3-2^3=x^3-8.\end{aligned}$$

$\therefore$ the given expression $=(x^3-27)-(x^3-8)$
$$\begin{aligned}&=x^3-27-x^3+8.\\&=-19.\end{aligned}$$

Write down the product of :

16. $(x-2)(x^2+2x+4)$.

17. $(x-6)(x^2+6x+36)$.

18. $(x-8)(x^2+8x+64)$.

19. $(x-12)(x^2+12x+144)$.

20. $(3x-2)(9x^2+6x+4)$.

21. $(5x-3y)(25x^2+15xy+9y^2)$.

22. $(pq-2r)(p^2q^2+2pqr+4r^2)$.

23. $(a^2-bc)(a^4+a^2bc+b^2c^2)$.

Simplify :

24. $(3m-4n)(9m^2+12mn+16n^2)-(2m-3n)(4m^2+6mn+9n^2)$.

25. $(4p-3q)(16p^2+12pq+9q^2)-(3p-4q)(9p^2+12pq+16q^2)$.

26. $(m^2-6)(m^4+6m^2+36)-(m^2-3)(m^4+3m^2+9)$.

Fill in the gaps :

27. $(a)^3 + (\quad)^3 = \{(\quad)+(3b)\}\{(a)^2 - (\quad)(\quad)+(\quad)^2\}$.
28. $(\quad)^3-(5y)^3 = \{(2x)-(\quad)\}\{(\quad)^2+(\quad)(\quad)+(5y)^2\}$.
29. $(4p)^3+(\quad)^3 = \{(\quad)+(\quad)\}\{(\quad)^2-(4p)(3q)+(\quad)^2\}$.
30. $(\quad)^3-(6b)^3 = \{(\quad)-(\quad)\}\{49a^2+(\quad)(\quad)+36b^2\}$.

Formula X. Square of multinomial.

$$(a+b+c)^2 = \{(a+b)+c\}^2$$
$$=(a+b)^2+2(a+b)c+c^2$$
$$=a^2+2ab+b^2+2ac+2bc+c^2$$
$$=a^2+b^2+c^2+2ab+2ac+2bc.$$

Similarly, $(a+b+c+d)^2$

$$=\{(a+b)+(c+d)\}^2$$
$$=(a+b)^2+2(a+b)(c+d)+(c+d)^2$$
$$=a^2+b^2+2ab+2ac+2ad+2bc+2bd+c^2+d^2+2cd$$
$$=a^2+b^2+c^2+d^2+2ab+2ac+2ad+2bc+2bd+2cd.$$

From such examples, we establish the following rule :

The square of any multinomial is equal to the sum of the squares of its terms plus twice the sum of the products of the terms, taken two at a time.

Example 1. Find the square of $2a-3b+c$.

$$(2a-3b+c)^2=(2a)^2+(-3b)^2+c^2+2(2a)(-3b)+2(2a)(c)+2(-3b)(c)$$
$$=4a^2+9b^2+c^2-12ab+4ac-6bc.$$

Example 2. Find the square of $a-3b+c-2d$.

$$(a-3b+c-2d)^2=a^2+(-3b)^2+c^2+(-2d)^2+2a(-3b)+2ac+2a(-2d)+2(-3b)c+2(-3b)(-2d)+2c(-2d)$$
$$=a^2+9b^2+c^2+4d^2-6ab+2ac-4ad-6bc+12bd-4cd.$$

EXERCISE 28.

Write down the square of :

1. $a+b-c$.
2. $a-b-c$.
3. $2a-b-c$.

4. $2a+b-c$. 5. $2a-b+3c$. 6. $5x-2y+3z$.
7. $2x-3y+1$. 8. x^2-xy+y^2. 9. $px^2-qxy+ry^2$.
10. $2a+b+3c+d$. 11. $a-2b-3c+4d$.
12. $2p-3q+r-2s+t$. 13. $3x-4y+2z-w$.

Simplify :

14. $(2a-b+c)^2-(a+b-2c)^2$.
15. $(2a+b+c)^2-(a-b-c)^2-(a+b-c)^2$.
16. $(a+2b+c)^2-(a-b+2c)^2+(2a+b-c)^2$.
17. $(a+b+c+d)^2+(a-b+c-d)^2$.
18. $(a-b-c-d)^2+(a+b-c-d)^2$.
19. $(2a-b+c-d)^2-(a+b-c+d)^2-(a-b+c-d)^2$.

Example 3. Find the value of $a^2+9b^2-6ab-2a+6b+64$ when $a=33$ and $b=24$.

The given expression $=(a)^2+(-3b)^2+2(a)(-3b)+2(a)(-1)$
$$+2(-3b)(-1)+(-1)^2+63$$
$$=(a-3b-1)^2+63$$

Substituting the values of a and b, we have the expression
$$=(33-72-1)^2+63$$
$$=(-40)^2+63$$
$$=1600+63=1663.$$

Find the value of :

20. $x^2+y^2+z^2-2xy+2xz-2yz$
when $x=15$, $y=20$, $z=24$.
21. $p^2+4q^2+9z^2-4pq+6pz-12qz$
when $p=8$, $q=12$, $z=15$.
22. $x^2+4y^2-4xy+10x-20y+36$ when $x=16$, $y=9$.
23. $9a^2+b^2-6ab+6a-2b-28$ when $a=15$, $b=37$.

A few important results, which can be deduced from formulæ I and II, are given below. They may be learnt by heart.

Formula XI. $(\boldsymbol{a+b})^2=(\boldsymbol{a-b})^2+\boldsymbol{4ab}$.

Proof. $(a+b)^2=a^2+b^2+2ab$
$$=(a^2+b^2-2ab)+4ab$$
$$=(a-b)^2+4ab.$$

Formula XII. $(a-b)^2=(a+b)^2-4ab.$

Proof. $(a-b)^2=a^2+b^2-2ab$

$$=(a^2+b^2+2ab)-4ab$$
$$=(a+b)^2-4ab.$$

Formula XIII. $(a+b)^2-(a-b)^2=4ab.$

Proof. $(a+b)^2-(a-b)^2=(a^2+2ab+b^2)-(a^2-2ab+b^2)$

$$=4ab.$$

Formula XIV. $\frac{1}{2}\{(a-b)^2+(b-c)^2+(c-a)^2\}$

$$=a^2+b^2+c^2-ab-ac-bc.$$

Proof. $\frac{1}{2}\{(a-b)^2+(b-c)^2+(c-a)^2\}$

$$=\tfrac{1}{2}\{(a^2-2ab+b^2)+(b^2-2bc+c^2)+(c^2-2ca+a^2)\}$$
$$=\tfrac{1}{2}\{2a^2+2b^2+2c^2-2ab-2bc-2ca\}$$
$$=a^2+b^2+c^2-ab-ac-bc.$$

Example 1. Find the value of $(a+b)^2$ when $a-b=3$ and $ab=10$.

$$(a+b)^2=(a-b)^2+4ab$$
$$=3^2+4\times10=9+40=49.$$

Example 2. Find the value of ab when $a+b=9$ and $a-b=3$.

Since $4ab=(a+b)^2-(a-b)^2,$

$\therefore$ $ab=\frac{1}{4}\{(a+b)^2-(a-b)^2\}$

$$=\tfrac{1}{4}(9^2-3^2)$$
$$=\tfrac{1}{4}(81-9)=18.$$

Example 3. Find the value of $a^2+b^2+c^2-ab-ac-bc$ when $a=32$, $b=30$ and $c=27$.

$$a^2+b^2+c^2-ab-ac-bc=\tfrac{1}{2}\{(a-b)^2+(b-c)^2+(c-a)^2\}$$
$$=\tfrac{1}{2}\{2^2+3^2+(-5)^2\}=\tfrac{1}{2}(4+9+25)$$
$$=19.$$

EXERCISE 29.

Find the value of :

1. $(a+b)^2$ when $a-b=5$, $ab=36$.
2. $(a+b)^2$ when $a-b=4$, $ab=77$.
3. $(a-b)^2$ when $a+b=12$, $ab=35$.
4. $(a-b)^2$ when $a+b=15$, $ab=36$.
5. ab when $a+b=11$, $a-b=3$.

6. ab when $a+b=22$, $a-b=8$.
7. $a^2+b^2+c^2-ab-ac-bc$ when $a=15$, $b=16$, $c=17$.
8. $a^2+b^2+c^2-ab-ac-bc$ when $a=1{\cdot}8$, $b=1{\cdot}3$, $c={\cdot}9$.
9. $a^2+b^2+c^2-ab-ac-bc$ when $a=p+1$, $b=p+2$, $c=p+3$.
10. Simplify $(x+1)^2+(x+2)^2+(x+3)^2-(x+1)(x+2)-(x+1)(x+3)-(x+2)(x+3)$.

[*Hint.* Put a for $x+1$, b for $x+2$ and c for $x+3$.]

11. Simplify $(x+3)^2+(x+5)^2+(x+7)^2-(x+3)(x+5)-(x+3)(x+7)-(x+5)(x+7)$.

Example 4. If $x+\dfrac{1}{x}=6$, find the value of :

(i) $x^2+\dfrac{1}{x^2}$, (ii) $x^4+\dfrac{1}{x^4}$.

$$\text{(i)}\ \left(x^2+\frac{1}{x^2}\right)=\left(x^2+\frac{1}{x^2}+2\right)-2$$
$$=\left(x+\frac{1}{x}\right)^2-2$$
$$=6^2-2=34.$$

$$\text{(ii)}\ \left(x^4+\frac{1}{x^4}\right)=\left(x^4+\frac{1}{x^4}+2\right)-2$$
$$=\left(x^2+\frac{1}{x^2}\right)^2-2$$
$$=34^2-2=1154.$$

Example 5. If $x-\dfrac{1}{x}=4$, find the value of $x^3-\dfrac{1}{x^3}$.

$$x^3-\frac{1}{x^3}=(x)^3-\left(\frac{1}{x}\right)^3$$
$$=\left(x-\frac{1}{x}\right)^3+3(x)\left(\frac{1}{x}\right)\left(x-\frac{1}{x}\right)$$
$$=\left(x-\frac{1}{x}\right)^3+3\left(x-\frac{1}{x}\right)$$
$$=4^3+3\times4$$
$$=64+12=76.$$

Find the value of :

*12. $x^2+\dfrac{1}{x^2}$ and $x^4+\dfrac{1}{x^4}$ when $x+\dfrac{1}{x}=5$.

*13. $x^2+\dfrac{1}{x^2}$ and $x^4+\dfrac{1}{x^4}$ when $x-\dfrac{1}{x}=3$.

*14. $x^4+\dfrac{1}{x^4}$ when $x-\dfrac{1}{x}=p$.

*15. $x^3+\dfrac{1}{x^3}$ when $x+\dfrac{1}{x}=5$.

*16. $x^3-\dfrac{1}{x^3}$ when $x-\dfrac{1}{x}=2$.

*17. $x^3+\dfrac{1}{x^3}$ when $x+\dfrac{1}{x}=m$.

*18. $x^3-\dfrac{1}{x^3}$ when $x-\dfrac{1}{x}=a$.

Example 6. Express $(x+5y+z)^2-4(x+3y)(2y+z)$ as a perfect square.

Let $x+3y=a$ and $2y+z=b$, then $x+5y+z=a+b$.

$\therefore$ the given expression $=(a+b)^2-4ab=(a-b)^2$

$$=\{(x+3y)-(2y+z)\}^2$$
$$=(x+y-z)^2.$$

Example 7. If $x=a+m$, $y=b+m$ and $z=c+m$, prove that

$$x^2+y^2+z^2-xy-xz-yz=a^2+b^2+c^2-ab-ac-bc.$$

$$x^2+y^2+z^2-xy-xz-yz=\tfrac{1}{2}\{(x-y)^2+(y-z)^2+(z-x)^2\}$$
$$=\tfrac{1}{2}\{(a+m-b-m)^2+(b+m-c-m)^2+(c+m-a-m)^2\}$$
$$=\tfrac{1}{2}\{(a-b)^2+(b-c)^2+(c-a)^2\}$$
$$=a^2+b^2+c^2-ab-ac-bc.$$

Express as a perfect square :

*19. $(3x-y+z)^2+4(3x+y)(2y-z)$.

*20. $(2x+y-z)^2+4(2x+3y)(2y+z)$.

*21. $(x+3y+3z)^2-4(x+y+2z)(2y+z)$,

*22. $(x-z)^2-4(x-y+z)(y-2z)$.

*23. If $a=p+q$, $b=q+r$ and $c=r+p$, prove that $a^2+b^2+c^2-ab-ac-bc=p^2+q^2+r^2-pq-pr-qr$.

*24. If $x=a+b-c$, $y=b+c-a$ and $z=c+a-b$, prove that $x^2+y^2+z^2-xy-xz-yz=4(a^2+b^2+c^2-ab-ac-bc)$.

*25. Find the value of $x^2+y^2+z^2-xy-xz-yz$ when $x=a(a+7)$, $y=(a+2)(a+5)$, $z=(a+3)(a+4)$.

*26. Find the value of $8xy(x^2+y^2)$ when $x+y=15$ and $x-y=11$.

Formula XV. $(\mathbf{a}+\mathbf{b}+\mathbf{c})(\mathbf{a}^2+\mathbf{b}^2+\mathbf{c}^2-\mathbf{ab}-\mathbf{ac}-\mathbf{bc})$
$=\mathbf{a}^3+\mathbf{b}^3+\mathbf{c}^3-3\mathbf{abc}$.

The formula can be easily established by actual multiplication; another form of the formula, which is sometimes useful, is $\frac{1}{2}(a+b+c)\{(a-b)^2+(b-c)^2+(c-a)^2\}=a^3+b^3+c^3-3abc$, for $a^2+b^2+c^2-ab-ac-bc=\frac{1}{2}\{(a-b)^2+(b-c)^2+(c-a)^2\}$ by formula XIV.

Example 1. Simplify

$$(2x+3y-z)(4x^2+9y^2+z^2-6xy+2xz+3yz).$$

Reducing the given expression to the standard form, we have

$$\{(2x)+(3y)+(-z)\}\{(2x)^2+(3y)^2+(-z)^2-(2x)(3y)-(2x)(-z)-(3y)(-z)\}$$
$$=(2x)^3+(3y)^3+(-z)^3-3(2x)(3y)(-z)$$
$$=8x^3+27y^3-z^3+18xyz.$$

Example 2. Find the value of $a^3+b^3+c^3-3abc$ when $a+b+c=9$ and $ab+ac+bc=26$.

Since $a+b+c=9$, $\therefore (a+b+c)^2=81$

$$\therefore (a^2+b^2+c^2)+2(ab+ac+bc)=81.$$

Substituting the value of $ab+ac+bc$, we have

$$(a^2+b^2+c^2)+2\times 26=81,$$
$$\therefore a^2+b^2+c^2=29.$$

Thus $a^3+b^3+c^3-3abc=(a+b+c)\{(a^2+b^2+c^2)-(ab+ac+bc)\}$

$$=9(29-26)$$
$$=9\times 3=27.$$

EXERCISE 30.

Simplify :

1. $(x+y-1)(x^2+y^2+1-xy+x+y)$.
2. $(x-y-1)(x^2+y^2+1+xy+x-y)$.
3. $(2x+y+1)(4x^2+y^2+1-2xy-2x-y)$.
4. $(x+2y-z)(x^2+4y^2+z^2-2xy+xz+2yz)$.
5. $(x+2y-3z)(x^2+4y^2+9z^2-2xy+3xz+6yz)$.
6. $(2x-y-3z)(4x^2+y^2+9z^2+2xy+6xz-3yz)$.

Complete the following identities :

7. $(3x-y-z)(\qquad)=27x^3-y^3-z^3-9xyz$.
8. $(\qquad)(25x^2+9y^2+z^2-15xy+5xz+3yz)$
$$=125x^3+27y^3-z^3+45xyz.$$

Find the value of $a^3+b^3+c^3-3abc$ when

9. $a+b+c=13,\qquad ab+ac+bc=71$.
10. $a+b+c=21,\qquad a^2+b^2+c^2=149$.
11. $ab+bc+ca=121,\quad a^2+b^2+c^2=158$.
12. If $x=a^2-bc$, $y=b^2-ca$, $z=c^2-ab$, shew that
$$ax+by+cz=(a+b+c)(x+y+z).$$

***Example 3.** If $x=a+b$, $y=b+c$, $z=c+a$, prove that
$$x^3+y^3+z^3-3xyz=2(a^3+b^3+c^3-3abc).$$
$$x^3+y^3+z^3-3xyz=\tfrac{1}{2}(x+y+z)\{(x-y)^2+(y-z)^2+(z-x)^2\}.$$

Substituting the values of x, y, z, we get the expression
$$=\tfrac{1}{2}\{2(a+b+c)\}\{(a-c)^2+(b-a)^2+(c-b)^2)\}$$
$$=2[\tfrac{1}{2}(a+b+c)\{(a-b)^2+(b-c)^2+(c-a)^2\}]$$
$$=2(a^3+b^3+c^3-3abc).$$

*13. If $a=y+z-x$, $b=z+x-y$ and $c=x+y-z$, prove that $a^3+b^3+c^3-3abc=4(x^3+y^3+z^3-3xyz)$.

*14. If $a=y+z-3x$, $b=z+x-3y$ and $c=x+y-3z$, prove that $a^3+b^3+c^3-3abc=-16(x^3+y^3+z^3-3xyz)$.

*15. If $2s=a+b+c$, prove that $(s-a)^3+(s-b)^3+(s-c)^3-3(s-a)(s-b)(s-c)=\tfrac{1}{4}(a^3+b^3+c^3-3abc)$.

2. We add here a few important results, known as **division formulae**, which should be carefully learnt by heart.

(i) $a^n - b^n$ is exactly divisible by $a-b$, if n is *any* integer.

Thus, $\dfrac{a-b}{a-b}=1$, $\qquad \dfrac{a^2-b^2}{a-b}=a+b$,

$\dfrac{a^3-b^3}{a-b}=a^2+ab+b^2$, $\dfrac{a^4-b^4}{a-b}=a^3+a^2b+ab^2+b^3$,

$\dfrac{a^5-b^5}{a-b}=a^4+a^3b+a^2b^2+ab^3+b^4$, and so on.

(ii) $a^n - b^n$ is exactly divisible by $a+b$, if n is an *even* integer.

Thus, $\dfrac{a^2-b^2}{a+b}=a-b$, $\qquad \dfrac{a^4-b^4}{a+b}=a^3-a^2b+ab^2-b^3$,

$\dfrac{a^6-b^6}{a+b}=a^5-a^4b+a^3b^2-a^2b^3+ab^4-b^5$, and so on.

(iii) $a^n + b^n$ is exactly divisible by $a+b$, if n is an *odd* integer.

Thus, $\dfrac{a+b}{a+b}=1$, $\qquad \dfrac{a^3+b^3}{a+b}=a^2-ab+b^2$,

$\dfrac{a^5+b^5}{a+b}=a^4-a^3b+a^2b^2-ab^3+b^4$, and so on.

(iv) $a^n + b^n$ is never exactly divisible by $a+b$ or $a-b$, if n is an *even* integer.

Example 1. Simplify $\dfrac{8x^3+27y^3}{2x+3y}$.

$$\frac{8x^3+27y^3}{2x+3y}=\frac{(2x)^3+(3y)^3}{2x+3y}=(2x)^2-(2x)(3y)+(3y)^2$$
$$=4x^2-6xy+9y^2.$$

Example 2. Simplify $\dfrac{x^5+32y^5}{x+2y}$.

$$\frac{x^5+32y^5}{x+2y}=\frac{(x)^5+(2y)^5}{x+2y}$$
$$=x^4-x^3(2y)+x^2(2y)^2-x(2y)^3+(2y)^4$$
$$=x^4-2x^3y+4x^2y^2-8xy^3+16y^4.$$

Example 3. Simplify $\dfrac{a^3+(b+c)^3}{a+b+c}$.

$$\frac{a^3+(b+c)^3}{a+b+c}=\frac{(a)^3+(b+c)^3}{a+(b+c)}=a^2-a(b+c)+(b+c)^2$$
$$=a^2-ab-ac+b^2+2bc+c^2$$
$$=a^2+b^2+c^2-ab-ac+2bc.$$

EXERCISE 31.

Write down by *inspection* the quotient of :

1. $(8a^3-x^3)\div(2a-x)$.
2. $(a^3+64b^3)\div(a+4b)$.
3. $(64a^6-x^6)\div(2a+x)$.
4. $(27m^3-1)\div(3m-1)$.
5. $(x^4-256y^4)\div(x-4y)$.
6. $(1-16m^4)\div(1+2m)$
7. $(32a^5+b^5)\div(2a+b)$.
8. $(m^{18}-n^{12})\div(m^3+n^2)$.
9. $\left(\frac{1}{27}a^3+b^3\right)\div\left(\frac{1}{3}a+b\right)$.
10. $(x^8-y^4z^4)\div(x^2+yz)$.
11. $\{p^2-(q-r)^2\}\div(p-q+r)$.
12. $\{a^3-(b-c)^3\}\div(a-b+c)$.
13. $(m^{18}-n^{12})\div(m^3-n^2)$.
14. $(x^9+a^9)\div(x+a)$.
15. Write down by *inspection*, the continued product of $(x+a)$, (x^2+a^2), (x^4+a^4).

[*Hint.* Include $x-a$ in the continued product and divide the product so obtained by $x-a$.]

A systematised list of multiplication formulae :

1. $(a+b)^2=a^2+2ab+b^2$.
2. $(a-b)^2=a^2-2ab+b^2$.
3. $a^2+b^2=(a+b)^2-2ab$.
4. $a^2+b^2=(a-b)^2+2ab$.
5. $(a+b)^2=(a-b)^2+4ab$.
6. $(a-b)^2=(a+b)^2-4ab$.
7. $(a+b)^2-(a-b)^2=4ab$.
8. $(a+b)(a-b)=a^2-b^2$.
9. $(a+b)(a^2-ab+b^2)=a^3+b^3$.
10. $(a-b)(a^2+ab+b^2)=a^3-b^3$.
11. $(x+a)(x+b)=x^2+(a+b)x+ab$.
12. $(x+a)(x+b)(x+c)=x^3+(a+b+c)x^2+(ab+ac+bc)x + abc$.
13. $(x+a)^3=x^3+3x^2a+3xa^2+a^3$, or $=x^3+a^3+3xa(x+a)$.
14. $(x-a)^3=x^3-3x^2a+3xa^2-a^3$, or $=x^3-a^3-3xa(x-a)$.
15. $x^3+a^3=(x+a)^3-3xa(x+a)$.
16. $x^3-a^3=(x-a)^3+3xa(x-a)$.
17. $(a+b+c)^2=a^2+b^2+c^2+2ab+2ac+2bc$.
18. $(a+b+c+d)^2=a^2+b^2+c^2+d^2+2ab+2ac+2ad +2bc+2bd+2cd$.
19. $\frac{1}{2}\{(a-b)^2+(b-c)^2+(c-a)^2\}=a^2+b^2+c^2-ab-ac -bc$.
20. $(a+b+c)(a^2+b^2+c^2-ab-ac-bc)=a^3+b^3+c^3-3abc$, or $\frac{1}{2}(a+b+c)\{(a-b)^2+(b-c)^2+(c-a)^2\}=a^3+b^3 +c^3-3abc$.

CHAPTER V

FACTORS

1. Type I. Factors of expressions of the form $\boldsymbol{ab+ac}$.

Example 1. Resolve into factors $3a^2-6a$.

Here $3a$ is common to both the terms: hence if we divide the given expression by $3a$, the quotient will be another factor. Thus:

$$3a\overline{)3a^2-6a}\quad a-2$$

$$\therefore \quad 3a^2-6a=3a(a-2).$$

Example 2. Resolve into factors $2a^3b^2c-4ab^3c^2+6a^2bc^3$.

Here we notice that $2abc$ is a factor in every term,

$$\therefore \quad 2a^3b^2c-4ab^3c^2+6a^2bc^3$$
$$=2abc(a^2b-2b^2c+3ac^2).$$

[Multiply $a^2b-2b^2c+3ac^2$ by $2abc$ and see if the product is $2a^3b^2c-4ab^3c^2+6a^2bc^3$.]

EXERCISE 32. [Mainly oral]

Resolve into factors:

1. $ma+mb$.	2. $pr+ps$.
3. x^2+2x.	4. $3x^2-6$.
5. m^2n+mn^2.	6. $abc-bcd$.
7. $-8x+24$.	8. $-11a^2-44a$.
9. $75.11+75.3$.	10. $49.15-49.12$.
11. $pa+qa-ra$.	12. $-2x^2+4x-6$.
13. ax^3+bx^2+cx.	14. $-3p^3-6p^2+9p$.
15. $a^2x^3-2a^3x^2+a^2x^2$.	16. $2mp^2+4mp-6m$.
17. $4a^5b-6a^4b+8a^3b^3$.	18. $-36x^{10}y^8+72x^8y^{10}$.
19. $ax^3y^2+a^2xy^3+a^3x^2y$.	20. $33a^3b^5-55a^5b^3$.

Instead of a simple expression, we may have a compound expression as a factor of every term, as in the next example.

Example 3. Resolve into factors

$$a^2(x-y)+3ab\,(x-y)+5b^2(x-y).$$

Here $x-y$ is a factor of every term,

$$\therefore\ a^2(x-y)+3ab(x-y)+5b^2(x-y)$$
$$=(x-y)(a^2+3ab+5b^2).$$

Resolve into factors:

21. $c(a+b)+d(a+b)$.
22. $p(p-q)+q(p-q)$.
23. $7m(x-1)+2(x-1)$.
24. $2a(x^2+x+1)+3b(x^2+x+1)$.
25. $3p(ax+b)+q(ax+b)+2r(ax+b)$.
26. $m^2(pq+r)+mn(pq+r)+n^2(pq+r)$.
27. $a(x-y)+b(y-x)+(c+d)(x-y)$.
28. $(x^2+7x)^2-25(x^2+7x)$.
29. $15(p^2-qr)^3-10(p^2-qr)^2$
30. $(2p+3q)(x-y)+(p-2q)(x-y)+(3p+q)(x-y)$.
31. $(x+y)^3-3xy(x+y)$.

2. **Type II.** Factors of expressions of the form

$$\boldsymbol{ac+bc+ad+bd}.$$

We notice that no factor is common to all the terms, but c is common to the first two terms and d is common to the last two terms. Enclosing the first two terms in one pair of brackets and the last two in another pair, we have

$$ac+bc+ad+bd$$
$$=(ac+bc)+(ad+bd)$$
$$=c(a+b)+d(a+b)$$
$$=(a+b)(c+d).$$

Example 1. Resolve into factors $3ab+6bc+4ad+8cd$.

$$3ab+6bc+4ad+8cd$$
$$=(3ab+6bc)+(4ad+8cd)$$
$$=3b(a+2c)+4d(a+2c)$$
$$=(a+2c)(3b+4d).$$

Example 2. Factorise $3x^2+4yz-4xy-3xz$.

Re-arranging the terms, the expression

$$=3x^2-3xz-4xy+4yz$$
$$=3x(x-z)-4y(x-z)$$
$$=(x-z)(3x-4y).$$

EXERCISE 33.

Factorise :

1.	$5x+5y+ax+ay$.	**2.**	$ax-ay-bx+by$.
3.	$mx-3y-my+3x$.	**4.**	$xy+3x+2y+6$.
5.	$x^2-3ax+bx-3ab$.	**6.**	$a^3+3a^2+9a+27$.
7.	$3p^3-5p^2+6p-10$.	**8.**	$6p^2-9pq+4pr-6qr$.
9.	$pq+p+q+1$.	**10.**	a^3+a^2-a-1.
11.	$2a^3-5a^2+2a-5$.	**12.**	$5a^3-2a^2+5a-2$.
13.	$x^2-xy-3x+3y$.	**14.**	$6a^2+3ab-2ac-bc$.
15.	$2a^4-a^3+4a-2$.	**16.**	$y^3-y^2-ay+y+a-1$.
17.	$11x^3+7x+35+55x^2$.	**18.**	ax^3+b+bx^2+ax.
19.	$6xy+6-9y-4x$.	**20.**	$x^2-2ax+2ab-bx$.

Sometimes we have to remove the brackets and re-arrange the terms, as in the next example.

Example 3. $a^3x-a^2b(x-y)-ab^2(y-z)-b^3z$.

The expression $=a^3x-a^2bx+a^2by-ab^2y+ab^2z-b^3z$

$$=(a^3x-a^2bx)+(a^2by-ab^2y)+(ab^2z-b^3z)$$
$$=a^2x(a-b)+aby(a-b)+b^2z(a-b)$$
$$=(a-b)(a^2x+aby+b^2z).$$

Resolve into factors :

21. $ax^2+(a-1)x-1$.

22. $abx^2+(ay-b)x-y$.

23. $a^3x+a^2(x-y)-a(y+z)-z$.

24. $a(x^2-1)+x^2(bx-c)+x(cx^3-b)$.

25. $x^3(kx+l)-x(kx+l)+m(x^2-1)$.

26. $(a-2b)^2+3a-6b$.

27. $ab(c^2+1)+c(a^2+b^2)$.

28. $ab(x^2+y^2)-xy(a^2+b^2)$.

3. **Type III.** Factors of expressions of the form
$$\boldsymbol{a^2+2ab+b^2} \text{ or } \boldsymbol{a^2-2ab+b^2}.$$

Example 1. Factorise $x^2+14x+49$.
$$x^2+14x+49=x^2+2.x.7+7^2$$
$$=(x+7)^2.$$

Example 2. Factorise $36a^2-12a+1$.
$$36a^2-12a+1=(6a)^2-2.(6a).1+1^2$$
$$=(6a-1)^2.$$

Example 3. Factorise $9+42xy+49x^2y^2$.
$$9+42xy+49x^2y^2=3^2+2.3.(7xy)+(7xy)^2.$$
$$=(3+7xy)^2.$$

EXERCISE 34. [Mainly oral]

Resolve into factors :

1. $4a^2+28a+49$.
2. $25p^2-30p+9$.
3. $25a^2+40ab+16b^2$.
4. $49m^2-56mn+16n^2$.
5. $36a^2+60ab+25b^2$.
6. $36a^2-84ab+49b^2$.
7. $49a^2+42ab+9b^2$.
8. $25x^2-60xy+36y^2$.
9. $16x^2+56xy+49y^2$.
10. $81a^2-126ab+49b^2$.
11. $81x^2+90xy+25y^2$.
12. $9m^2-66mn+121n^2$.
13. $64x^2+16xy+y^2$.
14. $16p^2-48pq+36q^2$.
15. $9x^2+54xy+81y^2$.
16. $25m^2-70mn+49n^2$.
17. $x^2+2+\frac{1}{x^2}$.
18. $\frac{x^2}{y^2}-2+\frac{y^2}{x^2}$.
19. $a^2x^2+6abxy+9b^2y^2$.
20. $16a^2b^2-8abc^2+c^4$.
21. $(2a+3b)^2+2(2a+3b)(c+d)+(c+d)^2$.
22. $(3p-q)^2-2(3p-q)(r+s)+(r+s)^2$.

4. **Type IV.** Factors of expressions of the form $\boldsymbol{a^2-b^2}$.

Example 1. Resolve into factors $16a^2-25b^2$.
$$16a^2-25b^2=(4a)^2-(5b)^2$$
$$=(4a+5b)(4a-5b).$$

NOTE. The first step, *i.e.*, the reduction of the given expression to the standard form, is essential.

Example 2. Resolve into factors $1-49x^2$.

$$1-49x^2=(1)^2-(7x)^2$$
$$=(1+7x)(1-7x).$$

EXERCISE 35.

Resolve into factors (*mentally*):

1. $16x^2-9$.	2. $81a^2-25b^2$.	3. $1-36x^2$.
4. $64a^2-1$.	5. $9x^2-100$.	6. $49-25a^2$.
7. $121-c^2$.	8. $1-49a^2c^2$.	9. a^4-25.
10. $16x^6-49$.	11. $1-81x^4y^2$.	12. x^6-36y^2.

Example 3. Resolve into factors $16x^5-x$.

$$16x^5-x=x(16x^4-1)$$
$$=x\{(4x^2)^2-(1)^2\}$$
$$=x(4x^2+1)(4x^2-1)$$
$$=x(4x^2+1)\{(2x)^2-(1)^2\}$$
$$=x(4x^2+1)(2x+1)(2x-1).$$

Resolve into factors:

13. $81x^4-1$.	14. $16x^5-81x$.
15. $1-16x^4$.	16. a^2-81a^6.
17. $36x^7-25x^3a^4$.	18. $192x^9-243x^5y^4$.
19. $484a^{17}b^9-324a^5b^3$.	20. $(2x-3y)^3-8x+12y$.

Example 4. Resolve into factors $4(a-b)^2-9(c-d)^2$

$$4(a-b)^2-9(c-d)^2=2^2(a-b)^2-3^2(c-d)^2$$
$$=(2a-2b)^2-(3c-3d)^2$$
$$=(2a-2b+3c-3d)(2a-2b-3c+3d).$$

Resolve into factors:

21. $(a+5b)^2-49c^2$.
22. $x^2-(2y-3z)^2$.
23. $(a+b)^2-(a-b)^2$.
24. $(3x-2y)^2-(2x+y)^2$.
25. $16(x-y)^2-9(x+y)^2$.
26. $49(a+b)^2-36(a-b)^2$.
27. $81(m+2n)^2-25(2m-n)^2$.
28. $4(3m-2n)^2-9(2m-3n)^2$.
29. $(8a+5)^2-(2a-7)^2$.
30. $(a+b-c)^2-(a-b+c)^2$.
31. $16(a+3b-4c)^2-9(2a-b+3c)^2$.
32. $(3a^2-5a+7)^2-(3a^2-5a-7)^2$.
33. $(x+a)^4-(x-a)^4$.

Simplify :

34. $2396^2 - 2391^2$. 35. $857 \times 857 - 143 \times 143$.

36. $\cdot 738 \times \cdot 738 - \cdot 262 \times \cdot 262$.

5. **Type V.** Factors of expressions of the form $\boldsymbol{a^3 + b^3}$ or $\boldsymbol{a^3 - b^3}$.

Example 1. Resolve into factors $8a^3 + 27b^3$.

$$\begin{aligned} 8a^3 + 27b^3 &= (2a)^3 + (3b)^3 \\ &= (2a + 3b)\{(2a)^2 - (2a)(3b) + (3b)^2\} \\ &= (2a + 3b)(4a^2 - 6ab + 9b^2). \end{aligned}$$

NOTE. The first step, *i.e.* the reduction of the given expression to the standard form, is essential.

Example 2. Resolve into factors $27x^3 - 1$.

$$\begin{aligned} 27x^3 - 1 &= (3x)^3 - (1)^3 \\ &= (3x - 1)\{(3x)^2 + (3x)(1) + (1)^2\} \\ &= (3x - 1)(9x^2 + 3x + 1). \end{aligned}$$

EXERCISE 36.

Resolve into factors (*mentally*) :

1. $m^3 + 1$. 2. $1 - a^3$. 3. $8 - x^3$.
4. $p^3 + 64$. 5. $125a^3 - 8$. 6. $p^3q^3 + 216$.
7. $64a^3 + 1000$. 8. $729x^3 - 216$. 9. $125x^6 + 64y^3$.
10. $8x^3 - 1000b^3$. 11. $64m^3n^3 + 1$. 12. $729x^3y^3 - 512z^3$.

Example 3. Resolve into factors $27a^7 - ab^6$

$$\begin{aligned} 27a^7 - ab^6 &= a(27a^6 - b^6) \\ &= a\{(3a^2)^3 - (b^2)^3\} \\ &= a(3a^2 - b^2)\{(3a^2)^2 + (3a^2)(b^2) + (b^2)^2\} \\ &= a(3a^2 - b^2)(9a^4 + 3a^2b^2 + b^4). \end{aligned}$$

Resolve into factors :

13. $a^6 - 8b^6$. 14. $27m^6 + 125n^6$.
15. $a^9 - b^9$. 16. $a^{12} - 8b^6c^6$.
17. $343a^{13}b + 64ab^{13}$. 18. $(3x + 2y)^3 - z^3$.
19. $(2a - 3b)^3 + 27c^3$. 20. $(2a + b)^3 - (a + 2b)^3$.
21. $(a - 2b)^3 + 1$ 22. $x^3 - 3x^2y + 3xy^2 - y^3 - 1$.
23. $125(a - 2b)^3 - 27(3a - 4b)^3$. 24. $64(x - 3y)^3 + 343(2x + y)^3$.

Simplify :

25. 12^3+8^3. 26. 25^3-5^3. 27. 32^3-12^3.

Example 4. Resolve into factors x^6-64.

$$x^6-64=(x^3)^2-(8)^2$$
$$=(x^3+8)(x^3-8)$$
$$=(x+2)(x^2-2x+4)(x-2)(x^2+2x+4).$$

NOTE. If you do this example by reducing the expression to the form $(x^2)^3-(4)^3$ and then factorise it, it will be found that the method given above is more convenient than this one.

Resolve into factors :

28. a^6-b^6. 29. $a^{12}-b^{12}$. 30. $64a^{12}-1$.

31. $729a^6-b^6$. 32. a^7-ab^6. 33. $3x^{13}-192x^7y^6$.

6. Type VI. Factors of quadratic expressions of the form $\boldsymbol{x^2+px+q}$.

Analysis. Since the first term of x^2+px+q is x^2, therefore the first term of each factor will be x.

Let $(x+m)(x+n)$ be the factors of x^2+px+q.

Now, $(x+m)(x+n)$ will be the factors of x^2+px+q if $x^2+(m+n)x+mn \equiv x^2+px+q$.

or, if $m+n=+p$ and $mn=+q$.

Thus, in order to find out the 2nd term in each of the above factors, we have *to think out* two numbers

(i) whose product may be equal to $(+q)$ and

(ii) whose algebraie sum may be equal to $(+p)$.

The application of these principles is illustrated in the following examples :

Example 1. Resolve into factors $x^2+11x+24$.

Obviously the first term of each factor will be x and the 2nd terms will be such that they are factors of $+24$ and their sum $=+11$.

The pairs of possible factors in which the product of the second terms $= +24$ are :

(i) $x + 1$ (ii) $x - 1$ (iii) $x + 2$ (iv) $x - 2$

$x + 24$ $\quad$ $x - 24$ $\quad$ $x + 12$ $\quad$ $x - 12$

(v) $x + 3$ (vi) $x - 3$ (vii) $x + 4$ (viii) $x - 4$

$x + 8$ $\quad$ $x - 8$ $\quad$ $x + 6$ $\quad$ $x - 6$

We notice that it is only in (v) that the sum of such factors $= +11$,

$\therefore$ $x+3$ and $x+8$ are the required factors.

Example 2. Resolve into factors $x^2 - 11x + 24$.

Here we have to find two numbers whose product $= +24$ and whose sum $= -11$.

Since the product is positive, the required numbers will be either both positive or both negative, and as their sum is negative, therefore both the required numbers will be negative.

Hence the pairs of possible factors are :

(i) $x - 1$ (ii) $x - 2$ (iii) $x - 3$ (iv) $x - 4$

$x - 24$ $\quad$ $x - 12$ $\quad$ $x - 8$ $\quad$ $x - 6$.

We notice that it is only in (iii) that the sum of 2nd terms is -11,

$\therefore$ $x-3$ and $x-8$ are the required factors.

Example 3. Resolve into factors $x^2 + 5x - 24$.

Here we have to find two numbers whose product $= -24$ and whose sum $= +5$.

Since the product $= -24$, one of the numbers must be negative and the other positive.

The pairs of the possible factors in which the product of the 2nd terms $= -24$ are;

(i)	(ii)	(iii)	(iv)
$x - 1$	$x + 1$	$x - 2$	$x + 2$
$x + 24$	$x - 24$	$x + 12$	$x - 12$

(v)	(vi)	(vii)	(viii)
$x - 3$	$x + 3$	$x - 4$	$x + 4$
$x + 8$	$x - 8$	$x + 6$	$x - 6.$

We notice that it is only in (v) that the sum of the 2nd terms $= +5$,

$\therefore \quad x-3$ and $x+8$ are the required factors.

Example 4. Resolve into factors $x^2-5x-24$.

Here we have to find two numbers whose product $= -24$ and whose sum $= -5$.

Since the product $= -24$, one of the numbers must be negative and the other positive.

Proceeding as in example 3, we find that it is only in (vi) that the sum of the 2nd terms $= -5$.

$\therefore \quad x+3$ and $x-8$ are the required factors.

NOTE.—(i) In types of examples 3 and 4, it is worth noting that the greater number has the same sign as the sign of the 2nd term in the original expression.

(ii) After some practice, it is not difficult to attempt such exercises *mentally*.

EXERCISE 37.

Resolve into factors:

1. $x^2+13x+42.$
2. $a^2-18a+65.$
3. $x^2+18x+65.$
4. $p^2-15p+54.$
5. $x^2+18x+72.$
6. $x^2+20x+91.$
7. $x^2-22x+117.$
8. $x^2+21x+104.$
9. $a^2+5a-104.$
10. $x^2+21x+108.$

11. $a^2+3a-108.$
12. $x^2+27x+180.$
13. $p^2+p-156.$
14. $a^2-3a-180.$
15. $a^2-2a-143.$
16. $a^2-a-240.$
17. $x^2+8x-105.$
18. $a^2-26a-120.$
19. $1-19x+60x^2.$
20. $1-49a-102a^2.$

Resolve into factors (*mentally*):

21. $x^2+21x+98.$
22. $a^2-17a+72.$
23. $p^2-5p-176.$
24. $m^2+6m-91.$
25. $p^2-10p+24.$
26. $x^2+7x-30.$
27. $m^2+9m-36.$
28. $x^2-11x-80.$
29. $a^2-10a-56.$
30. $m^2+5m-84.$
31. $x^2-20x+96.$
32. $x^2-6x-72.$
33. $x^2-25x+84.$
34. $x^2+17x-84.$
35. $a^2-9a-90.$
36. $x^2-22x-48.$
37. $x^2-x-42.$
38. $a^2+a-72.$
39. $1-19a-120a^2.$
40. $1+23x-78x^2.$

Example 5. Resolve into factors $x^2-2xy-15y^2$.

The factors will evidently be $(x+py)$, and $(x+qy)$, where p and q are such that their sum $=-2$ and their product $=-15$.

Proceeding as before, it is easy to see that -5 and $+3$ are the numbers whose sum $=-2$ and whose product $=-15$.

$\therefore\ x^2-2xy-15y^2=(x-5y)(x+3y)$

Resolve into factors:

41. $p^2-12pq+20q^2.$
42. $m^2-12mn+32n^2.$
43. $a^2+ab-30b^2.$
44. $a^2-4ab-45b^2.$
45. $p^2-13pq-48q^2.$
46. $a^2+20ab+96b^2.$
47. $m^2+3mn-28n^2.$
48. $x^2-11xy-80y^2.$
49. $a^2-6ab-91b^2.$
50. $x^2-6xy-135y^2.$

Example 6. Resolve into factors x^4+x^2-20.

Putting a for x^2, the given expression becomes

a^2+a-20, and when factorised $=(a+5)(a-4)$.

Hence $x^4+x^2-20=(x^2+5)(x^2-4)$

$$=(x^2+5)(x+2)(x-2).$$

Example 7. Resolve into factors $(a^2+2a)^2-4(a^2+2a)-21$.

Putting x for a^2+2a, the given expression becomes $x^2-4x-21$, and when factorised $=(x-7)(x+3)$.

Hence the given expression $=(a^2+2a-7)(a^2+2a+3)$.

Example 8. Resolve into factors

$$(2a+b)^2+(2a+b)(a-b)-12(a-b)^2.$$

Putting x for $2a+b$ and y for $a-b$, the given expression becomes $x^2+xy-12y^2$, and when factorised

$$=(x+4y)(x-3y).$$

Hence the given expression

$$= \{ (2a+b)+4(a-b) \} \{ 2a+b)-3(a-b) \}$$
$$=(2a+b+4a-4b)(2a+b-3a+3b)$$
$$=(6a-3b)(-a+4b)$$
$$=-3(2a-b)(a-4b).$$

Resolve into factors :

51. a^4-a^2-12.
52. x^4+3x^2-28.
53. x^4-18x^2-175.
54. a^4-25a^2+136.
55. $x^4-2x^2y^2-63y^4$.
56. a^6+2a^3-3.
57. p^6+26p^3-27.
58. x^6-27x^3+180.
59. $m^8-11m^4n^4-80n^8$.
60. $p^{12}-7p^6q^6-8q^{12}$
61. $(a^2-2a)^2-(a^2-2a)-30$.
62. $(x^2+x)^2-2(x^2+x)-35$.
63. $(m^2-3m)^2-8(m^2-3m)+12$.
64. $(p^2-5p)^2+5(p^2-5p)-36$.
65. $(a^2-4a)^2+16(a^2-4a)+48$.
66. $(x^2-6x)^2-3(x^2-6x)-180$.
67. $(m^2+7m)^2-(m^2+7m)-156$.
68. $(p^2+4p)^2+21(p^2+4p)+98$.
69. $(x^2-8x)^2-(x^2-8x)-240$.
70. $(m^2-7m)^2-2(m^2-7m)-195$.
71. $(x+y)^2+7(x+y)(x-y)+12(x-y)^2$.
72. $(x+2y)^2-2(x+2y)(x-y)-15(x-y)^2$.

73. $(2a-3b)^2-11(2a-3b)(a+2b)+24(a+2b)^2$.
74. $(3m-n)^2+3(3m-n)(m+n)-28(m+n)^2$.
75. $(3a+4b)^2-3(3a+4b)(a-2b)-10(a-2b)^2$.
76. $(4m-3n)^2-2(4m-3n)(m+2n)-35(m+2n)^2$
77. $(2a-5b)^2-5(2a-5b)(a-3b)-6(a-3b)^2$.
78. $(a^2-ab)^2-2(a^2-ab)(ab-b^2)-8(ab-b^2)^2$.
79. $(2a^2+3b^2)^2+5(2a^2+3b^2)(a^2-2b^2)-14(a^2-2b^2)^2$.
80. $(a^2-4ab)^2-9(a^2-4ab)(ab-4b^2)+18(ab-4b^2)^2$.

Simplify by factors :

81. $\dfrac{x^2+7x+10}{x^2+5x}$. 82. $\dfrac{x^2+2x+1}{x^2+5x+4}$.

83. $\dfrac{x^2-4x-5}{x^2-2x-15} \div \dfrac{x^2-x-2}{x^2-x-12}$.

84. What is the quotient if $x^2+7x+12$ is divided by $x+4$?

85. If $x+2$ is a factor of $x^2+mx+14$, what is the value of m ?

86. If $x-3$ is a factor of $x^2-px-15$, what is the value of p ?

87. If $x+4$ is a factor of x^2+9x+c, what is the value of c ?

7. **Type VII.** Factors of expressions of the form $\boldsymbol{ax^2+bx+c}$.

Analysis. $ax^2+bx+c=a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)$.

Let $(x+p)(x+q)$ be the factors of $x^2+\frac{b}{a}x+\frac{c}{a}$.

Here $p.q$ must be equal to $\frac{c}{a}$

and $p+q$,, ,, ,, ,, $\frac{b}{a}$

or $a(p.q)$,, ,, ,, ,, c

i.e., $(ap).(aq)$,, ,, ,, ,, ac

and $ap+aq$,, ,, ,, ,, b.

In other words, we have to find out (ap) and (aq) such two numbers whose product $= ac$ and whose algebraic sum $= b$.

The application of these principles is illustrated in the following examples :

Example 1. Resolve into factors $5x^2+33x-14$.

The *first* terms of the factors must be $5x$ and x.

The *second* ,, ,, ,, have different signs and are the factors of -14.

The possible pairs of factors are

(i) $\left.\begin{array}{l} 5x+\ 1 \\ \quad\times \\ x\ -14 \end{array}\right\}$ co-eff. of x would be $-70+1=-69$.

(ii) $\left.\begin{array}{l} 5x-\ 1 \\ \quad\times \\ x+14 \end{array}\right\}$,, ,, ,, $+70-1=+69$.

(iii) $\left.\begin{array}{l} 5x+14 \\ \quad\times \\ x-\ 1 \end{array}\right\}$,, ,, ,, $-5+14=+9$.

(iv) $\left.\begin{array}{l} 5x-14 \\ \quad\times \\ x+\ 1 \end{array}\right\}$,, ,, ,, $+5-14=-9$.

(v) $\left.\begin{array}{l} 5x+2 \\ \quad\times \\ x-7 \end{array}\right\}$,, ,, ,, $-35+2=-33$.

(vi) $\left.\begin{array}{l} 5x-2 \\ \quad\times \\ x+7 \end{array}\right\}$,, ,, ,, $+35-2=+33$.

$$\text{(vii)}\quad \left.\begin{array}{c} 5x+7 \\ \times \\ x-2 \end{array}\right\}\text{co-eff. of } x \text{ would be } -10+7=-3.$$

$$\text{(viii)}\quad \left.\begin{array}{c} 5x-7 \\ \times \\ x+2 \end{array}\right\}\quad ,, \quad ,, \quad ,, \qquad +10-7=+3.$$

As in case (vi) alone the co-eff. of $x=+33$, therefore the required factors are $5x-2$ and $x+7$.

Example 2. Resolve into factors $8x^2-6xy-9y^2$.

Here we have to find two numbers whose product $=-72$ and whose algebraic sum $=-6$.

On trial, we find that -12 and $+6$ are the numbers which satisfy the above conditions.

$$\begin{aligned} \therefore \quad 8x^2-6xy-9y^2 &= 8x^2-12xy+6xy-9y^2 \\ &= (8x^2-12xy)+(6xy-9y^2) \\ &= 4x(2x-3y)+3y(2x-3y) \\ &= (2x-3y)(4x+3y). \end{aligned}$$

Another Method.

When the factors cannot be found so easily, the following method may be employed:

Example 3. Resolve into factors $18x^2-9x-2$.

$$18x^2-9x-2=\frac{1}{18}\left\{(18x)^2-9(18x)-36\right\}$$

(Writing y for $18x$)

$$=\frac{1}{18}\left\{y^2-9y-36\right\}=\frac{1}{18}(y-12)(y+3)$$

$$=\frac{1}{18}(18x-12)(18x+3)$$

$$=\left(\frac{18x-12}{6}\right)\left(\frac{18x+3}{3}\right)$$

$$=(3x-2)(6x+1).$$

EXERCISE 38.

Resolve into factors:

1. $2x^2+11x+14.$
2. $3x^2+14x-5.$
3. $5x^2+8x+3.$
4. $6x^2+5x-6.$
5. $6x^2-4x-2.$
6. $8x^2-14x+3.$
7. $8x^2+2x-3.$
8. $12m^2+7m-10.$
9. $14a^2+29a+15.$
10. $20p^2+44p-15.$
11. $3-5p+2p^2.$
12. $2+7m-15m^2.$
13. $9a^2-15ab+4b^2.$
14. $2m^2-mn-21n^2.$
15. $4p^2+5pq-9q^2.$
16. $15x^2-77xy+10y^2.$
17. $12a^2+28ab-5b^2.$
18. $10x^2-41xy+21y^2.$

Simplify by factors:

19. $\dfrac{6x^2+x-35}{12x^2-19x-21}\times\dfrac{4x^2-5x-6}{2x^2-x-15}.$
20. $\dfrac{6x^2-7x-20}{3x^2-14x-24}\div\dfrac{21x^2+41x+10}{7x^2-40x-12}.$

Resolve into factors:

21. $2(x+y)^2-3z(x+y)-2z^2.$ [*Hint.* Put a for $(x+y)$.]
22. $2(a^2+b^2)^2+5ab(a^2+b^2)+2a^2b^2.$
23. $x^2(x+1)^2-14(x+1)x+24.$
24. $8(a+1)^2+2(a+1)(b+2)-15(b+2)^2.$

[*Hint.* Put x for $(a+1)$ and y for $(b+2)$.]

25. $8(3m-4n)^2-6(3m-4n)(4m-3n)-35(4m-3n)^2.$
26. $6x^4-7x^2-20.$
27. $6x^4-7x^2-3.$
28. $5x^6-7x^3-6.$
29. $18x^4+25x^2y^2-3y^4.$
30. $8x^6-65x^3+8.$
31. $4x^8-17x^4y^4+4y^8.$
32. $5x^2-13xy+6y^2.$
33. $5x^2-13xy-6y^2.$
34. $7x^2-25x+12.$
35. $7x^2-25x-12.$
36. $x^3y^3-9x^2y^2+20xy.$
37. $8a^3-2a^2b-15ab^2.$
38. $(a+7)(a-10)+16.$
39. $(x^2-4x)(x^2-4x-1)-20.$ [*Hint.* Put a for (x^2-4x).]
40. $(x^2+x-6)(x^2+x-20)-15.$

41. $(x^2+8x+7)(x^2+8x+15)-9.$

42. $\frac{x^2}{4}+2x-5.$ [*Hint.* Put a for $\frac{x}{2}$.]

8. **Type VIII.** Factors of expressions of the form

$$a^3+b^3+c^3-3abc.$$

By a well-known formula, $a^3+b^3+c^3-3abc$

$$=(a+b+c)(a^2+b^2+c^2-ab-ac-bc). \quad \text{(i)}$$

As $a^2+b^2+c^2-ab-ac-bc$

$$=\tfrac{1}{2}\{2a^2+2b^2+2c^2-2ab-2ac-2bc\}$$
$$=\tfrac{1}{2}\{(a^2-2ab+b^2)+(b^2-2bc+c^2)+(c^2-2ac+a^2)\}$$
$$=\tfrac{1}{2}\{(a-b)^2+(b-c)^2+(c-a)^2\}$$

$\therefore$ the formula can be put in the following form :

$$a^3+b^3+c^3-3abc=\tfrac{1}{2}(a+b+c)\{(a-b)^2+(b-c)^2+(c-a)^2\}. \quad \text{(ii)}$$

Example 1. Resolve into factors $8a^3-27b^3+1+18ab$.

Reducing the given expression to the standard form, we have

$$(2a)^3+(-3b)^3+(1)^3-3(2a)(-3b)(1)$$
$$=(2a-3b+1)\{(2a)^2+(-3b)^2+(1)^2-(2a)(-3b)-(2a)(1)-(-3b)(1)\}$$
$$=(2a-3b+1)(4a^2+9b^2+1+6ab-2a+3b).$$

Example 2. Factorise $(x+y)^3+(y+z)^3+(z+x)^3-3(x+y)(y+z)(z+x)$.

Putting a for $(x+y)$, b for $(y+z)$ and c for $(z+x)$, we get

$$a^3+b^3+c^3-3abc$$
$$=\tfrac{1}{2}(a+b+c)\{(a-b)^2+(b-c)^2+(c-a)^2\}.$$

Substituting the values of a, b and c, we get the expression

$$=\tfrac{1}{2}\{(x+y)+(y+z)+(z+x)\}\{(x-z)^2+(y-x)^2+(z-y)^2\}$$
$$=(x+y+z)(2x^2+2y^2+2z^2-2xy-2xz-2yz)$$
$$=2(x+y+z)(x^2+y^2+z^2-xy-xz-yz).$$

EXERCISE 39.

Factorise :

1. $a^3-8b^3+27c^3+18abc.$
2. $x^3-y^3-z^3-3xyz.$
3. $a^3-b^3-1-3ab.$
4. $x^3-3xy+y^3+1.$
5. $a^3+b^3+18ab-216.$
6. $x^3-8y^3-24xy-64.$
7. $64a^3-27b^3+1+36ab.$
8. $27x^3-125y^3-180xy-64.$
9. $8a^3-27b^3-c^3-18abc.$
10. $x^3-\dfrac{1}{x^3}-36.$ [*Hint.* $-36=-27-9.$]
11. $a^6+32a^3-64.$ [*Hint.* $32a^3=8a^3+24a^3.$]
12. $(x+y)^3+(y+z)^3-(z+x)^3+3(x+y)(y+z)(z+x).$
13. $(a-b)^3-(b-c)^3+(c-a)^3+3(a-b)(b-c)(c-a).$
14. $(x+1)^3+(x+2)^3+(x+3)^3-3(x+1)(x+2)(x+3).$
15. $(x+y+2z)^3+(y+z+2x)^3+(z+x+2y)^3$
$-3(x+y+2z)(y+z+2x)(z+x+2y).$

9. Type IX. **Factors of expressions of the form**

$$(x+a)(x+b)(x+c)(x+d)+k.$$

Example 1. Factorise $(x+1)(x+3)(x-4)(x-6)+13.$

The expression $=\{(x+1)(x-4)\}\{(x+3)(x-6)\}+13.$
$=(x^2-3x-4)(x^2-3x-18)+13.$

Putting a for x^2-3x, we get the expression

$$=(a-4)(a-18)+13$$
$$=a^2-22a+72+13$$
$$=a^2-22a+85$$
$$=(a-5)(a-17).$$

Substituting x^2-3x for a, the expression

$$=(x^2-3x-5)(x^2-3x-17).$$

Example 2. Factorise $(a+4)(a-6)(a+2)(a-3)-6a^2.$

The expression $=\{(a+4)(a-3)\}\{(a-6)(a+2)\}-6a^2$
$=(a^2+a-12)(a^2-4a-12)-6a^2$
$=(a^2-12+a)(a^2-12-4a)-6a^2.$

Putting x for a^2-12, the expression

$$\begin{aligned}&=(x+a)(x-4a)-6a^2\\&=x^2-3ax-4a^2-6a^2\\&=x^2-3ax-10a^2\\&=(x-5a)(x+2a).\end{aligned}$$

Substituting a^2-12 for x, the expression

$$=(a^2-5a-12)(a^2+2a-12).$$

NOTE. In factorising expressions of this type, we take the four factors in the expression in two pairs so that their products may contain the same *first two terms*, as illustrated in the above two examples.

EXERCISE 40.

Factorise :

1. $(x+1)(x+2)(x+3)(x+4)-3.$
2. $(x-1)(x-2)(x+3)(x+4)+4.$
3. $(x+2)(x+3)(x-5)(x-6)+12.$
4. $(x+1)(x+4)(x+7)(x+10)-40.$
5. $(x-3)(x-2)(x-6)(x-4)-12x^2.$
6. $(x-8)(x+4)(x+2)(x-4)+9x^2.$
7. $x(2x+1)(x-2)(2x-3)-63.$
8. $x(3x+2)(x-2)(3x-4)-21.$
9. $(2x+3)(2x+7)(2x-1)(2x-5)+175.$
10. $3x(3x-1)(3x+1)(3x+2)-3.$
11. $(x+3)(x^2-1)(x+5)-9.$
12. $(x^2-4)(x+1)(x+5)-45.$
13. $4(x+1)(2x+3)(x+2)(2x+5)-360.$
14. $9(3x+1)(x+1)(3x-4)(x-2)+13.$
15. Prove that the product of *any* four consecutive numbers increased by one is a perfect square.
16. Prove that the product of *any* four consecutive even numbers increased by 16 is a perfect square.
17. Prove that the product of *any* four consecutive odd numbers increased by 16 is a perfect square.

10. Type X. Factors of expressions which *can be reduced* to the form a^2-b^2.

Example 1. Resolve into factors $4a^2-b^2-9c^2+6bc$.

Since $6bc$ seems to be twice the product of two terms containing b and c, therefore terms containing b and c must be grouped together.

Thus the expression

$$=4a^2-(b^2+9c^2-6bc)$$
$$=(2a)^2-(b-3c)^2$$
$$=(2a+b-3c)(2a-b+3c).$$

Example 2. Resolve into factors $a^2-b^2-c^2+d^2-2(ad-bc)$.

The expression $=a^2-b^2-c^2+d^2-2ad+2bc$.

Since $2ad$ and $2bc$ appear to be twice the products of terms containing a, d and b, c respectively, therefore in this expression terms containing a, d and b, c are to be grouped within two pairs of brackets respectively.

Thus the expression

$$=(a^2-2ad+d^2)-(b^2-2bc+c^2)$$
$$=(a-d)^2-(b-c)^2$$
$$=\{(a-d)+(b-c)\}\ \{(a-d)-(b-c)\}$$
$$=(a+b-c-d)(a-b+c-d).$$

EXERCISE 41.

Resolve into factors:

1. $x^2+y^2-z^2+2xy$.
2. $16x^2-4y^2+12yz-9z^2$.
3. $4x^2-9y^2-25z^2+30yz$.
4. $16x^2+40xz-36y^2+25z^2$.
5. $36z^2-60xz-49y^2+25x^2$.
6. $2(ab+cd)-a^2-b^2+c^2+d^2$.
7. $a^2+4b^2-9c^2-16d^2-2(2ab-12cd)$.

8. $16a^2-9b^2-49c^2+25d^2+2(21bc+20ad)$.
9. $1-4a^2+9b^2-25c^2+20ac-6b$.
10. $4a^2-9b^2-25c^2+30bc-24a+36$.
11. $25c^2-49b^2-a^2+14ab-20c+4$.
12. $49a^2-1+16b^2-64c^2+16c-56ab$.

Example 3. Resolve into factors x^4-11x^2+1.

$$\begin{aligned} x^4-11x^2+1 &= x^4-2x^2+1-9x^2. \\ &= (x^2-1)^2-(3x)^2. \\ &= (x^2-1+3x)(x^2-1-3x) \\ &= (x^2+3x-1)(x^2-3x-1). \end{aligned}$$

Example 4. Resolve into factors $4x^4+81$.

$$\begin{aligned} 4x^4+81 &= (2x^2)^2+(9)^2+2(2x^2)(9)-2(2x^2)(9) \\ &= (2x^2+9)^2-(6x)^2 \\ &= (2x^2+6x+9)(2x^2-6x+9). \end{aligned}$$

Example 5. Resolve into factors $(a^2-2ab)-(c^2-2bc)$.

$$\begin{aligned} (a^2-2ab)-(c^2-2bc) &= a^2-2ab-c^2+2bc \\ &= a^2-2ab+b^2-c^2+2bc-b^2 \\ &= (a^2-2ab+b^2)-(c^2-2bc+b^2) \\ &= (a-b)^2-(c-b)^2 \\ &= \{(a-b)+(c-b)\}\ \{(a-b)-(c-b)\} \\ &= (a-b+c-b)(a-b-c+b) \\ &= (a-2b+c)(a-c). \end{aligned}$$

Resolve into factors :

13. a^4+2a^2+9.
14. a^4+7a^2+16.
15. $4a^4+8a^2b^2+9b^4$.
16. $9a^4-a^2b^2+16b^4$.
17. $4a^4-37a^2+9$.
18. $9a^4-33a^2b^2+16b^4$.
19. $25a^4-19a^2b^2+9b^4$.
20. $a^4-35a^2b^2+25b^4$.
21. $a^4-16a^2b^2+36b^4$.
22. $49a^4-60a^2b^2+16b^4$.
23. $a^4+a^2b^2+b^4$.
24. $625+4a^4$.
25. a^4+4.
26. a^4+64.
27. $9a^4+36b^4$.
28. $64p^4+81q^4$.

29. a^4+a^2+1.
30. a^8+a^4+1.
31. $a^8+a^4b^4+b^8$.
32. $25a^4+35a^2b^2+36b^4$.
33. $(9p^2-6pq)-(r^2-2qr)$.
34. $(a^2-6ab)-(25c^2-30bc)$.
35. $(25a^2-20ab)-3c(3c-4b)$.
36. $7a(7a-4b)-3c(3c-4b)$.

11. Type XI. Factorisation by suitable arrangement and grouping of terms.

Example 1. Resolve into factors :

$$9a^2-24ab+16b^2-6a+8b.$$

Grouping together terms of the same degree, we have the expression

$$=(9a^2-24ab+16b^2)-(6a-8b)$$
$$=(3a-4b)^2-2(3a-4b)$$
$$=(3a-4b)(3a-4b-2).$$

Example 2. Resolve into factors

$$a^2-b^2-9c^2+6bc+a+b-3c.$$

Grouping together terms of the same degree, we have the expression

$$=(a^2-b^2-9c^2+6bc)+(a+b-3c)$$
$$=\{(a^2)-(b-3c)^2\}+(a+b-3c)$$
$$=(a+b-3c)(a-b+3c)+(a+b-3c)$$
$$=(a+b-3c)(a-b+3c+1).$$

EXERCISE 42.

Resolve into factors :

1. $a^2+2ab+b^2+a+b$.
2. $a^2-2ab+b^2-a+b$.
3. $4a^2+12ab+9b^2-8a-12b$.
4. $a^2-10ab+25b^2-3a+15b$.
5. $a^2-b^2-c^2+2bc+a+b-c$.
6. $a^2-b^2-c^2-2bc+a-b-c$.
7. $4a^2-b^2-9c^2+6bc-6a+3b-9c$.
8. $9a^2-4c^2-24ab+16b^2-15a+20b+10c$.

Example 3. Resolve into factors $a^4+a^2c^2-b^2c^2-b^4$.

Combining similar terms, we have the expression

$$=(a^4-b^4)+(a^2c^2-b^2c^2)$$
$$=(a^2-b^2)(a^2+b^2)+c^2(a^2-b^2)$$
$$=(a^2-b^2)(a^2+b^2+c^2)$$
$$=(a-b)(a+b)(a^2+b^2+c^2).$$

Example 4. Resolve into factors $a^3+5a^2+15a+27$.

Combining 1st and 4th terms and 2nd and 3rd, we get

$$(a^3+27)+(5a^2+15a)$$
$$=\{(a)^3+(3)^3\}+5a(a+3)$$
$$=(a+3)(a^2-3a+9)+5a(a+3)$$
$$=(a+3)(a^2-3a+9+5a)$$
$$=(a+3)(a^2+2a+9).$$

Example 5. Resolve into factors

$$a^3x^2+x^5-2a^3bx+a^3b^2+b^2x^3-2bx^4$$

Grouping together terms containing a^3 within one pair of brackets and the rest within another pair of brackets, we have the expression

$$=(a^3x^2-2a^3bx+a^3b^2)+(x^5+b^2x^3-2bx^4)$$
$$=a^3(x^2-2bx+b^2)+x^3(x^2-2bx+b^2)$$
$$=(x^2-2bx+b^2)(a^3+x^3)$$
$$=(x-b)^2(a+x)(a^2-ax+x^2).$$

Resolve into factors :

9. $a^2+ab-bc-c^2$.
10. $4a^2+8ac-12bc-9b^2$.
11. $9a^2-21ac+35bc-25b^2$.
12. a^3+a^2+a+1.
13. a^3+a^2-a-1.
14. $a^3-2a^2b+2ab^2-b^3$.
15. $a^3+5a^2-10a-8$.
16. $8a^3+18a^2b-27ab^2-27b^3$.
17. $8a^3+14a^2b-21ab^2-27b^3$.
18. $27a^3-15a^2b-20ab^2+64b^3$.
19. $xy(a^2+b^2)+ab(x^2+y^2)$.
20. $a^2(a+2b)+b^2(2a+b)$.
21. $a^4-5a^2c^2+20b^2c^2-16b^4$.
22. $16a^4-28a^2c^2+63b^2c^2-81b^4$.
23. $a^4-2a^3b+2a^2b^2-2ab^3+b^4$.

24. $a^4-5a^3b+4a^2b^2-10ab^3+4b^4$.

25. $a^2p^2-3a^2q^2-b^2p^2+2a^2pq-2b^2pq+3b^2q^2$.

12. Type XII. To factorise an expression, when one of its factors is known.

Example 1. Factorise $x^3+5x^2a+2xa^2-8a^3$, when one of its factors is $x+2a$.

Here we decompose the expression in such a way as to take out $x+2a$ as a factor.

$$
\begin{aligned}
\text{The expression} &= x^2(x+2a)+3x^2a+2xa^2-8a^3 \\
&= x^2(x+2a)+3xa(x+2a)-4xa^2-8a^3 \\
&= x^2(x+2a)+3xa(x+2a)-4a^2(x+2a) \\
&= (x+2a)(x^2+3xa-4a^2) \\
&= (x+2a)(x+4a)(x-a).
\end{aligned}
$$

EXERCISE 43.

Factorise the following expressions, having given one factor of each :

	Expression.	Given factor.
1.	$2a^3+9a^2-8a-15$.	$a+5$.
2.	$4a^3-15a^2+3a+18$.	$a-3$.
3.	$2a^3-9a^2-27a+54$.	$2a-3$.
4.	$2a^3-7a^2-7a+12$.	$2a+3$.
5.	$a^3+a^2b+ab^2-3b^3$.	$a-b$.
6.	$a^3-4a^2b+5ab^2-2b^3$.	$a-2b$.
7.	$a^3+3a^2b-18ab^2-40b^3$.	$a+2b$.
8.	$2a^3-3a^2b+7ab^2-3b^3$.	$2a-b$.
9.	$6a^3-17a^2b+6ab^2+8b^3$.	$3a-4b$.
10.	$12a^3+5a^2b-11ab^2-6b^3$.	$4a+3b$.

The steps employed in the solution of example 1 can be abbreviated by practice, as illustrated in the next example.

Example 2. Factorise $2x^4+x^3-8x^2+11x-12$, when one of its factors is $2x-3$.

The expression, when properly decomposed

$$=2x^4-3x^3+4x^3-6x^2-2x^2+3x+8x-12$$
$$=x^3(2x-3)+2x^2(2x-3)-x(2x-3)+4(2x-3)$$
$$=(2x-3)(x^3+2x^2-x+4).$$

Factorise the following expressions, having given one factor of each :

	Expression.	Given factor.
11.	$2x^4-x^3-x^2-x-3.$	$2x-3.$
12.	$2x^4-x^3-4x^2-x-6.$	$2x+3.$
13.	$x^4-6x^3+7x^2+5x-4.$	$x-4.$
14.	$2x^4+7x^3-x^2+11x-4.$	$x+4.$
15.	$2x^4+5x^3-x^2-5x+2.$	$2x-1.$
16.	$2x^4+3x^3+2x^2-1.$	$x^2+x+1.$
17.	$x^4+x^3+2x^2-x+3.$	$x^2-x+1.$
18.	$2x^4-9x^3+11x^2-7x+3.$	$2x^2-x+1.$
19.	$x^4-6x^3+13x^2-16x+12.$	$x^2-x+2.$
20.	$x^4-4x^2+12x-9.$	$x^2-2x+3.$

EXERCISE 44.

[Revision]

Factorise :

1. $\frac{a^4}{81}-\frac{b^4}{16}.$

2. $1-(a+b)^2.$

3. $a^3-b^3-a+b.$

4. $a^2-4b^2-4bc-c^2.$

5. $1-\frac{x^3}{y^3}.$

6. $a^{16}+a^8+1.$

7. $a^{12}-1.$

8. $a^3+7a^2+7a+1.$

9. $x^2-2ax+(a-b)(a+b).$

10. $a^2-6ab+9b^2-a+3b-12$.
11. $(5a+2)^4-(3a-1)^4$.
12. $(a^2+3a)^2+5(a^2+3a)-84$.
13. $(a+b)^2-2(a^2-b^2)+(a-b)^2$.
14. $(a^2+ab+b^2)^2-(a^2-ab-b^2)^2$.
15. $49(a-b)^2-25(a+b)^2$.
16. $(a+b)^2+2c(a+b)+c^2-d^2$.
17. $(ax+by)^2-(ax-by)^2$.
18. $3a^3-6a^2-105a$.
19. $ax^2+(a+b)xy+by^2$.
20. $a(a+c)-b(b+c)$. 21. $(ax-by)^2+(bx+ay)^2$.
22. $6a^3+4a^2b+9ab^2+6b^3$.
23. $a^2b^2-a^2-b^2+1$. 24. $a^2b^2c^2-a^2c-b^2c+1$.
25. $3-12(a-b)^2$. 26. $1-12(x-y)+(6x-6y)^2$.
27. $x^3-\dfrac{4}{x}$. 28. x^3+64. 29. x^4+64.
30. x^8-y^8. 31. x^9-y^9. 32. $a^{3m}-b^{3n}$.
33. $27a^3b-48ab^3$. 34. $\dfrac{x^3}{27}-\dfrac{y^3}{64}$.
35. $9x^2-41xy+20y^2$. 36. $9x^2+41xy-20y^2$.
37. $1-a+b-90(a-b)^2$.
38. $1+a+b+c+ab+ac+bc+abc$.
39. $x^3-\cdot001$. 40. $12x^4+243y^4$.
41. $21(1-x^2)+40x$. 42. $9x^2-20-(x-1)^2$.
43. $2-3x-2x^2+(2x-1)(3x-4)$.
44. $(x+y)^2-x^2+y^2-2(x-y)^2$.
45. $(x+y)^4+(x^2-y^2)^2+(x-y)^4$.
46. $2(p+q)^4+5(p^2-q^2)^2+2(p-q)^4$.
47. $ab(x+y)^2-(a+b)(x^2-y^2)+(x-y)^2$.
48. $(a^2-b^2)+\left(\dfrac{1}{a^2}-\dfrac{1}{b^2}\right)$. 49. $ax^2+(a^2+1)x+a$.
50. $(2a-3b)^2-(2a-3b)(a+2b)-20(a+2b)^2$.
51. $(a^2-5a)^2-3(a^2-5a)-28$.
52. $(2x^2-3xy)^2-5(2x^2-3xy)(2xy-y^2)-24(2xy-y^2)^2$.
53. $x^3-6x^2y+18xy^2-27y^3$. 54. $\dfrac{a^3}{8}+b^3+\dfrac{c^3}{27}-\dfrac{abc}{2}$.

55. $x^2+\left(p-\frac{1}{p}\right)xy-y^2$. 56. $4x^4+(7a)^4$.

57. x^4-22x^2+9. 58. $(a+b+c)^2-a^2+b^2-c^2$.

59. $(2a+b+c)^2-a^2-(a+b+c)^2$.

60. $(a+b)^2+(b+c)^2-(c+d)^2-(d+a)^2$.

61. $(a+b)^3+(b+c)^3+(c+d)^3+(d+a)^3$.

62. $4(ab+cd)^2-(a^2+b^2-c^2-d^2)^2$.

63. $(s-a)^3+(s-b)^3+(s-c)^3-3(s-a)(s-b)(s-c)$.

64. $(x+2)(x+3)(x+4)(x+5)-360$.

65. $(a-3)(a-2)(a+6)(a+9)+3a^2$.

66. $(3n+1)(3n+4)(3n+7)(3n+10)+81$.

67. $a^2(a+b-c)^2-c^2(b+c-a)^2$.

68. $11x^3y^3-61x^2y^2-30xy$.

69. $(a^2-b^2)x^2-2bx-1$.

70. $(a^2-b^2)(c^2-d^2)+4abcd$.

71. $(a^2-b^2)(c^2+d^2)+2(a^2+b^2)cd$.

72. $x^4(x^4-1)-a^4(a^4-1)$.

73. $x^2(x^4-1)+y^2(y^4-1)$.

74. $y^2(y-1)-x^2(x-1)$.

75. $1+x-x^2+x^4$. 76. $z(x+y)^2-y(z+x)^2$.

77. $a(a-2)+b(b-2)+2ab$. 78. $(x^3-y^3)+2(x^4-y^4)$.

79. $(25p^2-10pq)-(r^2-2rq)$.

80. $(3x+1)(x+2)^2-(x+2)(x-1)^2$.

81. $(x+1)(x+2)(x+3)-2x-2$.

82. $x(x-2y)^3-y(y-2x)^3$.

83. $a^2px+a^2rz-b^2qy+b^2rz-a^2qy+b^2px$.

84. $x-3$ is one factor of $x^3-7x^2-9x-63$; find the others.

85. $2x-3$ is one factor of $2x^3-11x^2+22x-15$; find the others.

86. Prove that $(x^2+x-2)(x^2-4x+3)(x^2-x-6)$ is a perfect square.

CHAPTER VI

APPLICATION OF FORMULÆ AND FACTORS

By a judicious use of factors and formulæ the solution of many questions in various rules of Algebra can be simplified as illustrated below.

First Four Rules

Example 1. Subtract $(x^2-7x+15)^2$ from $(x^2-7x-15)^2$.

$$\begin{aligned}
\text{The result} &= (x^2-7x-15)^2-(x^2-7x+15)^2 \\
&= [(x^2-7x-15)+(x^2-7x+15)]\,[(x^2-7x-15) \\
&\qquad -(x^2-7x+15)] \\
&= (x^2-7x-15+x^2-7x+15) \\
&\qquad (x^2-7x-15-x^2+7x-15) \\
&= (2x^2-14x)(-30) = -60x(x-7).
\end{aligned}$$

EXERCISE 45.

Subtract :

1. $(x^2-5x+12)^2$ from $(x^2-5x-12)^2$.
2. $(x^2-3x+8)^2$ from $(x^2+3x+8)^2$.
3. $x^2(z+x-y)^2$ from $y^2(y+z-x)^2$.
4. $(5x^2-5xy-11y^2)^2$ from $(6x^2-5xy+11y^2)^2$.

Example 2. Find the continued product of

$$(a-b),\ (a+b),\ (a^2-ab+b^2) \text{ and } (a^2+ab+b^2).$$

$$\begin{aligned}
\text{The result} &= \{(a-b)(a^2+ab+b^2)\}\,\{(a+b)(a^2-ab+b^2)\} \\
&= (a^3-b^3)(a^3+b^3) = a^6-b^6.
\end{aligned}$$

5. Multiply $1-a^2+a^4$ by $1+a^2+a^4$.
6. Multiply $(a+b)^2-(a+b)+1$ by $(a+b)^2+(a+b)+1$.

Find the continued product of :

7. $(a-b)^2$, $(a+b)^2$ and $(a^2+b^2)^2$.
8. a^2-ab+b^2, a^2+ab+b^2 and $a^4-a^2b^2+b^4$.
9. $1+a(1+a)$, $1-a(1+a)$, $1+a^2(1+a)^2$.
10. $(a+b+c)$, $(a+b-c)$, $(b+c-a)$, $(c+a-b)$.
11. $(x+y+z)$, $(x-y-z)$, $(y-z-x)$, $(z-x-y)$.

12. $(a+b+c-d)$, $(a+b-c+d)$, $(a-b+c+d)$, $(-a+b+c+d)$.
13. $a^2+2ab-3b^2$, $a^2+3ab+2b^2$, $a^2-5ab+6b^2$.
14. $2a^2+a-3$, $2a^2-a-3$, a^4+a^2+1.
15. $a^2-2ab+4b^2$, $a^2+2ab+4b^2$, $a^4-4a^2b^2+16b^4$.
16. $(x+3)^2$, $(x^2+9)^2$, (x^2-6x+9).
17. (x^2+4x+4), (x^2-4x+4), (x^4+8x^2+16).

Example 3. Divide $64a^3-27b^3-8c^3-72abc$ by $4a-3b-2c$.

$$\begin{aligned}\text{The dividend} &= (4a)^3+(-3b)^3+(-2c)^3-3(4a)(-3b)(-2c)\\ &= (4a-3b-2c)\times\{(4a)^2+(-3b)^2+(-2c)^2-(4a)(-3b)\\ &\qquad -(4a)(-2c)-(-3b)(-2c)\}\\ &= (4a-3b-2c)(16a^2+9b^2+4c^2+12ab+8ac-6bc).\end{aligned}$$

$\therefore$ the quotient $=16a^2+9b^2+4c^2+12ab+8ac-6bc$.

Divide :

18. $(a+b)^2-(b-c)^2$ by $a+c$.
19. $a^8+a^4b^4+b^8$ by $(a^2-ab+b^2)(a^2+ab+b^2)$.
20. $a^8-b^8+a^2b^2(a^4-b^4)$ by $(a^2-ab+b^2)(a^2+ab+b^2)$.
21. $(a^2-3a+2)(a-3)$ by a^2-5a+6.
22. $(a^2-3a+2)(a+4)$ by a^2+3a-4.
23. $(a^2-5ab+6b^2)(a-4b)$ by $a^2-7ab+12b^2$.
24. $(a^2+ab+b^2)(a^3+b^3)$ by $a^4+a^2b^2+b^4$.
25. a^6-b^6 by $a^3-2a^2b+2ab^2-b^3$.
26. $(a-b)^3+c^3$ by $a-b+c$.
27. $(a+b)^3-8c^3$ by $a+b-2c$.
28. $a^3+4a-39$ by $a-3$.
29. $a^3-4a^2b+24b^3$ by $a+2b$.
30. a^6-26a^3-27 by $(a^2+3a+9)(a^2-a+1)$.
31. $27a^3-b^3-8c^3-18abc$ by $3a-b-2c$.
32. $125x^3-216y^3+z^3+90xyz$ by $5x-6y+z$.
33. $(a^6+b^6-2a^3b^3)$ by $(a-b)^2$.
34. $(2x^2-6y^2-4z^2+11yz+2zx+xy)$ by $(x+2y-z)$.
35. $(a^2-b^2)^2+4ab-1$ by $(a+b)^2-1$.
36. Divide the product of $2x^2+11x-21$ and $3x^2-20x-7$ by x^2-49.

37. The product of two expressions is $(4y-3x)^3-(4x-3y)^3$ and one of them is $7(y-x)$; find the other.

Shew that

38. $(3x^2-5x+4)^2-(2x^2+x-4)^2$ is exactly divisible by $(x-4)(x-2)$.

39. $(ax+by)^2-(bx+ay)^2$ is exactly divisible by $(a+b)(x+y)$.

40. $(x-4y)^3-(y-4x)^3+(2y-3x)^3-(2x-3y)^3$ is exactly divisible by $5(x-y)$.

41. $(ax+by+cz)^3+(bx+cy+az)^3$ is exactly divisible by $(a+b)x+(b+c)y+(c+a)z$.

Numerical Fractions and Evaluation

Example 4. Simplify by factors $\dfrac{(1{\cdot}53)^2-({\cdot}47)^2}{1{\cdot}06}$.

$$\frac{(1{\cdot}53)^2-({\cdot}47)^2}{1{\cdot}06}=\frac{(1{\cdot}53+{\cdot}47)(1{\cdot}53-{\cdot}47)}{1{\cdot}53-{\cdot}47}$$
$$=1{\cdot}53+{\cdot}47=2.$$

Simplify by factors :

42. $\dfrac{{\cdot}345\times{\cdot}345-{\cdot}26\times{\cdot}26}{{\cdot}085}$. **43.** $\dfrac{{\cdot}72\times{\cdot}72-{\cdot}48\times{\cdot}48}{1{\cdot}2}$.

44. $\dfrac{(137\frac{3}{4})^2-(128\frac{1}{4})^2}{9\frac{1}{2}}$. **45.** $\dfrac{(125\frac{5}{9})^2-(12\frac{4}{9})^2}{138}$. **46.** $\dfrac{(73)^3-(57)^3}{73-57}$.

47. $\dfrac{({\cdot}43)^3+({\cdot}35)^3}{{\cdot}78}$. **48.** $\dfrac{{\cdot}841\times{\cdot}841\times{\cdot}841+{\cdot}159\times{\cdot}159\times{\cdot}159}{{\cdot}841\times{\cdot}841-{\cdot}841\times{\cdot}159+{\cdot}159\times{\cdot}159}$.

49. $\dfrac{11{\cdot}42\times11{\cdot}42\times11{\cdot}42-1{\cdot}42\times1{\cdot}42\times1{\cdot}42}{11{\cdot}42\times11{\cdot}42+11{\cdot}42\times1{\cdot}42+1{\cdot}42\times1{\cdot}42}$.

50. $\dfrac{62\times62-38\times38}{62\times62+2\times62\times38+38\times38}$. **51.** $\dfrac{(2{\cdot}35)^3-(1{\cdot}65)^3}{(2{\cdot}35)^2-(1{\cdot}65)^2}$.

52. $\dfrac{{\cdot}73\times{\cdot}78\times{\cdot}73+{\cdot}59\times{\cdot}59\times{\cdot}59}{{\cdot}073\times{\cdot}073-{\cdot}073\times{\cdot}059+{\cdot}059\times{\cdot}059}$.

Example 5. Find the value of $x^3+y^3+z^3-3xyz$ when $x=329$, $y=334$ and $z=337$.

The expression $=\frac{1}{2}(x+y+z)\{(x-y)^2+(y-z)^2+(z-x)^2\}$

$$=\tfrac{1}{2}(329+334+337)\{(329-334)^2+(334-337)^2+(337-329)^2\}$$

$$=\tfrac{1}{2}\times 1000(25+9+64)$$

$$=500\times 98=49000.$$

Example 6. Express $x^4+x^2y^2+y^4$ in terms of a and b when $x+y=2a$ and $x-y=2b$.

From the given relations, we have $x=a+b$, $y=a-b$ and $xy=a^2-b^2$.

Now $x^4+x^2y^2+y^4=(x^2+xy+y^2)(x^2-xy+y^2)$

$$=\{(x+y)^2-xy\}\{(x-y)^2+xy\}$$

$$=\{(2a)^2-(a^2-b^2)\}\{(2b)^2+(a^2-b^2)\}$$

$$=(3a^2+b^2)(3b^2+a^2).$$

Find the value of:

53. $a^3+b^3-c^3+3abc$ when $a=\cdot457$, $b=\cdot236$, $c=\cdot693$.

54. $a^3+b^3+c^3-3abc$ when $a=357$, $b=367$, $c=377$.

55. $x^3+8y^3-27+18xy$ when $x+2y=3$.

56. $8x^3+y^3-1+6xy$ when $x=\cdot357$, $y=\cdot286$.

57. $a^3+b^3+c^3-3abc$ when $a=\cdot326$, $b=\cdot336$, $c=\cdot338$.

Express:

58. a^3+b^3 in terms of x and y if $x=a+b$, $y=a-b$.

59. $x^4-6x^2y^2+y^4$ in terms of a and b if $x+y=2a$, $x-y=2b$.

60. Find the value of $x^2-y^2+4x+14y-45$ when $x+y=23$ and $x-y=7$. [*Hint.* The expression $=(x+2)^2-(y-7)^2$.]

61. If $x=b+c-2a$, $y=c+a-2b$, $z=a+b-2c$, find the value of $y^2+z^2-x^2+2yz$.

H. C. F. by Factors

Example 7. Find the highest common factor of:

$$a^3+2a^2b+ab^2,\ 6a^3+4a^2b-2ab^2,\ 3(a^2+ab)^2.$$

We have $a^3+2a^2b+ab^2=a(a^2+2ab+b^2)=a(a+b)^2$ ... (i)

$$6a^3+4a^2b-2ab^2=2a(3a^2+2ab-b^2)$$

$$=2a(a+b)(3a-b) \quad \text{... (ii)}$$

$$3(a^2+ab)^2 = 3a^2(a+b)^2 \quad \text{... (iii)}$$

$\therefore$ From (i), (ii) and (iii) the H. C. F. is obviously

$$=a(a+b).$$

Find by factors the H. C. F. of :

62. $a^2-b^2,\ a(a-b)$.
63. $a^3-3a^2b,\ 3ab^2-9b^3$.
64. $a^4-4a^2b^2,\ ab^2+2b^3$.
65. $a^3+8,\ a^2+5a+6$.
66. $a^3-27,\ (a-3)^3$.
67. $a^2-a, (a-1)^2, a^3-1$.
68. $a^2-3a-18,\ a^2+5a+6$.
69. $9x^4y^4-36x^2y^6,\ 24x^4y^4-48x^3y^3$.
70. $48a^2x^2(a+x)^2(a^2x^2-a^2x),\ 64(a^2x^5-a^5x^2)(ax^3+a^2x^2)$.
71. $4a^2b(x^2-1),\ 6ab^2(x+1)^2,\ 8ab(x^2-2x+1)$.
72. $a^5b^2-a^4b^3+a^3b^4,\ 15a^3b+30a^2b+15ab$.
73. $a^2+5a+6,\ a^2+9a+14,\ a^2-7a-18$.
74. $a^2-c^2+b^2-2ab,\ a^2-b^2-c^2-2bc$.
75. $x^2+7x+12,\ x^2+9x+20,\ x^3+64$.
76. $a^2-6ab+8b^2,\ a^2-8ab+16b^2,\ a^3-64b^3$.
77. $12a^2-a-20,\ 15a^2-38a+24,\ 21a^2-52a+32$.
78. $x^3-y^3-z^3-3xyz,\ x^2-y^2-z^2-2yz$.
79. $21a^4-8a^3-45a^2,\ 42a^5+26a^4-36a^3$.

Sometimes one of the expressions can be easily resolved into factors. Then we find by trial which of these factors is common to both.

Example 8. Find the H. C. F. of $5x^2-3x-8$ and x^4-2x^3-4x-7.

$$\begin{aligned} 5x^2-3x-8 &= 5x^2-8x+5x-8 \\ &= x(5x-8)+(5x-8) \\ &= (5x-8)(x+1). \end{aligned}$$

Obviously, $5x-8$ cannot be a factor of the 2nd expression as it begins with $5x$ and ends with 8.

Thus $x+1$ may be a factor, of the 2nd expression. Arranging the terms of the 2nd expression so as to take out, if possible, $x+1$ as a factor, we have the expression $=x^4+x^3-3x^3-3x^2+3x^2+3x-7x-7$

$$=x^3(x+1)-3x^2(x+1)+3x(x+1)-7(x+1)$$

$=(x+1)(x^3-3x^2+3x-7)$.

$\therefore$ $x+1$ is the H. C. F.

Find by factors the H. C. F. of:

80. $2a^2-7a+6,\ a^3-5a^2+a+10$.

81. $3a^2+a-14,\ a^3+2a^2-3a-10$.

82. $a^2-4a-21,\ a^3+3a^2-3a-9$.

83. $a^4-1,\ 3a^5+2a^4+4a^3+2a^2+a$.

84. $a^2+a-6,\ a^2+3a-10,\ a^3+a^2-5a-2$.

85. $a^2+a-12,\ a^2-2a-3,\ a^3-4a^2-2a+15$.

86. $8x^3-1,\ 10x^3-19x^2+5x+1$.

87. $x^3-y^3-1-3xy,\ x^2-4xy+3y^2+2x-3$.

L. C. M. by Factors

Example 9. Find by factors the L. C. M. of:

$(a^2+2a)^2,\ 2a^4+3a^3-2a^2$ and $2a^3-3a^2-14a$.

$(a^2+2a)^2=\{a(a+2)\}^2=a^2(a+2)^2$,

$2a^4+3a^3-2a^2=a^2(2a^2+3a-2)=a^2(a+2)(2a-1)$.

$2a^3-3a^2-14a=a(2a^2-3a-14)=a(a+2)(2a-7)$.

$\therefore$ the L. C. M. $=a^2(a+2)^2(2a-1)(2a-7)$.

Find by factors the L. C. M. of:

88. $(a-1)^2,\ (a^3-a)$.

89. $a^2-4,\ a^3+8$.

90. $(a-1)^3,\ a^3-1$.

91. $a^2+4a+3,\ a^2+5a+6$.

92. $a^2-ab-2b^2,\ a^2-5ab+6b^2,\ a^2-2ab-3b^2$.

93. $x^2+5ax+4a^2,\ x^2+11ax+28a^2,\ x^2+20ax+91a^2$.

94. $x^2+3x+2,\ x^2+4x+3,\ x^2+5x+6$.

95. $a^2-1,\ a^3+1,\ a^3-1$.

96. $a^2-1,\ a^2+1,\ a^4+1,\ a^8-1$.

97. $a^2-1,\ a^3+1,\ a^3-1,\ a^6+1$.

98. $a^2-4b^2,\ (a+2b)^2,\ (a-2b)^2$.

99. $6b^2(a-b)^2,\ 9a^3(a-b)^3,\ 12ab(a-b)^5$.

100. $18(a+b)^2(a^3-b^3),\ 24(a-b)^2(a+b)^3,\ 36(a^2-b^2)^2$.

101. $6a^2b^2(a^3-b^3),\ 15b^4(a-b)^3,\ 12a^3b(a-b)(a^2-b^2)$.

102. $12(a-b)^2(a^2+b^2),\ 9(a-b)^3(a+b),\ 6(a^4-b^4),$
$18(a+b)^3(a-b)$.

103. $a^4+a^2b^2+b^4,\ a^3b+b^4,\ (a^2-ab)^3$.

104. $x^3-3x^2+3x-1,\ x^3-x^2-x+1,\ x^4-2x^3+2x-1,$
$x^4-2x^3+2x^2-2x+1.$

105. $3a^3-18a^2b+27ab^2,\ 4a^4+24a^3b+36a^2b^2,\ 6a^4-54a^2b^2.$

106. $8a^3+27b^3,\ 8a^3-27b^3,\ 16a^4+36a^2b^2+81b^4.$

107. $6a^2-a-1,\ 3a^2+7a+2,\ 2a^2+3a-2.$

108. $2a^2+11a-21,\ 3a^2+25a+28,\ a^4+5a^3-14a^2-5a-35.$

109. $x^3-a^3,\ x^2-ax+a^2,\ x^3+a^3.$

110. $x^2+(b-c)x-bc,\ x^2+(c-a)x-ca,\ x^2+(a-b)x-ab$

111. $1+4a+4a^2-16a^4,\ 1+2a-8a^3-16a^4.$

Simplification of Fractions by Factors

Example 10 Reduce to the lowest terms $\dfrac{2x^2+3xy-2y^2}{2x^2-7xy+3y^2}.$

$$\frac{2x^2+3xy-2y^2}{2x^2-7xy+3y^2}=\frac{2x^2+4xy-xy-2y^2}{2x^2-6xy-xy+3y^2}$$
$$=\frac{2x(x+2y)-y(x+2y)}{2x(x-3y)-y(x-3y)}$$
$$=\frac{(x+2y)(2x-y)}{(x-3y)(2x-y)}=\frac{x+2y}{x-3y}.$$

Example 11. Simplify by factors

$$\frac{x^3-1}{x^2+x-6}\times\frac{x^2-4x+4}{x^2-4x+3}\times\frac{x^2-9}{x^4+x^2+1}.$$

The expression

$$=\frac{(x-1)(x^2+x+1)}{(x+3)(x-2)}\times\frac{(x-2)^2}{(x-3)(x-1)}\times\frac{(x+3)(x-3)}{(x^2+x+1)(x^2-x+1)}$$
$$=\frac{x-2}{x^2-x+1}.$$

Reduce to the lowest terms;

112. $\dfrac{x^2+3x+2}{x^2+6x+5}.$ **113.** $\dfrac{4(a+b)^2}{5(a^2-b^2)}.$ **114.** $\dfrac{x^2-16x-17}{x^2-22x+85}.$

115. $\dfrac{x^2+2x-15}{x^2+9x+20}.$ **116.** $\dfrac{1-7x+12x^2}{1-8x+15x^2}.$ **117.** $\dfrac{1-9x^2+14x^4}{1-4x^2-21x^4}.$

118. $\dfrac{6x^2-49x+65}{14x^2-93x+13}$.

119. $\dfrac{3abc-3b^2c}{a-bc-b(1-c)}$.

120. $\dfrac{x^2-(a-b)x-ab}{x^3+bx^2+ax+ab}$.

121. $\dfrac{2x^2+xy-y^2}{x^3+x^2y-x-y}$.

122. $\dfrac{a^2-b^2-2bc-c^2}{a^2+2ab+b^2-c^2}$.

123. $\dfrac{x^3-x^2-x+1}{x^3+x^2+x+1}$.

124. $\dfrac{4a^4-5a^2+1}{4a^4-6a^3+2a^2}$.

125. $\dfrac{(x+a)^2-(b+c)^2}{(x+b)^2-(a+c)^2}$.

126. $\dfrac{x-1}{x^3-1}$.

127. $\dfrac{x^3-1}{x^4-1}$.

128. $\dfrac{x^4-1}{x^6-1}$.

129. $\dfrac{x^8-1}{x^{12}-1}$.

130. $\dfrac{(x+1)^3-(x-1)^3}{(x+1)^4-(x-1)^4}$.

131. $\dfrac{(x^2+6x+11)^2-9(x+3)^2}{(x^2+7x+9)^2-(2x+5)^2}$.

Simplify by factors:

132. $\dfrac{ac+bc}{a^2+ab+b^2}\times\dfrac{a^3-b^3}{a^2-b^2}$.

133. $\dfrac{x(a^2-ab+b^2)}{a^2y-b^2y}\times\dfrac{(a^3+b^3)}{a^4+a^2b^2+b^4}$.

134. $\dfrac{x^2-3x+2}{x^2+x-12}\times\dfrac{x^2+6x+8}{x^2-4}$.

135. $\dfrac{x^2-4x-12}{x^2+2x-35}\times\dfrac{x^2+5x-14}{x^2-8x+12}$.

136. $\dfrac{ax+by}{a^2x^2-b^2y^2}\times\dfrac{a^3x^3+b^3y^3}{a^4x^4+a^2b^2x^2y^2+b^4y^4}$.

137. $\dfrac{x^2-1}{x^2+x-2}\times\dfrac{x^3+8}{x^4+4x^2+16}\div\dfrac{x^2+x}{x^3+2x^2+4x}$.

138. $\dfrac{6x^2-5xy-6y^2}{4x^2-11xy+6y^2}\times\dfrac{5x^2-7xy-6y^2}{6x^2-11xy+3y^2}\times\dfrac{12x^2-13xy+3y^2}{15x^2+19xy+6y^2}$.

139. $\dfrac{x^2-ax-2a^2}{x^2+ax-5cx-5ac}\times\dfrac{x^2+2bx-5cx-10bc}{x^2-2ax+2cx-4ac}\times\dfrac{x+2c}{x+2b}$.

140. $\dfrac{x^3-y^3}{(x+y)^2}\times\dfrac{(x^3+y^3)}{x^2+xy-2y^2}\div\dfrac{x^4+x^2y^2+y^4}{x^2+3xy+2y^2}$.

141. $\dfrac{x^3-a^3}{x^2-ax+bx-ab}\times\dfrac{x^2-b^2}{x^2+ax-bx-ab}\div\dfrac{x^2+ax+a^2}{x^2+ax+bx+ab}$.

142. $\dfrac{a^2-b^2+c^2-2ac}{a^2+b^2-c^2-2ab}\times\dfrac{c^2-a^2-b^2+2ab}{b^2-c^2-a^2+2ac}$.

SECTIONAL REVISION II

TEST PAPERS

PAPER I

1. (i) State $(a \pm b)^2 = a^2 \pm 2ab + b^2$ in words and apply it in expanding $(5x \pm 4y)^2$.

(ii) Simplify $(3a - b)^2 + (a + 3b)^2$.

2. Find the value of $x^2 + \frac{1}{x^2}$ when $x + \frac{1}{x} = 6$.

3. Prove that (i) the difference between two numbers which are formed by the same two digits is exactly divisible by 9; (ii) the difference of their squares is exactly divisible by 99.

4. Factorise the following:

(i) $(2a-1)^2 - (a-2)^2$; (ii) $1 - 4a^2 - b^2 - 4ab$;

(iii) $24x^2 - 4x - 48$; (iv) $(x-1)(x-5)(x-9) + (x-3)(x-4)(x-5)$.

5. Find the L. C. M. of $6a^2 - 7a + 2$, $3a^2 - 11a + 6$, $2a^2 - 7a + 3$.

6. Simplify $\dfrac{3x^2 - xy - 2y^2}{6x^2 - 5xy - 6y^2} \div \dfrac{x^2 + 3xy - 4y^2}{2x^2 + 5xy - 12y^2}$.

PAPER 2

1. (i) Find the missing term in the perfect squares:

(*a*) $81x^2 - 126xy + (\quad)$, (*b*) $36x^2 + (\quad) + 121y^2$.

(ii) Find by $(a \pm b)^2 = a^2 \pm 2ab + b^2$, the value of

(*a*) $(504)^2$, (*b*) $(495)^2$.

2. Find the value of $x^2 + \frac{1}{x^2}$ when $x - \frac{1}{x} = 8$.

3. (i) A class is required to work out examples 2, 5, 8, 11, 14, 17, 20, etc. in a set of examples. Find a formula which gives these numbers.

(ii) Write down the quotient of $x^5 - y^5$ by $x - y$.

4. (i) Shew that $(4x^2-5x+7)^2-(5x^2+14x+2)^2$ is divisible by x^2+x+1 and find the quotient.

(ii) Prove the identity

$$(x-1)(2x+3)+(2x+1)(x-5)\equiv 3x+2)(x-3)+(x-2)(x+1).$$

5. Factorise (i) x^3+x^2-42x.

(ii) $(a+b)^2-2(a^2-b^2)-15(a-b)^2$.

6. Find the L. C. M. of

$$35a^2-11a-6 \text{ and } 40a^2-29a+3.$$

PAPER 3

1. State $(a+b)(a-b)=a^2-b^2$ in words; apply it in finding the product of $(p-2q+3r)(p-2q-3r)$ and the value of $(328)^2-(323)^2$.

2. Find the value of $x^4+\dfrac{1}{x^4}$ when $x-\dfrac{1}{x}=4$.

3. Prove the identity $(n+3)^2\equiv 3(n+2)^2-3(n+1)^2+n^2$ and use it to express the value of d^2 in terms of a^2, b^2, c^2 where a, b, c, d are four consecutive numbers, of which a is the least.

4. Shew that $(4x^2-8x-1)^2-(2x^2-5x+7)^2$ is divisible by $2x^2-3x-8$, and express the quotient as the product of two factors.

5. Simplify the following:

(i) $\dfrac{(2a-9)^2-(a-6)^2}{(a-5)^2}$.

(ii) $\dfrac{3(a-8)^3(a-9)^2-(a-8)^2(a-9)^3}{2a^2-31a+120}$.

6. Find the L. C. M. of $a^2-3ab-10b^2$, $a^2-8ab+15b^2$ and $a^2+2ab-35b^2$.

PAPER 4

1. (i) State in words the formula

$(x+a)(x+b)(x+c)=x^3+(a+b+c)x^2+(ab+ac+bc)x+abc$, and complete the following:

$$(p+2)(\quad -3)(p+\quad)\equiv p^3+(\quad)p^2+(\quad)p\quad -24.$$

(ii) Write down the product of $(x+3)$, $(x-4)$, $(x+5)$.

2. Find the value of $x^3+\frac{1}{x^3}$ when $x+\frac{1}{x}=4$.

3. If $a+b+c=0$, shew that

(i) $(a+b)(b+c)(c+a)=-abc$,

and (ii) $a^3+b^3+c^3=3abc$.

4. If $a=1$, $b=2$, $c=3$, find the value of

(i) $\dfrac{(a+b+c)^3}{a^3+b^3+c^3}$, (ii) $a^{bc}+b^{ca}+c^{ab}$.

(iii) $(bc)^a+(ca)^b+(ab)^c$.

5. Factorise (i) $16x^8-1$, (ii) $27a^5b^2-8a^2b^5$,

(iii) $4(2a+3b)^2-9(a-b)^2$,

(iv) $(2x^2-5x+3)(2x^2-5x+4)-2$.

6. Simplify $\dfrac{3x^2-6xy}{4xy-12y^2}\div\dfrac{4x^2-6xy}{6xy-9y^2}\div\dfrac{4x^2-13xy+3y^2}{4x^2-9xy+2y^2}$.

PAPER 5

1. State in words $(a\pm b)^3=a^3\pm 3a^2b+3ab^2\pm b^3$

$=a^3\pm b^3\pm 3ab(a\pm b)$,

and fill in the blanks in the following

$(5m-\quad)^3\equiv(\quad)-3(\quad)(\quad)+3(5m)(\quad)-(2n)^3$.

2. Find the value of :

(i) a^3+b^3 if $a+b=11$ and $ab=30$,

(ii) a^3-b^3 if $a-b=3$ and $ab=40$,

(iii) $8m^3-36m^2+54m-50$ when $m=3$

3. (i) Shew that x^2+y^2 is less than $(x+y)^2$

(ii) Find v in terms of u and k from $\frac{1}{u}-\frac{1}{v}=\frac{1}{k}$.

4. Factorise (i) $(x-3)(x+1)^3+(x-1)^4-2(x-1)^3$.

(ii) $x^4+x^3-30x^2$,

(iii) $x^2-xz+yz-y^2$.

5. (i) Make n the subject of the formula $I=\dfrac{E}{\frac{r}{n}+R}$.

(ii) Find the co-efficient of x^3 in the expansion of $(6x^3-5x^2-4x+2)(3x^3+2x^2+5x-1)$.

6. (i) Find the value of $a^2+b^2+c^2$ when $a+b+c=15$ and $ab+ac+bc=85$.

(ii) Simplify $\dfrac{x^2+xy+y^2}{x^2+2xy+y^2} \times \dfrac{x^2-xy+y^2}{x^2-2xy+y^2} \div \dfrac{x^6-y^6}{(x^2-y^2)^3}$.

PAPER 6

1. State in words $(a+b)(a^2+ab+b^2)=a^3+b^3$; apply it in writing down the product of

$(3+4a)(9-12a+16a^2)$ and simplify

$(p^2+5)(p^4-5p^2+25)-(p^2-2)(p^4+2p^2+4)$.

2. Find the value of $x^3-\dfrac{1}{x^3}$ when $x-\dfrac{1}{x}=p$.

3. Factorise (i) $(2x+y)^2+6y(2x+y)+9y^2$,

(ii) $x^2(x+a)-x(x+a)(a+b)+ab(x+a)$.

4. (i) Find k when $(x-a)(x-3a)(x+a)(x+3a)+k$ is a perfect square.

(ii) Substitute $(m-1)$ for a in the expression $2a^3-3a^2+4a-5$ and arrange the result in descending powers of m.

5. Find the H. C. F. of :

(i) $15a^2-60b^2$, $3a^2-3ab-18b^2$, $6a^2+2ab-20b^2$.

(ii) $16a-15(1+a)(1-a)$, $(3a+1)^2-4(a-2)^2$, $5a^3-3a^2-10a+6$.

6. Simplify $\dfrac{x^2-13x+40}{x^2-5x+4} \times \dfrac{x^2-4x}{x^2-2x-48} \div \dfrac{x^2-5x}{x^2+5x-6}$.

PAPER 7

1. State in words $(a+b+c)^2=a^2+b^2+c^2+2ab+2ac+2bc$; apply it in expanding $(3x-2y+4z)^2$, and simplify $(2p-q+r)^2-(p+2q-r)^2-(2r+q-p)^2$.

2. Find the value of $a^2+b^2+c^2$ when $a+b+c=12$ and $ab+ac+bc=47$.

3. (i) Find the number which is half-way between $a-b$ and $b-a$.

(ii) Given any three consecutive numbers; prove that the difference of the squares of the greatest and the least is equal to four times the other number.

4. Factorise (i) $81a^4+64b^4$, (ii) $35(x-y)^2-41(x-y)+12$.

(iii) $4x^2-9y^2-6x-9y$,

(iv) $(2x+1)(2x+3)(2x+5)(2x+7)+16$.

5. Find the L. C. M. of $12a^2+8a-15$, $6a^2+7a-3$ and $18a^2-21a+5$.

6. Find the value of x which will make the expression $x^5-8x^3+11x^2+7x-1789$ exactly divisible by x^2+7x-1.

PAPER 8

1. State in words $(a+b+c)(a^2+b^2+c^2-ab-ac-bc)$ $=a^3+b^3+c^3-3abc$; apply it in finding the product of $(2m-n-3)(4m^2+n^2+2mn+6m-3n+9)$.

2. Use the identity $a^2-b^2=(a+b)(a-b)$ to find the difference between the squares of 587047 and 312953.

3. Resolve into factors:

(i) $512(x-\frac{1}{8})^3-8(x-\frac{1}{2})^3$,

(ii) $1+2x+x(x+2)+(x+1)(x+2)$,

(iii) $4a^2+12ab+9b^2-8a-12b$,

(iv) $(a^2-1)(a+2)+(a^2+2a)(a+1)$.

4. Shew by means of a formula that $(ax+by+cz)^3+(cx-by+az)^3$ is divisible by $(a+c)(x+z)$.

5. Simplify $\dfrac{a^4+a^2b^2+b^4}{a^2-4ab-21b^2}\times\dfrac{a^2+2ab-3b^2}{a^3-b^3}\div\dfrac{1}{a-7b}$.

6. Find the value of $27\times51+27\times49+73\times51+73\times49$.

PAPER 9

1. (i) Simplify by factors $\dfrac{\cdot254\times\cdot254-\cdot16\times\cdot16}{\cdot094}$.

(ii) Write by inspection the continued product of $(x-a)$, (x^2+a^2), (x^4+a^4).

2. If $2s=a+b+c$, prove that

$$(s-a)^3+(s-b)^3+(s-c)^3-3(s-a)(s-b)(s-c)$$
$$=\tfrac{1}{2}(a^3+b^3+c^3-3abc).$$

3. Divide the product of $2m^2+11m-21$ and $3m^2-20m-7$ by m^2-49.

4. Find the H. C. F. of $x^2-7x+10$ and $4x^3-25x^2+20x+25$.

5. Factorise (i) a^4+4. (ii) a^3+a^2-a-1.
(iii) a^8+a^4+1.

6. Simplify $\dfrac{6x^2+x-1}{2x^2-5x-12} \times \dfrac{6x^2+11x+3}{9x^2-1} \div \dfrac{2x^2+9x+4}{x^2-16}$.

PAPER 10

1. Simplify $\dfrac{\cdot71\times\cdot71\times\cdot71-\cdot29\times\cdot29\times\cdot29}{\cdot71\times\cdot71+\cdot71\times\cdot29+\cdot29\times\cdot29}$.

2. Factorise (i) $(a^4-b^4)-(a+b)^2(a-b)^2+2b(a^3+b^3)$,
(ii) $x^3-10x^2+31x-30$.

3. Find the H. C. F. of $x^2-(a-c)x-ac$ and $x^2-(a+c)x+ac$.

4. Simplify. $\dfrac{x^4+x^2+1}{x^2-1}+\dfrac{(x-1)^3}{x^3-1} \div \dfrac{x^3+8x^2-9x}{x+1}$.

5. Write down the continued product of
$(x-y)$, $(x+y)$, (x^2-xy+y^2), (x^2+xy+y^2), and (x^6+y^6).

6. If $x=a^2-bc$, $y=b^2-ca$, $z=c^2-ab$, prove that
$x^3+y^3+z^3-3xyz=(a^3+b^3+c^3-3abc)^2$.

CHAPTER VII

H. C. F. BY DIVISION

The H. C. F. of x^3-y^3 and x^2-y^2 is obviously $x-y$. If we multiply the first expression by a^3 or (a^3-b^3) and the second by a^2 or a^2-b^2, the H. C. F. of the new expressions is $a^2(x-y)$ or $(a-b)(x-y)$ and is thus altered.

This alteration is due to the fact that the multipliers contain a common factor.

If we multiply the first expression by a^3 or (a^3-b^3) and the second by m^2 or (m^2-n^2), the H. C. F. of the new expressions is the same as that of the original.

If the original expressions be $(a^3-b^3)(x^3-y^3)$ and x^2-y^2, the H. C. F. is $x-y$, and if the second expression be multiplied by $(a-b)$ or a^2-b^2, the H. C. F. is altered. This alteration is due to the fact that the multiplier is either contained in the first expression as a factor or contains a factor which is also a factor of the first expression.

This can be briefly expressed as follows :

The H. C. F. of two expressions A and B is the same as the H. C. F. of mA and nB provided that m does not contain a factor which is contained in B or n. and n does not contain a factor which is contained in A or m.

Again, if the original expressions be $p^3(x^3-y^3)$ and $q^2(x^2-y^2)$ and if the first expression be divided by p^3 and the second by q^2, the H. C. F. remains unaltered, because p^3 and q^2 do not contain a common factor. If the original expressions be $p^3(x^3-y^3)$ and $p^2(x^2-y^2)$ and if the first be divided by p^3 and the second by p^2, the H. C. F. is altered, for p^3 and p^2 do contain p^2 as the common factor.

If A and B stand for two expressions whose factors are $a.H$ and $b.H$ respectively, (where a and b are prime to one another), then obviously $A \pm B = H\ (a \pm b)$, or the H.C.F. of two expressions is contained in their *sum* as well as *difference.*

Again, since $mA \pm nB = H\ (ma \pm nb)$, the H. C. F. of two expressions is contained in the sum and difference of their multiples, provided that the multiplying factors (m and n) satisfy the conditions laid down above.

Example 1. Find the H.C.F. of a^3+a^2-2 and a^3+2a^2-3.

The process adopted in finding out H. C. F. is similar to one used in finding out the G. C. M. of large numbers in arithmetic.

a	a^3+a^2-2	a^3+2a^2-3	1
	a^3-a	a^3+a^2-2	
1	a^2+a-2	a^2-1	a
	a^2-1	a^2-a	
	$a-1$	$a-1$	1
		$a-1$	

$\therefore$ the H. C. F. $= a-1$.

Steps in the process. Divide a^3+2a^2-3 by a^3+a^2-2, the remainder $= a^2-1$. This remainder contains the required H. C. F.

Divide a^3+a^2-2 by a^2-1, the remainder in this case is $a-1$. This remainder contains the required H. C. F.

Divide a^2-1 by $a-1$. Since in this case there is no remainder, $a-1$ is the H. C. F.

Example 2. Find the H. C. F. of $a^5-a^3-4a^2-3a-2$ and $2a^5-a^4-a^3-3a^2$.

The second expression $= a^2(2a^3-a^2-a-3)$.

Since a^2 is a factor of the second expression but not of the first, therefore it may be neglected. The second expression thus left $= 2a^3-a^2-a-3$.

Now the first expression, as it is, cannot be divided by $2a^3-a^2-a-3$, unless we multiply it by 2, which is not a factor of the second expression.

Thus we have

		$a^5 \quad - a^3 - 4a^2 - 3a - 2$	
		2	
	$2a^3 - a^2 - a - 3$	$2a^5 \quad - 2a^3 - 8a^2 - 6a - 4$	
-2	$2a^3 + 18a^2 + 18a + 16$	$2a^5 - a^4 - a^3 - 3a^2$	a^2
-19	$-19a^2 - 19a - 19$	$a^4 - a^3 - 5a^2 - 6a - 4$	
	$a^2 + a + 1$	2	
		$2a^4 - 2a^3 - 10a^2 - 12a - 8$	
		$2a^4 - a^3 - a^2 - 3a$	a
		$- a^3 - 9a^2 - 9a - 8$	
		$- a^3 - a^2 - a$	$-a$
		$- 8a^2 - 8a - 8$	
		$- 8a^2 - 8a - 8$	-8

$\therefore$ the H. C. F. $= a^2 + a + 1$.

NOTE. The method illustrated above can be used with special advantage, where the expressions cannot be easily factorised. In this method it is *necessary* to arrange the terms in the expressions according to ascending or descending powers of a certain letter.

EXERCISE 46.

Find by the method of division the H. C. F. of :

1. $a^3+6a^2+13a+12$, $\quad a^3+7a^2+16a+16$.
2. $a^3+7a^2+17a+15$, $\quad a^3+8a^2+19a+12$.
3. $a^3-10a^2+26a-8$, $\quad a^3-9a^2+23a-12$.
4. $6a^3-5a^2+4a-1$, $\quad 3a^3+2a^2+5a-2$.
5. $2a^3-7a^2-8a-35$, $\quad 2a^3+9a^2+16a+21$.
6. $3a^3+4a^2-6a-8$, $\quad 36a^3+27a^2-16a+16$.
7. $24a^4-72a^3+54a^2$, $\quad 16a^5-48a^4+36a^3$.
8. $2a^3+5a^2+a-2$, $\quad 2a^3-5a^2-4a+3$.
9. $a^3+4a^2b-8ab^2+24b^3$, $\quad a^5-a^4b+8a^2b^3-8ab^4$.
10. a^3-4a^2+2a+3, $\quad 2a^4-9a^3+12a^2-7$.
11. a^4+a^2-6, $\quad a^4-3a^2+2$.
12. $a^4-9a^2-30a-25$, $\quad a^5+a^4-7a^2+5a$.
13. $2a^4-6a^3+3a^2-3a+1$, $\quad a^7-3a^6+a^5-4a^2+12a-4$.

14. $2a^4+9a^3+14a+3,$ $\quad 3a^4+15a^3+5a^2+10a+2.$

15. $12a^4-30a^3+156a-210,$ $\quad 15a^4-25a^3+145a-75.$

To find the H. C. F. of more than two expressions by this method, first we find the H. C. F. of two of the expressions, and then find the H.C.F. of this H.C.F. and the third expression; and so on. The last H.C.F. is the one required.

Find by the method of division the H. C. F. of:

16. $x^3-4x^2+4x,\ x^3+x^2-7x+2,\ 2x^3-x^2-7x+2.$

17. $x^3-x^2-7x+3,\ x^3-5x^2+9x-9,\ x^3-4x^2+4x-3.$

18. $2x^3-7x^2+7x-2,\ 4x^3-13x^2+11x-2,\ x^4-3x^3+6x-4.$

19. $x^3+6x^2y+9xy^2+4y^3,\ x^4+9x^3y+28x^2y^2+36xy^3+16y^4$
and $x^4+8x^3y+21x^2y^2+22xy^3+8y^4.$

20. $3x^3+28x^2y+52xy^2-48y^3,\ 3x^3+4x^2y-28xy^2+16y^3$
and $3x^3+10x^2y-44xy^2+24y^3.$

In certain cases the questions on H. C. F. can be done more easily by the *alternate destruction of the highest and the lowest terms*, as illustrated in the next example:

Example 3. Find the H. C. F. of x^5-x^3+8 and x^5-x^2+4

Let $A=x^5-x^3+8$

and $B=x^5-x^2+4.$

Then $A-B=-x^3+x^2+4$ [Divide by -1 and call the result C.]

or $C=x^3-x^2-4.$

Again, $2B-A=x^5+x^3-2x^2$, [Divide by x^2 and call the result D.]

or $D=x^3+x-2.$

Now the H. C. F. of A and B is the same as that of C and D.

$D-C=x^2+x+2.$ [Call it E.]

$C-2D=-x^3-x^2-2x$, [Divide by $-x$ and call the result F.]

or $F=x^2+x+2.$

Now the H. C. F. of E and F is the same as that of C and D or that of A and B, but E and F are identical, therefore x^2+x+2 is the required H. C. F.

NOTE. This method is of universal application and ***especially useful*** when the co-efficients of the given expressions are cross-wise equal or related as multiples and sub-multiples.

Find by the method of alternate destruction of the highest and the lowest terms, the H. C. F. of :

21. $2x^4+x^3-6x^2-2x+3$ and $2x^4-3x^3+2x-3$.
22. $x^4-3x+20$ and $5x^4-3x^3+64$.
23. $x^4+x^3-5x^2-3x+2$ and $x^4-3x^3+x^2+3x-2$.
24. $2x^5-11x^2-9$ and $4x^5+11x^4+81$.
25. $2x^5-5x^2+3$ and $3x^5-5x^3+2$.
26. x^5+11x^3-54 and $x^5+11x+12$.
27. $6x^3-11x^2+5x-3$ and $9x^3-9x^2+5x-2$.

Example 4. Reduce to the lowest terms $\dfrac{x^3+x^2+x-3}{x^3+3x^2+5x-3}$.

The H. C. F. of the numerator and the denominator when found by the *division method* or by the *method of alternate destruction of the highest and the lowest terms* is equal to x^2+2x+3.

$$\therefore \text{ the expression} = \frac{(x^3+x^2+x-3)\div(x^2+2x+3)}{(x^3+3x^2+5x-3)\div(x^2+2x+3)}$$

$$=\frac{x-1}{x+1}.$$

Reduce to the lowest terms :

28. $\dfrac{2x^2-13x+18}{2x^3-11x^2+3x+27}$.

29. $\dfrac{x^3-3x+2}{x^3+3x^2-4}$.

30. $\dfrac{x^3-13x+12}{x^3+x-2}$.

31. $\dfrac{4x^3-8x^2+5x-3}{12x^3+4x^2+x+5}$.

32. $\dfrac{x^4-4x^3+4x^2-4x+3}{x^4-2x^3-2x^2-2x-3}$.

33. $\dfrac{x^4-13x^2+36}{x^4-x^3-7x^2+x+6}$.

34. $\dfrac{3x^4-14x^3-9x+2}{2x^4-9x^3-14x+3}$.

35. $\dfrac{x^4-2x^3-25x^2+26x+120}{x^4-4x^3-19x^2+46x+120}$.

CHAPTER VIII

L. C. M. by H. C. F.

Let A and B stand for two expressions whose H. C. F. is H. Divide A and B by H and let a and b be their respective quotients.

Then
$$\left.\begin{array}{l} A=a.\ H \\ B=b.\ H. \end{array}\right\}$$

Since a and b have no common factor, the L. C. M. of A and B is obviously

$$=a.\ H.\ b.$$

Let L stand for the L. C. M. of A and B,

then
$$L=a.\ H.\ b, \qquad \text{(i)}$$
or
$$L.H.=(a.H).\ (b.H)$$
$$=A.\ B. \qquad \text{(ii)}$$

$$\left.\begin{array}{ll} \therefore & L=\dfrac{A}{H}.\ B \\ \text{or} & L=A.\dfrac{B}{H}. \end{array}\right\} \qquad \text{(iii)}$$

From (i), (ii) and (iii) we establish three important rules :

(i) *To find the L. C. M. of two expressions, we have to divide each by their H. C. F. and multiply the product of the quotients thus got by the H. C. F.*

(ii) *The product of the L. C. M. and the H. C. F. of two expressions is equal to the product of the expressions.*

(iii) *To find the L. C. M. of two expressions we have to divide one of them by their H. C. F. and multiply the quotient by the other.*

Example 1. Find the L. C. M. of $x^3-9x^2+26x-24$ and $x^3-6x^2+11x-6$.

First we find the H. C. F. of these expressions.

x	$x^3-6x^2+11x-6$	$x^3-9x^2+26x-24$	1
	x^3-5x^2+6x	$x^3-6x^2+11x-6$	
-1	$-x^2+5x-6$	$-3x^2+15x-18$	
	$-x^2+5x-6$	x^2-5x+6	-3

$\therefore$ the H.C.F. $=x^2-5x+6$ and when the second expression is divided by the H. C. F., the quotient $=x-1$.

$\therefore$ the L. C. M. $=(x-1)(x^3-9x^2+26x-24)$.

Example 2. The H. C. F. of two expressions is $x-7$ and their L. C. M. is $x^3-10x^2+11x+70$. One of the expressions is $x^2-5x-14$. Find the other.

$\because$ $(x-7)(x^3-10x^2+11x+70)=(x^2-5x-14)$ (2nd expression),

$$\therefore \text{ the 2nd expression}=\frac{(x-7)(x^3-10x^2+11x+70)}{x^2-5x-14}$$

$$=\frac{(x-7)(x^3-10x^2+11x+70)}{(x-7)(x+2)}$$

$$=\frac{x^3-10x^2+11x+70}{x+2}$$

$$=x^2-12x+35.$$

EXERCISE 47.

Find the H.C.F. and L.C.M. of the following expressions :

1. $x^3-5x^2+9x-9,\ x^3+x^2-3x+9$.
2. $x^3-2x^2-13x-10,\ x^3-x^2-10x-8$.
3. $4x^3-10x^2-18x+45,\ 6x^3+8x^2-27x-36$.
4. $2x^4+x^3-9x^2+8x-2,\ 2x^4-7x^3+11x^2-8x+2$.
5. $x^4+x^3-9x^2-3x+18,\ x^5+6x^2-49x+42$.
6. $8x^4-6x^3-8x^2+9x-6,\ 16x^4-12x^3+20x^2-9x+6$.
7. $x^4-2x^3+5x^2-4x+3,\ 2x^4-x^3+6x^2+2x+3$.
8. $15x^3-31ax^2+5a^2x+2a^3,\ 6x^4-25ax^3+26a^2x^2-a^4$.

9. The H. C. F. of two expressions is $5x^2-2x-1$, their L. C. M. is $30x^4+13x^3-11x^2-7x-1$, and one expression is $10x^3+x^2-4x-1$; find the other.

10. The H. C. F. of two expressions is x^3+1, their L. C. M. is $(x^3+1)(3x-1)(4x-1)$, and one expression is $3x^4-x^3+3x-1$; find the other.

11. The H. C. F. of two expressions is $1+x^3-x^4$, their L. C. M. is $(1+x)(1-x^3)(1+x^3-x^4)$, and one expression is $1+x+x^3-x^5$; find the other.

12. The H. C. F. of two expressions is x^2-2x+4, their L. C. M. is $x^6-2x^5+7x^4-6x^3-6x^2+36x-72$, and one expression is $x^4-2x^3+x^2+6x-12$; find the other.

To find the L. C. M. of more than two expressions, first we find the L. C. M. of two expressions, then the L. C. M. of that L. C. M. and the third expression, and so on. The last L. C. M. is the one required.

Find the L. C. M. of the following:

13. x^3+x+2, x^3+3x+4 and x^3+x^2+4.

***14.** $x^5+x^3+x^2+1$, x^4-x^3+x-1 and x^5+2x^4-x-2.

***15.** $x^5+x^4-4x^3+2x^2+6x-9$, x^4-x^2+6x-9 and $x^4+2x^3-5x^2-6x+9$.

CHAPTER IX

FRACTIONS

1. To reduce fractions to equivalent fractions with a common denominator.

Example 1. Reduce $\frac{2x}{a^2(a+x)}$, $\frac{3y}{b^2(a-x)}$, $\frac{4z}{c^2(a^2-x^2)}$ to equivalent fractions with the lowest common denominator.

The L. C. M. of the denominators $=a^2b^2c^2(a+x)(a-x)$ and the quotients obtained on dividing it by the denominators are $b^2c^2(a-x)$, $a^2c^2(a+x)$ and a^2b^2 respectively.

Hence we have

$$\frac{2x}{a^2(a+x)}=\frac{b^2c^2(a-x)2x}{a^2b^2c^2(a+x)(a-x)}=\frac{2b^2c^2x(a-x)}{a^2b^2c^2(a+x)(a-x)},$$

$$\frac{3y}{b^2(a-x)}=\frac{a^2c^2(a+x)3y}{a^2b^2c^2(a+x)(a-x)}=\frac{3a^2c^2y(a+x)}{a^2b^2c^2(a+x)(a-x)},$$

and $$\frac{4z}{c^2(a^2-x^2)}=\frac{a^2b^2(4z)}{a^2b^2c^2(a+x)(a-x)}=\frac{4a^2b^2z}{a^2b^2c^2(a+x)(a-x)}.$$

EXERCISE 48.

Reduce the following to the equivalent fractions with the lowest common denominator:

1. $\frac{1}{x+1}$, $\frac{3}{4x+4}$, $\frac{x}{x^2-1}$. 2. $\frac{3}{x^2-2x-3}$, $\frac{4}{x^2-5x+6}$.

3. $\frac{a}{a-b}$, $\frac{b}{a+b}$, $\frac{ab}{a^2-b^2}$, $\frac{b^2}{a^2+b^2}$.

4. $\frac{1}{x^2-ax+a^2}$, $\frac{1}{x^2+ax+a^2}$, $\frac{a^2}{x^4+a^2x^2+a^4}$.

5. $\frac{a}{a-b}$, $\frac{a+b}{a^2+ab+b^2}$, $\frac{ab}{a^3-b^3}$.

6. $\dfrac{1}{x^2-(a+b)x+ab}$, $\dfrac{1}{x^2-(a+c)x+ac}$, $\dfrac{1}{x^2-(b+c)x+bc}$.

7. $\dfrac{1}{(a-b)^2-c^2}$, $\dfrac{1}{(b-c)^2-a^2}$, $\dfrac{1}{(c-a)^2-b^2}$.

8. $\dfrac{a}{b(a-b-c)}$, $\dfrac{b}{a(a-b+c)}$, $\dfrac{c}{a^2+b^2-c^2-2ab}$.

Example 2. Reduce to the lowest common denominator

$$\frac{a}{x-a},\ \frac{x}{a-x},\ \frac{a^2}{x^2-a^2},\ \frac{ax}{a^2-x^2}.$$

Here we have to arrange the denominators in the same order.,

Thus $\dfrac{x}{a-x}=\dfrac{-1\times x}{-(a-x)}=\dfrac{-x}{x-a}$,

and $\dfrac{ax}{a^2-x^2}=\dfrac{-1\times ax}{-(a^2-x^2)}=\dfrac{-ax}{x^2-a^2}$.

The L. C. M. of the denominators when arranged as above $=x^2-a^2$ and the quotients obtained by dividing it by the denominators are $x+a$, $x+a$, 1, and 1 respectively.

Hence we have

$$\frac{a}{x-a}=\frac{a(x+a)}{x^2-a^2},$$

$$\frac{x}{a-x}=\frac{-x}{x-a}=\frac{-x(x+a)}{x^2-a^2},$$

$\dfrac{a^2}{x^2-a^2}$ stands as it is,

$$\frac{ax}{a^2-x^2}=\frac{-ax}{x^2-a^2}.$$

Reduce to the lowest common denominator :

9. $\dfrac{2}{a-b}$, $\dfrac{3}{b-a}$, $\dfrac{4}{a+b}$.

10. $\dfrac{a+b}{a-b}$, $\dfrac{a-b}{a+b}$, $\dfrac{2ab}{b^2-a^2}$.

11. $\dfrac{4a-b}{1-4ab}$, $\dfrac{4a+b}{1+4ab}$, $\dfrac{4b(1-8a^2)}{16a^2b^2-1}$.

12. $\dfrac{3}{x^2-5x+6}$, $\dfrac{4}{x^2-4x+3}$, $\dfrac{5}{3x-2-x^2}$.

2. Addition and Subtraction of Fractions

Example 1. Find the value of $\dfrac{x}{x-y}+\dfrac{y}{y-x}$.

Since $\dfrac{y}{y-x}=\dfrac{y\times(-1)}{(y-x)\times(-1)}=\dfrac{-y}{x-y}$,

we have $\dfrac{x}{x-y}+\dfrac{y}{y-x}=\dfrac{x}{x-y}+\dfrac{-y}{x-y}$

$$=\frac{x+(-y)}{x-y}=\frac{x-y}{x-y}=1.$$

Example 2. Find the value of $\dfrac{x}{x-y}-\dfrac{y}{x+y}$.

The L. C. M. of the denominators $=x^2-y^2$.

$$\therefore \frac{x}{x-y}-\frac{y}{x+y}=\frac{x(x+y)}{x^2-y^2}-\frac{y(x-y)}{x^2-y^2}$$

$$=\frac{x(x+y)-y(x-y)}{x^2-y^2}$$

$$=\frac{x^2+y^2}{x^2-y^2}.$$

Example 3. Simplify $\dfrac{1}{x^2-5x+6}-\dfrac{1}{x^2-6x+8}-\dfrac{2}{x^2-7x+12}$

Factorising the denominators, we have

the expression $=\dfrac{1}{(x-2)(x-3)}-\dfrac{1}{(x-2)(x-4)}-\dfrac{2}{(x-3)(x-4)}$

The L. C. M. of the denominators is $(x-2)(x-3)(x-4)$

$\therefore$ The given expression $=\dfrac{(x-4)-(x-3)-2(x-2)}{(x-2)(x-3)(x-4)}$

$$=\frac{x-4-x+3-2x+4}{(x-2)(x-3)(x-4)}$$

$$=\frac{-2x+3}{(x-2)(x-3)(x-4)}.$$

EXERCISE 49.

Find the value of :

1. $\dfrac{2}{x-2}+\dfrac{3}{x+3}$.
2. $\dfrac{4}{x-3}+\dfrac{5}{x+4}$.
3. $\dfrac{x+a}{x-a}-\dfrac{x-a}{x+a}$.
4. $1+\dfrac{4xy}{(x-y)^2}$.

5. $\frac{x-1}{x+1}-\frac{x-2}{x+2}$.

6. $(1+x)+\frac{x^2}{1-x}$.

7. $\frac{1}{a^2+ab}+\frac{1}{b^2+ab}$.

8. $\frac{x}{x^2-y^2}-\frac{y}{y^2-x^2}$.

9. $\frac{1}{2x+3y}-\frac{(2x-3y)^2}{8x^3+27y^3}$.

10. $\frac{1}{(a-b)(a-c)}+\frac{1}{(a-c)(b-c)}$.

11. $\frac{a+b}{a-b}-\frac{a-b}{a+b}+\frac{2ab}{b^2-a^2}$.

12. $\frac{1}{a^2-b^2}-\frac{2}{(a+b)^2}+\frac{3}{(a-b)^2}$.

13. $\frac{1}{x^2-5x+6}+\frac{1}{x^2-9x+20}$.

14. $\frac{1}{x^2+3x-10}-\frac{1}{2x^2-x-6}$.

15. $\frac{1}{x^2-3x+2}+\frac{1}{x^2-5x+6}+\frac{2}{x^2-8x+15}$.

16. $\frac{1}{x^2+7x+12}+\frac{1}{x^2+9x+20}-\frac{2}{x^2+8x+15}$.

17. $\frac{1}{x^2+3x+2}+\frac{1}{x^2+4x+3}-\frac{2}{x^2+5x+6}$.

18. $\frac{2x-3}{x^2+2x-3}-\frac{3x-4}{x^2+x-2}+\frac{x+5}{x^2+5x+6}$.

19. $\frac{1}{x^2+5ax+4a^2}+\frac{1}{x^2+11ax+28a^2}+\frac{2}{x^2+20ax+91a^2}$.

Example 4. Simplify $\frac{2a^2+a-1}{a+1}+\frac{3a^2+5a+2}{3a+2}+\frac{4-a^2}{a+2}$.

Factorising the numerators of these fractions, we have the expression

$$=\frac{(2a-1)(a+1)}{a+1}+\frac{(3a+2)(a+1)}{3a+2}+\frac{(2-a)(2+a)}{a+2}$$
$$=(2a-1)+(a+1)+(2-a)$$
$$=2a+2=2(a+1).$$

Simplify:

20. $\frac{x^2-1}{x-1}+\frac{x^2-7x+12}{x-4}+\frac{x^2+5x+6}{x+3}$.

21. $\frac{x^2-2x-3}{x^2-4x+3}+\frac{5x^2+5x-30}{x^2+2x-3}+\frac{x^2+x-2}{x^2-2x+1}$.

22. $\dfrac{a^2-(2b-3c)^2}{(3c+a)^2-4b^2}+\dfrac{4b^2-(3c-a)^2}{(a+2b)^2-9c^2}+\dfrac{9c^2-(a-2b)^2}{(2b+3c)^2-a^2}$.

23. $\dfrac{a^4+a^2b^2+b^4}{a^3+b^3}-\dfrac{a^2+ab}{a-b}+\dfrac{b^3}{a^2-b^2}$.

24. $\dfrac{a^2-1}{(a+1)^2}+\dfrac{a^2-5a+6}{a^2-4a+4}+\dfrac{1}{a+1}$.

Example 5. Find the value of $\dfrac{x^3+2x+3}{x+1}-\dfrac{x^3-2x+1}{x-1}$

Since it is not so easy to factorise the numerators, we actually divide each numerator by the corresponding denominator,

The expression $=(x^2-x+3)-(x^2+x-1)$

$$=x^2-x+3-x^2-x+1$$
$$=-2x+4=2(2-x).$$

25. Find the value of $\dfrac{x^3+3x^2-4}{x+2}+\dfrac{x^3+1}{x+1}$

$$+\dfrac{2x^3+6x^2+x+3}{x+3}.$$

26. Find the value of $\dfrac{3x^3-4x^2-2x+6}{x^2-1}-\dfrac{3x^3-4x^2+4x-3}{x^2+1}$.

Example 6. Simplify $\dfrac{1}{a^2-1}+\dfrac{1}{a^2-2}-\dfrac{1}{a^2+1}-\dfrac{1}{a^2+2}$.

Combining fractions having almost similar denominators, we have the expression

$$=\left(\frac{1}{a^2-1}-\frac{1}{a^2+1}\right)+\left(\frac{1}{a^2-2}-\frac{1}{a^2+2}\right)$$
$$=\frac{(a^2+1)-(a^2-1)}{a^4-1}+\frac{(a^2+2)-(a^2-2)}{a^4-4}$$
$$=\frac{2}{a^4-1}+\frac{4}{a^4-4}$$
$$=\frac{2(a^4-4)+4(a^4-1)}{(a^4-1)(a^4-4)}$$
$$=\frac{6(a^4-2)}{(a^4-1)(a^4-4)}.$$

Example 7. Simplify $\dfrac{1}{a+3}+\dfrac{1}{a-3}+\dfrac{6}{9-a^2}$.

Since the first two fractions have almost similar denominators, we combine them together and change the signs of the third.

$$\therefore \text{ the expression } = \left(\frac{1}{a+3}+\frac{1}{a-3}\right)-\frac{6}{a^2-9}$$
$$=\frac{a-3+a+3}{a^2-9}-\frac{6}{a^2-9}$$
$$=\frac{2a}{a^2-9}-\frac{6}{a^2-9}$$
$$=\frac{2(a-3)}{a^2-9}=\frac{2}{a+3}$$

Example 8. Simplify $\dfrac{1}{a-1}-\dfrac{a}{a^2-1}-\dfrac{a^2}{a^4-1}-\dfrac{a^4}{a^8-1}$.

In exercises of this type, it is useful to combine the first fraction with the second, the result so obtained with the third, and so on.

$$\text{The expression } =\left(\frac{1}{a-1}-\frac{a}{a^2-1}\right)-\frac{a^2}{a^4-1}-\frac{a^4}{a^8-1}$$
$$=\frac{(a+1)-a}{a^2-1}-\frac{a^2}{a^4-1}-\frac{a^4}{a^8-1}$$
$$=\frac{1}{a^2-1}-\frac{a^2}{a^4-1}-\frac{a^4}{a^8-1}$$
$$=\left(\frac{1}{a^2-1}-\frac{a^2}{a^4-1}\right)-\frac{a^4}{a^8-1}$$
$$=\frac{(a^2+1)-a^2}{a^4-1}-\frac{a^4}{a^8-1}$$
$$=\frac{1}{a^4-1}-\frac{a^4}{a^8-1}$$
$$=\frac{a^4+1-a^4}{a^8-1}=\frac{1}{a^8-1}$$

Simplify:

27. $\dfrac{1}{a-1}-\dfrac{1}{a+1}+\dfrac{1}{a-2}-\dfrac{1}{a+2}$.

28. $\dfrac{1}{a-b}-\dfrac{2}{2a+b}+\dfrac{1}{a+b}-\dfrac{2}{2a-b}.$

29. $\dfrac{1}{a-3b}+\dfrac{3}{a+b}+\dfrac{1}{a+3b}+\dfrac{3}{a-b}.$

30. $\dfrac{1}{a^2-1}+\dfrac{1}{a^2-3}-\dfrac{1}{a^2+1}-\dfrac{1}{a^2+3}.$

31. $\dfrac{3}{1+a}-\dfrac{2}{1-a}-\dfrac{5a}{a^2-1}.$ 32. $\dfrac{3-x}{1-3x}-\dfrac{3+x}{1+3x}-\dfrac{1-16x}{9x^2-1}.$

33. $\dfrac{a+b}{a-b}-\dfrac{2ab}{a^2-b^2}-\dfrac{2a^2b^2}{a^4-b^4}.$

34. $\dfrac{1}{1-a}+\dfrac{1}{1+a}+\dfrac{2}{1+a^2}+\dfrac{4}{1+a^4}.$

35. $\dfrac{1}{a-1}-\dfrac{1}{a+1}+\dfrac{a-2}{a^2-a+1}-\dfrac{a+2}{a^2+a+1}.$

36. $\dfrac{1}{a+b}+\dfrac{3b}{ab-a^2}+\dfrac{4b-a}{a^2-b^2}.$

37. $\dfrac{1}{x+a}+\dfrac{a}{x^2-a^2}+\dfrac{x}{x^2+a^2}+\dfrac{2x^3}{x^4+a^4}.$

38. $\dfrac{1}{a-1}-\dfrac{a}{a^2-1}-\dfrac{a^2}{a^4-1}-\dfrac{a^4}{a^8-1}.$

39. $\dfrac{1}{x-a}+\dfrac{2x}{x^2+a^2}+\dfrac{4x^3}{x^4+a^4}-\dfrac{8x^7}{x^8-a^8}.$

[*Hint.* Add and subtract $\dfrac{1}{x+a}$.]

40. $\dfrac{a-b}{a^2-ab+b^2}+\dfrac{a+b}{a^2+ab+b^2}+\dfrac{2b^3}{a^4+a^2b^2+b^4}.$

41. $\dfrac{1}{a^2-ab+b^2}-\dfrac{2ab}{a^4-a^2b^2+b^4}+\dfrac{4a^3b^3}{a^8-a^4b^4+b^8}-\dfrac{8a^7b^7}{a^{16}+a^8b^8+b^{16}}.$

42. $\dfrac{1}{x+3}+\dfrac{x+1}{x^2-3x+9}-\dfrac{2x^2+x+12}{x^3+27}.$

43. $\dfrac{1}{(n-1)n}+\dfrac{1}{n(n+1)}+\dfrac{1}{(n+1)(n+2)}+\dfrac{1}{(n+2)(n+3)}.$

Example 9. Simplify $\frac{1}{x+1}-\frac{3}{x+2}+\frac{3}{x+3}-\frac{1}{x+4}$.

Combining the first and the fourth terms and the second and the third, we have the expression

$$=\left(\frac{1}{x+1}-\frac{1}{x+4}\right)-3\left(\frac{1}{x+2}-\frac{1}{x+3}\right)$$

$$=\frac{x+4-x-1}{(x+1)(x+4)}-3.\frac{x+3-x-2}{(x+2)(x+3)}$$

$$=\frac{3}{(x+1)(x+4)}-\frac{3}{(x+2)(x+3)}$$

$$=3\left\{\frac{1}{(x+1)(x+4)}-\frac{1}{(x+2)(x+3)}\right\}$$

$$=3\left\{\frac{(x+2)(x+3)-(x+1)(x+4)}{(x+1)(x+2)(x+3)(x+4)}\right\}$$

$$=\frac{3(x^2+5x+6-x^2-5x-4)}{(x+1)(x+2)(x+3)(x+4)}$$

$$=\frac{6}{(x+1)(x+2)(x+3)(x+4)}.$$

Simplify:

44. $\frac{1}{x+2}-\frac{3}{x+3}+\frac{3}{x+4}-\frac{1}{x+5}$.

45. $\frac{1}{x+4}-\frac{3}{x+2}+\frac{3}{x}-\frac{1}{x-2}$.

46. $\frac{1}{x+1}-\frac{1}{x+3}-\frac{1}{x+5}+\frac{1}{x+7}$.

47. $\frac{1}{x+a}-\frac{1}{x+b}+\frac{1}{x-a}-\frac{1}{x-b}$.

48. $\frac{1}{a^2-2b^2}-\frac{1}{a^2+2b^2}+\frac{1}{a^2-3b^2}-\frac{1}{a^2+3b^2}$.

3. **Complex and Continued Fractions.** The process for the simplification of complex fractions in Algebra is similar to the one in Arithmetic. In Algebra, however, the process is generally shortened by the application of factors. A few examples given further illustrate the method of simplification.

Example 1. Simplify $\dfrac{\dfrac{a}{b}+\dfrac{b}{a}-2}{\dfrac{1}{b^2}-\dfrac{1}{a^2}}$.

$$\text{The fraction} = \frac{\dfrac{a^2+b^2-2ab}{ab}}{\dfrac{a^2-b^2}{a^2b^2}} = \frac{(a-b)^2}{ab}\times\frac{a^2b^2}{a^2-b^2}$$

$$=\frac{ab(a-b)}{a+b}.$$

Example 2. Simplify $\dfrac{\dfrac{a+b}{a-b}-\dfrac{a-b}{a+b}}{\dfrac{a+b}{a-b}+\dfrac{a-b}{a+b}}$.

$$\frac{a+b}{a-b}-\frac{a-b}{a+b}=\frac{(a+b)^2-(a-b)^2}{a^2-b^2}=\frac{4ab}{a^2-b^2}$$

$$\frac{a+b}{a-b}+\frac{a-b}{a+b}=\frac{(a+b)^2+(a-b)^2}{a^2-b^2}=\frac{2(a^2+b^2)}{a^2-b^2}.$$

$$\text{Hence the given fraction}=\frac{4ab}{a^2-b^2}\div\frac{2(a^2+b^2)}{a^2-b^2}$$

$$=\frac{4ab}{a^2-b^2}\times\frac{a^2-b^2}{2(a^2+b^2)}$$

$$=\frac{2ab}{a^2+b^2}.$$

EXERCISE 50.

Simplify:

1. $\dfrac{1-\dfrac{1}{1+x}}{\dfrac{1}{1+x}}$.

2. $\dfrac{\dfrac{1}{x-1}+\dfrac{x}{x+1}}{\dfrac{x}{x-1}-\dfrac{1}{x+1}}$.

3. $\dfrac{\dfrac{1+x}{1-x}-\dfrac{2x}{1+x}}{\dfrac{1-x}{1+x}+\dfrac{2x}{1-x}}$.

4. $\dfrac{\dfrac{x}{x-1}-\dfrac{x+1}{x}}{\dfrac{x}{x+1}-\dfrac{x-1}{x}}$.

5. $\dfrac{\dfrac{x+y}{x-y}-\dfrac{x^2+y^2}{x^2-y^2}}{\dfrac{x}{x+y}+\dfrac{y}{x-y}}.$

6. $\dfrac{\dfrac{x+2}{x+1}-\dfrac{1}{x+3}}{\dfrac{1}{x+1}+\dfrac{x+2}{x+3}}.$

7. $\dfrac{\dfrac{1}{x^2-xy+y^2}+\dfrac{1}{x^2+xy+y^2}}{\dfrac{x+y}{x-y}+\dfrac{x-y}{x+y}}.$

8. $\dfrac{x-\dfrac{2}{x}-1}{x-\dfrac{6}{x}+1}.$

9. $\dfrac{\left(\dfrac{x+a}{x-a}\right)^2-x^2}{\dfrac{x+a}{x-a}-x}.$

10. $\dfrac{\dfrac{x^3-y^3}{x-y}+\dfrac{x^3+y^3}{x+y}}{\dfrac{x+y}{x-y}-\dfrac{x-y}{x+y}}.$

11. $\dfrac{1+\dfrac{x-y}{x+y}}{1-\dfrac{x-y}{x+y}}\times\dfrac{1+\dfrac{y-z}{y+z}}{1-\dfrac{y-z}{y+z}}\times\dfrac{1+\dfrac{z-x}{z+x}}{1-\dfrac{z-x}{z+x}}.$

12. $\dfrac{\dfrac{x-y}{1+xy}+\dfrac{y-z}{1+yz}}{1-\dfrac{(x-y)(y-z)}{(1+xy)(1+yz)}}.$

13. $\dfrac{\dfrac{x+y}{1-xy}-\dfrac{x+z}{1-xz}}{1+\dfrac{(x+y)(x+z)}{(1-xy)(1-xz)}}.$

14. $\dfrac{1+\dfrac{1}{a}}{1-\dfrac{1}{a}}\div\dfrac{a+\dfrac{1}{a}}{a-\dfrac{1}{a}}-\dfrac{\dfrac{2}{a}}{1+\dfrac{1}{a^2}}.$

15. $\dfrac{\dfrac{a}{a+1}+\dfrac{a}{a-1}}{\dfrac{2}{a^2-1}}-\dfrac{4a-\dfrac{1}{a}}{2+\dfrac{1}{a}}.$

16. $\dfrac{1}{a-\dfrac{a}{a+1}}+\dfrac{1}{a+\dfrac{a}{a-1}}+\dfrac{1-\dfrac{3}{a}}{1-\dfrac{a}{3}}.$

17. $\left(\dfrac{1}{a}+\dfrac{\dfrac{1}{ab}}{\dfrac{1}{a}+\dfrac{1}{b}}\right)\left(\dfrac{1}{a}+\dfrac{\dfrac{1}{ab}}{\dfrac{1}{a}-\dfrac{1}{b}}\right)\div\dfrac{\dfrac{1}{a^2}+\dfrac{1}{b^2}}{\dfrac{1}{a^2}-\dfrac{1}{b^2}}.$

18. $\left\{\dfrac{y+\dfrac{x-y}{1+xy}}{1-\dfrac{(x-y)y}{1+xy}}-\dfrac{x-\dfrac{x-y}{1-xy}}{1-\dfrac{x(x-y)}{1-xy}}\right\}\div\left(\dfrac{x}{y}-\dfrac{y}{x}\right).$

In Algebra, for the simplification of a **continued fraction**, we begin from the bottom and go upward, step by step, as in Arithmetic.

Example 3. Simplify $\dfrac{a}{a-\dfrac{1}{a-\dfrac{a}{1+a}}}$.

The expression $=\dfrac{a}{a-\dfrac{1}{\dfrac{a+a^2-a}{1+a}}}=\dfrac{a}{a-\dfrac{1}{\dfrac{a^2}{1+a}}}$

$$=\frac{a}{a-1\times\dfrac{1+a}{a^2}}=\frac{a}{a-\dfrac{1+a}{a^2}}$$

$$=\frac{a}{\dfrac{a^3-1-a}{a^2}}=\frac{a\times a^2}{a^3-a-1}$$

$$=\frac{a^3}{a^3-a-1}.$$

Simplify :

19. $\dfrac{1}{x-\dfrac{1}{x+\dfrac{1}{x-\dfrac{1}{x}}}}$.

20. $\dfrac{x}{x-\dfrac{x-1}{1-\dfrac{1}{x+1}}}$.

21. $2-\dfrac{2}{1-\dfrac{3}{2-\dfrac{3}{1-x}}}$.

22. $\dfrac{1}{1-\dfrac{1}{1+\dfrac{1}{1+\dfrac{2b-a}{a-b}}}}$

23. $\dfrac{a}{a^2-\dfrac{a^3-1}{a+\dfrac{1}{a+1}}}$.

24. $\dfrac{1}{1-\dfrac{1}{a-\dfrac{1}{b-\dfrac{1}{a+b-\dfrac{a^2}{a-b}}}}}$.

25. $$\frac{1+x^2}{1-x+\dfrac{x^2}{1+x-\dfrac{x}{1+x}}}.$$ 26. $$\frac{x}{1-\dfrac{x}{1+x+\dfrac{x}{1-x+x^2}}}.$$

27. $$\frac{2x^3}{x^2+y^2-\dfrac{4xy(x-y)}{(x+y)-\dfrac{(x-y)^2}{x+y}}}.$$ 28. $$\frac{x}{1+\dfrac{x}{2-x+\dfrac{4x}{3-2x}}}.$$

29. $$\frac{3}{x+4-\dfrac{5(x+3)}{x+4-\dfrac{x^2+4}{x+1}}}.$$

30. $$\frac{x^2}{y-\dfrac{x^2}{y+\dfrac{x^2}{y-\dfrac{x^2}{y}}}}-\frac{x^2}{y+\dfrac{x^2}{y-\dfrac{x^2}{y+\dfrac{x^2}{y}}}}.$$

It is sometimes useful to decompose a given fraction into two parts, one integral and the other fractional. Such decomposition is illustrated by means of the examples given below:

Example 4. $$\frac{x+3}{x-2}=\frac{(x-2)+5}{(x-2)}=\frac{x-2}{x-2}+\frac{5}{x-2}$$
$$=1+\frac{5}{x-2}.$$

Example 5. $$\frac{2x-3}{x+5}=\frac{2(x+5)-13}{x+5}$$
$$=\frac{2(x+5)}{x+5}-\frac{13}{x+5}$$
$$=2-\frac{13}{x+5}.$$

Example 6. Prove that $\dfrac{2x^2-5x-7}{x-3} \equiv 2x+1-\dfrac{4}{x-3}$.

$$\begin{array}{r|l} x-3 \;) \; 2x^2-5x-7 & 2x+1 \\ 2x^2-6x & \\ \hline x-7 & \\ x-3 & \\ \hline -4 & \end{array}$$

By actual division we obtain $2x+1$ as the quotient and -4 as the remainder.

If the numerator be of a degree lower than that of the denominator, even then the decomposition is possible, as illustrated below :

Example 7. Prove that $\dfrac{3x}{2x^2+1} \equiv 3x-6x^3+12x^5-\dfrac{24x^7}{2x^2+1}$.

$$\frac{3x}{2x^2+1} \equiv \frac{3x}{1+2x^2}$$

$$\equiv 3x-6x^3+12x^5-\frac{24x^7}{1+2x^2}$$

[By actual division.]

$$\equiv 3x-6x^3+12x^5-\frac{24x^7}{2x^2+1}.$$

In such cases, the division can be carried on to any number of terms and the process can be stopped at any step by annexing to the quotient the fraction whose numerator is the last remainder and whose denominator is the divisor.

Thus if we had carried the quotient to four terms, in the above example, we would have got

$$\frac{3x}{2x^2+1} \equiv 3x-6x^3+12x^5-24x^7+\frac{48x^9}{2x^2+1}.$$

Express the following fractions in a form partly integral and partly fractional :

31. $\dfrac{x+7}{x+3}$. 32. $\dfrac{x+7}{x-3}$. 33. $\dfrac{2x+5}{x-1}$. 34. $\dfrac{3x-4}{x+2}$.

35. $\dfrac{3x-5}{x-3}$. 36. $\dfrac{4x-11}{2x-1}$. 37. $\dfrac{4x-9}{2x+1}$. 38. $\dfrac{6x-1}{3x+2}$.

39. $\dfrac{6x-7}{3x-2}$. 40. $\dfrac{x^4-1}{x+a}$. 41. $\dfrac{4x^3+4x^2+8x+1}{2x^2+2x+3}$.

Prove that

42. $\dfrac{1}{1-x} \equiv 1+x+x^2+x^3+\dfrac{x^4}{1-x}$.

43. $\dfrac{1}{x-1} \equiv \dfrac{1}{x}+\dfrac{1}{x^2}+\dfrac{1}{x^3}+\dfrac{1}{x^3(x-1)}$.

44. $\dfrac{1}{1+2x} \equiv 1-2x+4x^2-\dfrac{8x^3}{1+2x}$.

45. $\dfrac{1}{2x+1} \equiv \dfrac{1}{2x}-\dfrac{1}{4x^2}+\dfrac{1}{8x^3}-\dfrac{1}{8x^3(2x+1)}$.

46. $\dfrac{1+x^2}{(1-x)^2} \equiv 1+2x+4x^2+6x^3+\dfrac{8x^4-6x^5}{(1-x)^2}$.

47. $\dfrac{x+6}{x+4}+\dfrac{x+2}{x-4} \equiv 2+\dfrac{4x}{x^2-16}$.

48. $\dfrac{4x^2+7}{2x-1}+\dfrac{6x^2-8x+11}{3x-1} \equiv 4x-1+\dfrac{42x-17}{6x^2-5x+1}$.

*49. $$\frac{3x+6}{(x+1)(x+2)(x+3)} \equiv \frac{1}{(x+1)(x+2)}+\frac{1}{(x+1)(x+3)} + \frac{1}{(x+2)(x+3)}$$

*50. $$\frac{3x^2-14}{(x-1)(x-2)(x-3)} \equiv \frac{x+1}{(x-2)(x-3)}+\frac{x+2}{(x-1)(x-3)} + \frac{x+3}{(x-1)(x-2)}.$$

*51. Apply the identity

$\dfrac{1}{1-x} \equiv 1+x+x^2+\dfrac{x^3}{1-x}$ in finding the approximate value of $\dfrac{1}{0\cdot 9}$ to 3 decimal places.

CHAPTER X

SIMULTANEOUS EQUATIONS OF THE FIRST DEGREE AND PROBLEMS

1. Let us consider the equation $x+y=7$.

It is not an identity, for it is not satisfied by *any* values of x and y; *e.g.*, $x=8$ and $y=5$ will not satisfy it.

But if we assign any value to x and the corresponding value to y, as given in the following table,

x	0	1	2	3	4	5	6	7	8	9	10
y	7	6	5	4	3	2	1	0	−1	−2	−3

the equation is satisfied. As the number of such pairs of values is unlimited, such an equation is called an **indeterminate equation.**

Again, if we take another equation $x-y=3$, we find that it is not an identity, for it is not satisfied for *any* values of x and y; *e.g.*, $x=12$, $y=7$ will not satisfy it.

But if we assign any value to x and the corresponding value to y, as given in the following table,

x	0	1	2	3	4	5	6	7	8	9	10
y	−3	−2	−1	0	1	2	3	4	5	6	7

the equation is satisfied. As the number of such pairs of values is unlimited, this equation is also *indeterminate.*

It is important to note that in spite of the fact that both the equations $x+y=7$ and $x-y=3$ are indeterminate, *when considered separately*, yet there is *one value* of x, *i.e.*, 5 and *one value* of y, *i.e.*, 2 (see the above tables), for which they are *simultaneously* true.

Such equations for which we have to find a common solution are called **Simultaneous Equations.**

There are three methods of solving simultaneous equations involving two unknowns, as illustrated below :

Example 1. Solve the equations

$$3x+y=17 \quad \ldots \quad \text{(i)}$$
$$8x+11y=37 \quad \ldots \quad \text{(ii)}$$

From (i) we have $y=17-3x$ ... (iii)

,, (ii) ,, $y=\dfrac{37-8x}{11}$ ... (iv)

Equating the values of y from (iii) and (iv), we have

$$17-3x=\frac{37-8x}{11}$$

or $187-33x=37-8x$

or $-25x=-150$

or $x=6.$

Substituting the value of x in (i), we have

$$3\times 6+y=17$$

or $y=-1.$

Thus we have $x=6$ and $y=-1$.

This is the **first method.** It consists in expressing one variable in terms of the other in both the equations and equating the results.

EXERCISE 51.

Solve the following equations by the above method :

1. $\left.\begin{array}{l} x-2y=2 \\ 5x+5y=4. \end{array}\right\}$
2. $\left.\begin{array}{l} 4x-y=6 \\ 3x-2y=7. \end{array}\right\}$
3. $\left.\begin{array}{l} x+5y=13 \\ 2x-3y=0. \end{array}\right\}$
4. $\left.\begin{array}{l} 3x-4y=0 \\ 2x+3y=17. \end{array}\right\}$
5. $\left.\begin{array}{l} 2x+5y=7 \\ 8y+11x=19. \end{array}\right\}$
6. $\left.\begin{array}{l} 3x+4y=19 \\ 6x-5y=25. \end{array}\right\}$
7. $\left.\begin{array}{l} 5x+3y=47 \\ 3x-5y=1. \end{array}\right\}$
8. $\left.\begin{array}{l} 3x-2y=10 \\ 2x+4y=12. \end{array}\right\}$

9. $\left.\begin{aligned} x+ay&=b \\ ax-by&=c. \end{aligned}\right\}$ 10. $\left.\begin{aligned} \tfrac{1}{3}(x-y)&=\tfrac{1}{4}(y-1) \\ \tfrac{1}{7}(4x-5y)&=x-7. \end{aligned}\right\}$

Example 2. Solve the equations

$$5x+y=14 \quad \dots \quad \dots \quad \text{(i)}$$
$$15x+6y=54 \quad \dots \quad \dots \quad \text{(ii)}$$

From (i) $y=-5x+14$.

Now substituting this value of y in (ii), we have

$$15x+6(-5x+14)=54 \quad \dots \text{(iii)}$$

or $15x-30x+84=54$

or $-15x=-30$

or $x=2$.

To find y, substitute this value of x in (i)

$\therefore \quad 5\times 2+y=14$

$\therefore \quad y=4.$

$\therefore \quad \left.\begin{aligned} x&=2 \\ y&=4 \end{aligned}\right\}$ is the required solution.

This is the **second method.** It consists in expressing one variable in terms of the other in one equation only and substituting this value in the other equation.

Solve the following equations by the second method:

11. $\left.\begin{aligned} 3x+2y&=9 \\ x+3y&=10. \end{aligned}\right\}$ 12. $\left.\begin{aligned} 7x-y&=5 \\ 2x+3y&=8. \end{aligned}\right\}$

13. $\left.\begin{aligned} 4x+7y&=62 \\ 3y-2x&=8. \end{aligned}\right\}$ 14. $\left.\begin{aligned} 2x+5y+7&=0 \\ 3x+7y+10&=0. \end{aligned}\right\}$

15. $\left.\begin{aligned} 7x+4y&=5 \\ 5x+6y&=2. \end{aligned}\right\}$ 16. $\left.\begin{aligned} 3y&=7(x-10) \\ 2(x-5)&=3(y+10). \end{aligned}\right\}$

17. $\left.\begin{aligned} 3x+9y-42&=0 \\ 2x-4y+2&=0. \end{aligned}\right\}$ 18. $\left.\begin{aligned} \frac{7+x}{5}-\frac{2x-y}{4}&=3y-5 \\ \frac{5y-7}{2}+\frac{4x-3}{6}&=18-5x. \end{aligned}\right\}$

19. $\left.\begin{aligned} \tfrac{1}{2}(3x-2y)&=\tfrac{1}{4}(2x-y)+3 \\ \tfrac{1}{2}(5x-4y)&=\tfrac{1}{3}(4x-3y)+3. \end{aligned}\right\}$

20. $\left.\begin{aligned} ax+by&=c \\ a^2x+b^2y&=c^2. \end{aligned}\right\}$

Example 3. Solve the equations

$$3x+4y=20 \quad \cdots \quad \cdots \quad \text{(i)}$$

$$5x-3y=14 \quad \cdots \quad \cdots \quad \text{(ii)}$$

Multiplying (i) by 3 and (ii) by 4, we have

$$9x+12y=60 \quad \cdots \quad \cdots \quad \text{(iii)}$$

$$20x-12y=56 \quad \cdots \quad \cdots \quad \text{(iv)}$$

Adding (iii) and (iv), we have

$$29x=116$$

$$x=4.$$

Substituting this value of x in (i), we have

$$3\times 4+4y=20$$

or $$4y=8$$

or $$y=2.$$

$\therefore$ $\left.\begin{array}{l} x=4 \\ y=2 \end{array}\right\}$ is the required solution.

This is the **third method.** It consists in equalising the co-efficients of one and the same variable in both the equations, then by subtraction or addition deriving an equation involving the other variable only.

This method is more frequently used than the other two :

NOTE. It is important to note that in each of these methods it is essential to *eliminate* or get rid of one of the two variables and thus to derive an equation involving a *single* variable.

Solve the following equations by the third method :

21. $\left.\begin{array}{l} x+y=14 \\ x-y=6. \end{array}\right\}$

22. $\left.\begin{array}{l} x-y=12 \\ x+y=26. \end{array}\right\}$

23. $\left.\begin{array}{l} 2x+3y=28 \\ 3x-4y=-9. \end{array}\right\}$

24. $\left.\begin{array}{l} 4x-2y=12 \\ 5x+3y=26. \end{array}\right\}$

25. $\left.\begin{array}{l} 6x+7y=32 \\ 9x+y=29. \end{array}\right\}$

26. $\left.\begin{array}{l} 7x-8y=-22 \\ 11x-10y=4. \end{array}\right\}$

27. $\left.\begin{array}{l} 32x+15y=62 \\ 18x-13y=-8. \end{array}\right\}$

28. $\left.\begin{array}{l} 13x-11y=32 \\ 15x+7y=96. \end{array}\right\}$

29. $\left.\begin{array}{l} 2x + \cdot 4y = 1\cdot 2 \\ 3\cdot 4x - \cdot 02y = \cdot 01. \end{array}\right\}$

30. $\left.\begin{array}{l} \frac{1}{6}x + \frac{1}{16}y = 6 \\ \frac{1}{12}y - \frac{1}{9}x = 2 \end{array}\right\}$

31. $7x - 3y = 5x + y = 22.$

32. $14x + 3y = 6x + 15y = 48.$

33. $6x + 5y = 7x + 3y + 1 = 2(x + 6y - 1).$

34. $\left.\begin{array}{l} \frac{x-y}{3} = \frac{y-1}{4} \\ \frac{4x-5y}{7} = x - 7. \end{array}\right\}$

35. $\left.\begin{array}{l} \frac{32+x}{5} - \frac{2x-y}{4} = 3y \\ \frac{5y-43}{2} + \frac{4x-3}{6} = -5x \end{array}\right\}$

36. $\frac{x}{4} + \frac{y}{5} = \frac{x}{5} + \frac{y}{4} - 1 = 22.$

37. $3x - 2y + 2 = 5x - 3y + 1\frac{5}{6} = 6x - y - 4\frac{1}{2}.$

38. $7(x-5) = y - 2,\ \frac{4y-3}{2} = \frac{x+10}{6}.$

39. $7(2y - 3x) - 2 = 2(9x - y),\ 4x - 2y = 2(x+3) - 12.$

40. $ax + by = c,\ (a-b)x + (a-c)y = d.$

Simultaneous equations involving the reciprocals of the variables may be solved as they stand, without clearing them of the fractions.

Example 4. Solve the equations

$$\frac{2}{x} + \frac{3}{y} = 2 \quad \ldots \quad \ldots \quad \text{(i)}$$

$$\frac{5}{x} + \frac{2}{y} = 3\frac{1}{6} \quad \ldots \quad \ldots \quad \text{(ii)}$$

Multiplying (i) by 2 and (ii) by 3, we have

$$\frac{4}{x} + \frac{6}{y} = 4 \quad \ldots \quad \ldots \quad \text{(iii)}$$

$$\frac{15}{x} + \frac{6}{y} = \frac{19}{2} \quad \ldots \quad \ldots \quad \text{(iv)}$$

Subtracting (iv) from (iii), we have

$$-\frac{11}{x} = -\frac{11}{2} \text{ and } x = 2.$$

Substituting this value of x in (i), we have

$$\frac{2}{2}+\frac{3}{y}=2$$

or $$\frac{3}{y}=1 \text{ and } y=3.$$

$\therefore$ $\left.\begin{array}{l} x=2 \\ y=3 \end{array}\right\}$ is the required solution.

Solve the equations:

41. $\left.\begin{array}{l} 2x+\dfrac{3}{y}=5 \\ 5x-\dfrac{2}{y}=3. \end{array}\right\}$

42. $\left.\begin{array}{l} \dfrac{2}{x}+\dfrac{1}{y}=\dfrac{11}{12}. \\ \dfrac{1}{x}+\dfrac{3}{y}=1\frac{1}{12}. \end{array}\right\}$

43. $\left.\begin{array}{l} \dfrac{2}{x}+\dfrac{5}{y}=1 \\ \dfrac{3}{x}+\dfrac{2}{y}=\dfrac{19}{20}. \end{array}\right\}$

44. $\left.\begin{array}{l} \dfrac{6}{x}+\dfrac{4}{y}=3 \\ \dfrac{9}{x}-\dfrac{1}{y}=2\frac{3}{4}. \end{array}\right\}$

45. $\left.\begin{array}{l} \dfrac{1}{3x}-\dfrac{1}{7y}=\dfrac{2}{3} \\ \dfrac{1}{2x}-\dfrac{1}{3y}=\dfrac{1}{6}. \end{array}\right\}$

46. $\left.\begin{array}{l} \dfrac{2}{3x}-\dfrac{5}{4y}=\dfrac{1}{12} \\ \dfrac{3}{4x}-\dfrac{7}{6y}=\dfrac{1}{3}. \end{array}\right\}$

47. $\dfrac{m}{x}-\dfrac{n}{y}=a,\ px=qy.$

When the co-efficients of x and y are interchanged in two simultaneous equations, it is useful to employ the method of **addition** and **subtraction**, as illustrated below:

Example 5. Solve the equations

$$4x+7y=49 \quad \dots \quad \dots \quad \text{(i)}$$
$$7x+4y=61 \quad \dots \quad \dots \quad \text{(ii)}$$

Adding (i) and (ii), we have

$$11x+11y=110$$

or $$x+y=10 \quad \dots \quad \dots \quad \text{(iii)}$$

Subtracting (i) from (ii), we have

$$3x-3y=12$$

or $$x-y=4 \quad \dots \quad \dots \quad \text{(iv)}$$

Adding (iii) and (iv) and subtracting (iv) from (iii), we have

$$x=7 \text{ and } y=3.$$

Solve the equations:

48. $\left.\begin{array}{l}10x+11y=20\\ 11x+10y=22.\end{array}\right\}$

49. $\left.\begin{array}{l}9x+5y=60\\ 5x+9y=52.\end{array}\right\}$

50. $\left.\begin{array}{l}11x+3y=-21\\ 3x+11y=35.\end{array}\right\}$

51. $\left.\begin{array}{l}17y+6x=28\\ 17x+6y+5=0.\end{array}\right\}$

52. $\left.\begin{array}{l}3x+20=4y-10\\ 4(x-1)=3(y-3).\end{array}\right\}$

53. $\left.\begin{array}{l}\frac{1}{6}x+\frac{1}{4}y=\frac{13}{6}\\ \frac{1}{4}x+\frac{1}{6}y=2.\end{array}\right\}$

54. $\left.\begin{array}{l}ax+by=m\\ bx+ay=n.\end{array}\right\}$

55. $\left.\begin{array}{l}49x-57y=172\\ 57x-49y=252.\end{array}\right\}$

2. Simultaneous equations involving three unknown quantities.

Example 1. Solve $3x+y+2z=13$ ... ... (i)

$$x+4y+3z=14 \quad \ldots \quad \ldots \quad \text{(ii)}$$

$$2x+3y+4z=16 \quad \ldots \quad \ldots \quad \text{(iii)}$$

Multiplying (i) by 3 and (ii) by 2, we have

$$9x+3y+6z=39 \quad \ldots \quad \ldots \quad \text{(iv)}$$

$$2x+8y+6z=28 \quad \ldots \quad \ldots \quad \text{(v)}$$

Subtracting (v) from (iv), we have

$$7x-5y=11 \quad \ldots \quad \ldots \quad \text{(vi)}$$

Again multiplying (ii) by 4 and (iii) by 3, we have

$$4x+16y+12z=56 \quad \ldots \quad \ldots \quad \text{(vii)}$$

$$6x+9y+12z=48 \quad \ldots \quad \ldots \quad \text{(viii)}$$

Subtracting (viii) from (vii), we have

$$-2x+7y=8 \quad \ldots \quad \ldots \quad \text{(ix)}$$

Multiplying (vi) by 7 and (ix) by 5, we have

$$49x-35y=77 \quad \ldots \quad \ldots \quad \text{(x)}$$

$$-10x+35y=40 \quad \ldots \quad \ldots \quad \text{(xi)}$$

Adding (x) and (xi), we have

$$39x=117$$

$$\text{or} \quad x=3.$$

Substituting this value in (vi), we have

$$\therefore 21-5y=11, \quad \therefore y=2.$$

Substituting the values of x and y in (i), we have

$$9+2+2z=13, \quad \therefore z=1.$$

Thus $\quad x=3,\ y=2$ and $z=1$.

Example 2. Solve

$$y+z=17 \quad \ldots \quad \ldots \quad \text{(i)}$$
$$z+x=14 \quad \ldots \quad \ldots \quad \text{(ii)}$$
$$x+y=11 \quad \ldots \quad \ldots \quad \text{(iii)}$$

Adding (i), (ii) and (iii), we have

$$2(x+y+z)=42$$
$$x+y+z=21. \quad \ldots \quad \ldots \quad \text{(iv)}$$

Subtracting (i), (ii) and (iii), respectively from (iv), we have

$$x=4,\ y=7,\ z=10.$$

Example 3. Solve

$$\frac{xy}{x+y}=\frac{1}{3} \quad \ldots \quad \ldots \quad \text{(i)}$$
$$\frac{yz}{y+z}=\frac{1}{5} \quad \ldots \quad \ldots \quad \text{(ii)}$$
$$\frac{zx}{z+x}=\frac{1}{4} \quad \ldots \quad \ldots \quad \text{(iii)}$$

Taking the reciprocals of (i), (ii) and (iii), we have

$$\frac{x+y}{xy}=3, \text{ or } \frac{1}{y}+\frac{1}{x}=3 \quad \ldots \quad \ldots \quad \text{(iv)}$$
$$\frac{y+z}{yz}=5, \text{ or } \frac{1}{z}+\frac{1}{y}=5 \quad \ldots \quad \ldots \quad \text{(v)}$$
$$\frac{z+x}{zx}=4, \text{ or } \frac{1}{x}+\frac{1}{z}=4 \quad \ldots \quad \ldots \quad \text{(vi)}$$

Adding (iv), (v) and (vi), we get

$$2\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)=12$$
$$\text{or} \quad \frac{1}{x}+\frac{1}{y}+\frac{1}{z}=6 \quad \ldots \quad \ldots \quad \text{(vii)}$$

Subtracting (iv), (v) and (vi) respectively from (vii),

we have
$$\left.\begin{aligned} \frac{1}{z}&=3, \ \therefore z=\frac{1}{3} \\ \frac{1}{x}&=1, \ \therefore x=1 \\ \frac{1}{y}&=2, \ \therefore y=\frac{1}{2}. \end{aligned}\right\}$$

Example 4. Solve $x+ay+a^2z=a^3$ ... ... (i)

$x+by+b^2z=b^3$ ... ... (ii)

$x+cy+c^2z=c^3$ ... ... (iii)

Subtracting (ii) from (i), we have

$$y(a-b)+z(a^2-b^2)=a^3-b^3$$

or $$y+z(a+b)=a^2+ab+b^2 \quad \ldots \quad \text{(iv)}$$

Subtracting (iii) from (ii), we have

$$y(b-c)+z(b^2-c^2)=b^3-c^3$$

or $$y+z(b+c)=b^2+bc+c^2 \quad \ldots \quad \text{(v)}$$

Subtracting (v) from (iv), we have

$$z(a-c)=(a^2-c^2)+b(a-c)$$

or $$z=a+c+b$$
$$=a+b+c.$$

Substituting the value of z in (iv), we have

$$y+(a+b+c)(a+b)=a^2+ab+b^2$$

or $$y=-(ab+bc+ca).$$

Substituting the values of y and z in (i), we have

$$x-a(ab+bc+ca)+a^2(a+b+c)=a^3$$

or $$x=abc.$$

EXERCISE 52.

Solve the following equations :

1. $\left.\begin{aligned} x-y+z&=1 \\ x-2y+4z&=8 \\ x-3y+9z&=27. \end{aligned}\right\}$

2. $\left.\begin{aligned} x+2y+3z&=14 \\ 2x+3y+4z&=20 \\ 3x+y+6z&=23. \end{aligned}\right\}$

3. $\left.\begin{aligned} x-3y+2z&=1 \\ 2x-4y+z&=4 \\ 3x+y-5z&=-11. \end{aligned}\right\}$

4. $\left.\begin{aligned} x-y-z&=-2 \\ x+y+2z&=7 \\ 5x+6y+4z&=32. \end{aligned}\right\}$

5. $\left.\begin{aligned} x+2y+3z&=22 \\ 2x-3y+z&=5 \\ 3x+4y-2z&=7. \end{aligned}\right\}$

6. $\left.\begin{aligned} x+2y+3z&=20 \\ 2x+3y-5z&=-7 \\ 4x-5y+7z&=21. \end{aligned}\right\}$

7. $\left.\begin{aligned} \frac{1}{x}+\frac{1}{y}+\frac{1}{z}&=6 \\ \frac{2}{x}-\frac{3}{y}+\frac{4}{z}&=8 \\ \frac{3}{x}-\frac{4}{y}+\frac{5}{z}&=10 \end{aligned}\right\}$

8. $\left.\begin{aligned} \frac{1}{x}+\frac{1}{y}-\frac{1}{z}&=2 \\ \frac{3}{x}-\frac{4}{y}+\frac{5}{z}&=18 \\ \frac{2}{x}+\frac{3}{y}+\frac{4}{z}&=38 \end{aligned}\right\}$

9. $x+y=20$, $y+z=16$, $z+x=18$.

10. $x+y=2a$, $y+z=2b$, $z+x=2c$.

11. $\frac{1}{x}+\frac{1}{y}=1$, $\frac{1}{y}+\frac{1}{z}=2$, $\frac{1}{z}+\frac{1}{x}=4$.

12. $\frac{1}{x}+\frac{1}{y}=a$, $\frac{1}{y}+\frac{1}{z}=b$, $\frac{1}{z}+\frac{1}{x}=c$.

13. $\left.\begin{aligned} y+z-x&=8 \\ z+x-y&=12 \\ x+y-z&=16. \end{aligned}\right\}$

14. $\left.\begin{aligned} x-3y+z&=-3 \\ y-3z+x&=\ \ 1 \\ z-3x+y&=-7. \end{aligned}\right\}$

15. $ay+bx=c$, $cx+az=b$, $bz+cy=a$.

[*Hint.* Multiply first by c, second by b and third by a.]

16. $\frac{x+y}{xy}=\frac{y+z}{yz}=\frac{z+x}{zx}=\frac{2}{5}$.

17. $\left.\begin{aligned} 2x+3y&=5xy \\ 3y+4z&=7yz \\ 4z+5x&=9zx. \end{aligned}\right\}$

18. $\left.\begin{aligned} 3xy&=4(x+y) \\ 2xz&=3(x+z) \\ 5yz&=12(y+z). \end{aligned}\right\}$

19. $\left.\begin{aligned} x+y&=axy \\ y+z&=byz \\ z+x&=czx. \end{aligned}\right\}$

20. $\left.\begin{aligned} x-ay+a^2z&=a^3 \\ x-by+b^2z&=b^3 \\ x-cy+c^2z&=c^3. \end{aligned}\right\}$

21. $\left.\begin{aligned} x+ay+bcz&=a^2 \\ x+by+caz&=b^2 \\ x+cy+abz&=c^2. \end{aligned}\right\}$

22. $xy=20$, $yz=24$, $zx=30$.
[*Hint.* Multiply the three equations and take the square root.]

23. $\left.\begin{aligned} x(x+y+z)&=60 \\ y(x+y+z)&=75 \\ z(x+y+z)&=90. \end{aligned}\right\}$ [*Hint.* Add the three equations.]

24. $\left.\begin{aligned} (x+y)(y+z)&=96 \\ (y+z)(z+x)&=80 \\ (z+x)(x+y)&=120. \end{aligned}\right\}$

Example 5. Solve $x+y=10$ ... ... (i)

$xy=24$ ... ... (ii)

$$(x-y)^2=(x+y)^2-4xy$$

or $$x-y=\sqrt{(x+y)^2-4xy}$$

$$=\sqrt{10^2-4.24}$$

$$=\sqrt{100-96}=\sqrt{4}=2. \quad \ldots \quad \text{(iii)}$$

Adding (i) and (iii), we have

$$2x=12, \quad \therefore \quad x=6.$$

Subtracting (iii) from (i), we have

$$2y=8, \quad \therefore \quad y=4.$$

Thus $x=6$ and $y=4$.

Solve the equations:

25. $\left.\begin{aligned} x+y&=11 \\ xy&=28. \end{aligned}\right\}$

26. $\left.\begin{aligned} x+y&=17 \\ xy&=60. \end{aligned}\right\}$

27. $\left.\begin{aligned} x-y&=7 \\ xy&=120. \end{aligned}\right\}$

28. $\left.\begin{aligned} x-y&=5 \\ xy&=84. \end{aligned}\right\}$

29. $\left.\begin{aligned} x+y&=15 \\ xy&=56. \end{aligned}\right\}$

30. $\left.\begin{aligned} x-y&=4 \\ xy&=77. \end{aligned}\right\}$

31. $\left.\begin{aligned} x+y&=a \\ xy&=b^2. \end{aligned}\right\}$

32. $\left.\begin{aligned} x-y&=m \\ xy&=n^2. \end{aligned}\right\}$

33. $\left.\begin{aligned} \frac{1}{x}+\frac{1}{y}&=\frac{7}{12} \\ \frac{1}{xy}&=\frac{1}{12}. \end{aligned}\right\}$

34. $\left.\begin{aligned} \frac{1}{x}-\frac{1}{y}&=\frac{3}{10} \\ \frac{1}{xy}&=\frac{1}{10}. \end{aligned}\right\}$

3. Problems involving simultaneous equations.

Example 1. Find two numbers such that four times the first added to three times the second is 93 and the excess of three times the first over twice the second is 6.

Let x be the first number, and y the second.

Four times the first + three times the second $=4x+3y$,

$\therefore$ the first equation is $4x+3y=93$ ... ... (i)

The excess of three times the first over twice the second $=3x-2y$,

$\therefore$ the second equation is $3x-2y=6$ ... ... (ii)

Multiplying (i) by 2 and (ii) by 3, we have

$$8x+6y=186 \quad ... \quad ... \text{(iii)}$$
$$9x-6y=18 \quad ... \quad ... \text{(iv)}$$

Adding (iii) and (iv), we have

$$17x=204, \quad \therefore \quad x=12.$$

Substituting this value of x in (i), we have

$$48+3y=93, \quad \therefore \quad y=15.$$

Hence the two numbers are 12 and 15.

Example 2. A person sold 9 chairs and 6 tables for Rs. 90. Again he sold 8 chairs and 5 tables at the same rate, for Rs. 77. Find the price of one chair and one table.

Let Rs. x be the price of one chair and Rs. y the price of one table. Since the price of 9 chairs and 6 tables = Rs. $(9x+6y)$,

$\therefore$ the first equation is $9x+6y=90$ ... ... (i)

Since the price of 8 chairs and 5 tables = Rs. $(8x+5y)$,

$\therefore$ the second equation is $8x+5y=77$ ... ... (ii)

Multiplying (i) by 5 and (ii) by 6, we have

$$45x+30y=450 \quad ... \quad ... \text{(iii)}$$
$$48x+30y=462 \quad ... \quad ... \text{(iv)}$$

Subtracting (iii) from (iv), we have

$$3x=12, \quad \therefore \quad x=4.$$

Substituting this value of x in (i), we have

$$36+6y=90, \quad \therefore \quad y=9.$$

Hence the price of one chair is Rs. 4 and the price of one table is Rs. 9.

Example 3. A fraction becomes equal to $\frac{1}{3}$ when 1 is subtracted from its numerator and it becomes equal to $\frac{1}{4}$ when 8 is added to its denominator. Find the fraction.

Let x be the numerator and y the denominator of the fraction,

then $$\frac{x-1}{y}=\frac{1}{3}$$

or $$3x-y=3 \quad ... \quad ... \text{(i)}$$

$$\text{and} \quad \frac{x}{y+8}=\frac{1}{4}$$

or $$4x-y=8 \qquad \ldots \qquad \text{(ii)}$$

Subtracting (i) from (ii), we have

$$x=5.$$

Substituting this value of x in (i), we have

$$15-y=3$$

or $$y=12.$$

Hence the required fraction is $\frac{5}{12}$.

Example 4. A number consists of three digits whose sum is 17; the middle digit exceeds the sum of the other two by 1; if the digits be reversed, the number is diminished by 396. Find the number.

Let x, y and z be the digits in hundreds, tens and units place respectively.

Since the sum of the digits $=x+y+z$,

$$\therefore \quad x+y+z=17 \qquad \ldots \qquad \text{(i)}$$

Since the middle digit y exceeds the sum of the other digits x and z by 1,

$$\therefore \quad y=x+z+1$$

or $$x-y+z=-1 \qquad \ldots \qquad \text{(ii)}$$

Since the actual number is $100x+10y+z$ and it becomes $100z+10y+x$ when the digits are reversed,

$$\therefore \quad (100x+10y+z)-(100z+10y+x)=396.$$

Simplifying and dividing both sides by 99, we have

$$x-z=4 \qquad \ldots \qquad \text{(iii)}$$

Adding (i) and (ii), we have

$$2x+2z=16$$

or $$x+z=8 \qquad \qquad \text{(iv)}$$

From equations (iii) and (iv), we have

$$x=6 \text{ and } z=2.$$

Substituting the values of x and z in (i), we have

$$6+y+2=17, \qquad \therefore \quad y=9.$$

Hence the required number is 692.

EXERCISE 53.

1. Find two numbers whose sum is 39 and difference is 7.

2. Find two numbers such that three times their sum is 84 and twice their difference is 4.

3. Find two numbers such that four times the first added to the second is equal to 96 and the excess of three times the first over the second is equal to 51.

4. Find two numbers such that three times the first added to four times the second is equal to 114 and four times the first added to three times the second is equal to 117.

5. Two decimal fractions have their sum equal to ·56 and three times their difference equal to ·24. Find them.

6. A straight line 27″ in length is divided into two parts such that six times the first part exceeds four times the other by 12″ Find each part.

7. The sum of three numbers is 77. Three times the first added to the third is less than four times the second by 3. The excess of the third over the second is 18. Find the numbers.

8. The sum of two numbers is increased by 10 and the result divided by 8; the quotient is less than their difference by 1. Also $\frac{1}{2}$ of the greater is greater than $\frac{1}{4}$ of the less by 5. Find the two numbers.

9. Find three numbers such that the excess of the first over the second is 8, the excess of the second over the third is 4, and the sum of the first and the third is 28.

10. A and B have Rs. 320 between them. A gains from B $\frac{1}{5}$ of B's money. Then B gains from A Rs. 16. Each has now the same amount. Find the original sum of each.

11. Find three numbers x, y, z such that x increased by $\frac{1}{2}$ of y, y increased by $\frac{1}{3}$ of z, and z increased by $\frac{1}{4}$ of x may each be 250.

12. 25 horses and 12 cows together cost Rs. 2,350; also 18 horses and 15 cows cost Rs. 2,010. Find the cost price of a horse and a cow.

13. 16 maunds of wheat and 6 maunds of rice together cost Rs. 89, and 24 maunds of wheat and 15 maunds of rice cost Rs. 166 8 as. Find the rates of wheat and rice per maund.

14. The price of 12 tables and 20 chairs is Rs. 164 and the price of 15 tables and 24 chairs is Rs. 201. Find the price of a table and a chair.

15. A certain fraction becomes $\frac{2}{3}$ when its numerator is diminished by 1, and becomes $\frac{1}{2}$ when its denominator is increased by 5. Find the fraction.

16. What fraction is that which becomes 1 if 1 be added to its numerator and becomes $\frac{1}{2}$ if 1 be added to its denominator?

17. What fraction is that which becomes $\frac{2}{3}$ if its numerator be doubled and denominator increased by 9 and becomes $\frac{1}{2}$ if its numerator be increased by 7 and denominator doubled?

18. If the denominator of a fraction be added to the numerator and the numerator be subtracted from the denominator, it becomes $2\frac{2}{3}$. The denominator of the fraction exceeds twice its numerator by 1. Find the fraction.

19. The denominator of a fraction exceeds the numerator by 4, and if 5 be taken away from each, the sum of the reciprocal of the new fraction and four times the original fraction is 5. Find the original fraction. [B. U. 1892.]

20. A person bought a horse and a carriage for Rs. 810; if the price of the horse were 10% more and that of the carriage 10% less, he would have to pay Rs. 799. Find the cost price of the horse and also that of the carriage.

21. A person sells two articles together for Rs. 46, making 10% profit on one and 20% profit on the other. If he had sold each article at 15% profit, the result would have been the same. At what price does he sell each article?

(C. U. 1891.)

22. A person has two horses and a saddle worth Rs. 95; if the saddle be put on the first horse, his value becomes

double that of the second ; but if the saddle be put on the second horse, his value would be less than that of the first by Rs. 125. What is the value of each ?

23. A number of two digits when divided by 5 gives a certain quotient and a remainder 4 ; when divided by 8, gives another quotient and a remainder 7. The digit in the tens place is equal to the second quotient diminished by 2 and the other digit is less than the first quotient by 6. Find the number.

24. A man had a sum of 16*s.* 6*d.* in shillings and pence. If the number of shillings and pence were interchanged, he would gain 2*s.* 9*d.* Find how many shillings and pence he had.

25. The cost of 12 lbs. of tea and 15 lbs. of butter is £3 19*s.* 6*d.* ; the cost of 8 lbs. of tea and 6 lbs. of butter is £2 3*s.* Find the cost of each per lb.

26. A number consists of two digits. When it is divided by the sum of the digits, the quotient is 7. The sum of the reciprocals of the digits is 9 times the reciprocal of the product of the digits. Find the number. (M. M. 1887.)

27. Reverse the digits of a number, it becomes $\frac{5}{8}$ of what it was before; also the difference between the two digits is one. Find the number. (C. E. 1883.)

28. A certain number of 3 digits exceeds the sum of its digits by 414 and the number formed by reversing the digits exceeds the same sum by 612. If the given number and one formed by reversing the digits be in the ratio of 71 : 104, find the number.

29. A number consists of 3 digits whose sum is 18. The digit in the hundreds place is $\frac{1}{8}$ of the number formed by the remaining two digits, and the digit in the units place is $\frac{1}{8}$ of the number formed by the remaining two digits. Find the number.

30. A number consists of 3 digits whose sum is 10. The middle digit is equal to the sum of the other two. The number formed by reversing the digits is greater than the original number by 99. Find the number.

Example 5. Two persons A and B, 20 miles apart, starting at the same time, meet in $2\frac{1}{2}$ hours if they walk in opposite directions; but they meet in 10 hours, if they start at the same time and walk in the same direction. Find their speed per hour.

Let x miles per hour be the speed of A and y miles per hour the speed of B.

When they walk in opposite directions, their speed of approach is $(x+y)$ miles per hour and since they together cover 20 miles in $2\frac{1}{2}$ hours,

$$\therefore \quad 2\tfrac{1}{2}(x+y)=20$$
$$\text{or} \qquad x+y=8 \ldots\ldots\ldots\ldots\text{(i)}$$

When they walk in the same direction, their speed of approach is $(x-y)$ miles per hour. Since A overtakes B after making up a distance of 20 miles in 10 hours,

$$\therefore \quad 10(x-y)=20$$
$$\text{or} \qquad x-y=2 \ldots\ldots\ldots\ldots\text{(ii)}$$

From (i) and (ii), we have

$$x=5 \text{ and } y=3.$$

Hence the speed of A is 5 miles an hour and the speed of B is 3 miles an hour.

Example 6. A sum of money is divided equally among a certain number of men; had there been eight more, each would have received Re. 1 less, and had there been four fewer, each would have received Re. 1 more than he did. Find the number of men and the sum of money.

Let x be the number of men and y the number of rupees each gets.

Then the sum divided $=xy$ rupees.

By the question $\quad (x+8)(y-1)=xy \ldots\ldots\ldots\text{(i)}$

and $\quad (x-4)(y+1)=xy \ldots\ldots\ldots\text{(ii)}$

From (i), we have

$$xy-x+8y-8=xy$$
$$\therefore \quad -x+8y-8=0 \ldots\ldots\ldots\ldots\text{(iii)}$$

From (ii), we have

$$xy+x-4y-4=xy$$
$$x-4y-4=0 \ldots\ldots\ldots\ldots\text{(iv)}$$

Adding (iii) and (iv), we have

$$4y-12=0, \qquad \therefore \quad y=3.$$

Substituting this value of y in (iv), we have

$$x-12-4=0, \qquad \therefore \quad x=16.$$

Hence the number of men $=16$ and the sum divided $=$ Rs. $3\times16=$ Rs. 48.

***Example 7.** Two persons have together 5 mds. of luggage and are charged for the excess above the ordinary concession Rs. 5 and Rs. 3 respectively. If all the luggage had belonged to one man, he would have been charged Rs. 9. How much luggage had each passenger and what was the limit of concession per passenger?

Let x mds. be the luggage of the first passenger, then $(5-x)$ mds. would be the luggage of the second passenger.

Let y mds. of luggage be allowed to each to carry free of charge and let z rupees per md. be the rate of charge for the excess luggage.

Then the excess luggage carried by the first passenger is $(x-y)$ mds. and the charge on it $=$ Rs. $(x-y)z$,

$$\therefore \qquad (x-y)z=5 \qquad \ldots \qquad \ldots \qquad \text{(i)}$$

The excess luggage carried by the second passenger is $(5-x-y)$ mds. and the charge on it $=$ Rs. $(5-x-y)z$,

$$\therefore \qquad (5-x-y)z=3 \qquad \ldots \qquad \ldots \qquad \text{(ii)}$$

If all the luggage had belonged to one man, the excess luggage would have been $(5-y)$ mds. and the charge on it would have been $(5-y)z$ rupees.

$$\therefore \qquad (5-y)z=9 \qquad \ldots \qquad \ldots \qquad \text{(iii)}$$

Dividing (ii) by (i), we have

$$\frac{5-x-y}{x-y}=\frac{3}{5}$$

or $$5(5-x-y)=3(x-y) \qquad \ldots \qquad \ldots \qquad \text{(iv)}$$

Dividing (iii) by (i), we have

$$\frac{5-y}{x-y}=\frac{9}{5}$$

or $$5(5-y)=9(x-y) \qquad \ldots \qquad \ldots \qquad \text{(v)}$$

From (iv) and (v), we have

$$x=3 \text{ and } y=\tfrac{1}{2}.$$

Substituting the values of x and y in (i), we have

$$(3-\tfrac{1}{2})z=5, \qquad \therefore \quad z=2$$

Hence the luggage of one man $=3$ mds., of the second $=2$ mds., and the concession allowed per passenger $=\frac{1}{2}$ md.

31. A person goes a certain distance on bicycle. If he had gone 3 miles an hour faster, he would have taken $\frac{2}{3}$ hour less; but if he had gone 2 miles an hour slower, he would have taken $\frac{2}{3}$ hour longer. Find the distance.

32. A person walks on the first day a certain distance at the rate of 4 miles an hour and on the second day a certain distance at the rate of 3 miles an hour, and thus completes a journey of 30 miles. If he had walked the first distance at 3 miles an hour and the second distance at 4 miles an hour, he would have taken $\frac{1}{2}$ hour more. Find the time he took to walk the whole distance.

33. Two persons, 15 miles apart, start at the same time. If they walk in opposite directions, they meet in 2 hours; but if they walk in the same direction, they meet in 30 hours. Find their rates of walking.

34. A person rows down-stream 35 miles in 7 hours and up-stream the same distance in $17\frac{1}{2}$ hours. Find the rate at which he rows and the speed of the stream.

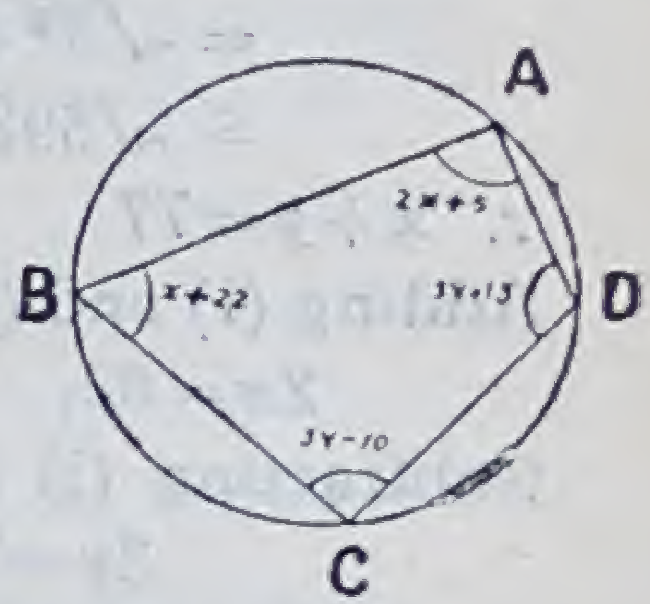
35. A messenger is allowed 16 hours for going from P to Q and back again, including 4 hours' rest at Q. But if he were to go 2 miles an hour faster each way, he would be able to take 5 hours' rest at Q. Find the speed and the distance from P to Q.

36. A person rows down-stream 26 miles and up-stream the same distance in $19\frac{1}{2}$ hours. Also the difference of time taken by him to row 39 miles up and down is 9 hours and 45 minutes. Find the rate at which he rows and the speed of the stream.

37. A boat goes up-stream 30 miles and down-stream 44 miles in 10 hours. It also goes up-stream 40 miles and down-stream 55 miles in 13 hours. Find the speeds of the boat and the stream.

38. The perimeter of a rectangle is 72 ft. If its length be decreased by 6 ft. and breadth increased by 4 ft., the area is unaltered. Find the area.

39. In a cyclic quadrilateral the sum of each pair of opposite angles is 180°. The number of degrees in its angles is given in the figure in x and y. Find x and y.



40. A merchant has two kinds of tea, one worth 2*s.* per lb. and the other worth 3*s.* 4*d.* per lb.; from these he prepares a mixture of 120 lbs. worth 2*s.* 4*d.* per lb. How much of each kind must he mix ?

41. Three ladies buy 3 kinds of cloth, *A*, *B* and *C*. The first buys 5 yds. of *A* and 3 yds. of *B*, spending the same amount on each; the second buys 10 yds. of *A* and 5½ yds. of *C*, spending £5; while the third buys 3 yds. of each kind of cloth for £3 6*s*. Find the price per yard of each kind of cloth.

*42. Two persons have together 6 mds. of luggage and are charged for the excess above the ordinary concession Re. 1 4 as. and Re 1 14 as. respectively. If all the luggage had belonged to one man, he would have been charged Rs. 3 7 as. How much luggage had each passenger and what is the limit of concession per passenger ?

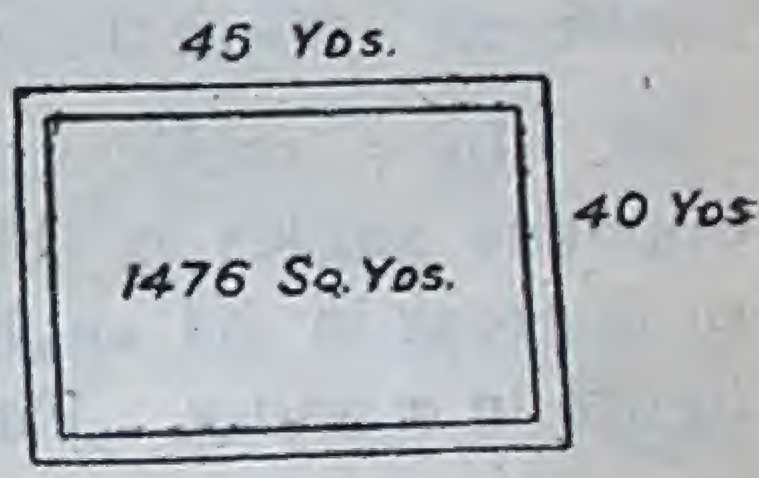
43. The monthly expenses of a family, when wheat is selling @ 18 srs. for a rupee, are Rs. 60; when it is selling @ 20 srs. for a rupee, expenses are Rs. 58, other expenses remaining unchanged. What will be the expenses when wheat is selling @ 24 srs. for a rupee ?

***Example 8.** A rectangular garden is 45 yds. long and 40 yds. broad. A road of uniform width is constructed inside all round it. The area of the remaining garden is 1476 sq. yds. Find the width of the road.

Let the length of the inner rectangle be x yds. and its breadth y yds.

Obviously, $x-y=45-40=5$ ... (i)

and $xy=1476$ ... (ii)



$\therefore\ x+y=\sqrt{(x-y)^2+4xy}$

$=\sqrt{5^2+4\times1476}$

$=\sqrt{5929}=77.$

$\therefore\ x+y=77$ ...(iii)

Adding (i) and (iii), we have

$2x=82, \qquad \therefore\ x=41.$

Subtracting (i) from (iii), we have

$2y=72, \qquad \therefore\ y=36.$

$\therefore$ width of the road $=\frac{1}{2}(45-41)=2$ yds.

NOTE. In examples of this type, when the path is of uniform width, the difference between the length and the breadth of the inner rectangle is the same as that of the outer rectangle.

*44. The area of a rectangle is 1728 sq. ft. and its semi-perimeter is 84 ft. Find its length and breadth.

*45. The area of a triangle is 80 sq. ft., the sum of its base and altitude is 28 ft. Find its base and altitude.

*46. The area of a rectangle is 3840 sq. ft. and its length is greater than its breadth by 32 ft. Find its length and breadth.

*47. The top of a rectangular table is 40″ long and 28″ wide. Leaving out a uniform margin all round, it is covered with oil cloth whose area is 640 sq. inches. Find the width of the margin.

*48. The difference between the length and breadth of a rectangle is 14 ft. and its area is 275 sq. ft.

(i) Find its semi-perimeter, (ii) find its length and breadth. [P. U. 1931.]

SECTIONAL REVISION III

TEST PAPERS

PAPER 1

1. Shew that the H.C.F. of two expressions A and B divides exactly their sum as well as difference.

2. Simplify (i) $\dfrac{a}{a^2-b^2}-\dfrac{b}{b^2-a^2}$.

(ii) $\dfrac{a^2-3ab+2b^2}{a-2b}+\dfrac{6a^2-5ab-6b^2}{2a-3b}-\dfrac{6a^2+ab-2b^2}{3a+2b}$.

3. Solve the equations $\left.\begin{aligned} 5x+\frac{3}{y}&=20\frac{1}{2} \\ 3x-\frac{4}{y}&=11\frac{1}{3}. \end{aligned}\right\}$

4. Find two numbers such that three times the first added to five times the second is equal to 53, and also such that three times the second subtracted from four times the first is equal to 3.

5. Find the value of $x^3+\dfrac{1}{x^3}$ when $x+\dfrac{1}{x}=4$.

PAPER 2

1. Find the H.C.F. of $6x^3-4x^2+3x-2$, $6x^3-31x^2+30x-8$ and $12x^3-2x^2-7x+2$.

2. (i) Express $\dfrac{x^3-1}{x+a}$ in a form partly integral and partly fractional.

(ii) Simplify $\dfrac{x+1}{2x^3-4x^2}+\dfrac{x-1}{2x^3+4x^2}-\dfrac{1}{x^2-4}$.

3. Solve the equations $\left.\begin{aligned}\frac{12}{x}-\frac{5}{y}&=3\\ \frac{9}{x}+\frac{2}{y}&=3\tfrac{2}{5}.\end{aligned}\right\}$

4. Eight years back A was twice as old as B. After 15 years their united ages will be 82. Find their present ages.

5. A man's monthly income is progressive and varies as below: Rs. 50, Rs. 56, Rs. 62, Rs. 68, Rs. 74, Rs. 80 &c. Find a formula which determines his monthly income in *general.*

PAPER 3

1. Find the H.C.F. of $2x^5-5x^2+3$ and $3x^5-5x^3+2$ by the method of alternate destruction of the highest and the lowest terms.

2. Simplify $\dfrac{5}{x^2-3x-28}+\dfrac{3}{x^2+x-12}+\dfrac{9}{x^2-10x+21}$.

3. Solve the equations $\left.\begin{aligned}13x+9y&=57\\ 9x+13y&=53.\end{aligned}\right\}$

4. A number of two digits is reversed if it is diminished by 9; the sum of its digits is 15. Find the number.

5. Shew that $(4x^2-2x-15)^2-(2x^2-5x-9)^2$ is divisible by $2x^2+3x-6$ and express the quotient in the form of factors.

PAPER 4

1. Reduce to the lowest terms $\dfrac{6x^4+25x^3+25x^2+13x+3}{3x^4+11x^3+13x^2+7x+2}$.

2. Simplify $\dfrac{1+x^2}{1-x^2}+\dfrac{4x^2}{1+x^4}-\dfrac{1-x^2}{1+x^2}$.

3. Solve the equations $\left.\begin{aligned}\frac{1}{x}+\frac{1}{y}&=\frac{9}{20}\\ \frac{1}{y}+\frac{1}{z}&=\frac{8}{15}\\ \frac{1}{z}+\frac{1}{x}&=\frac{7}{12}.\end{aligned}\right\}$

4. A person does a journey in a motor car at a uniform speed in 6 hours. On his return he stops at the half-way for half an hour, but quickening his speed by 3 miles an hour completes it in the same time. Find his original speed and the length of the journey.

5. Find the value of $a^3+b^3+c^3-3abc$ when $a+b+c=9$ and $a^2+b^2+c^2=29$.

PAPER 5

1. If A and B are two algebraic expressions and $A=a.H$, $B=b.H$. prove that the L.C.M. $=\frac{A}{H}.B$.

2. Simplify $\dfrac{6x^2-5xy-6y^2}{14x^2-23xy+3y^2}-\dfrac{15x^2+8xy-12y^2}{35x^2+47xy+6y^2}$.

3. Solve the equations $\left.\begin{aligned}2x+3y+z&=19\\ x+2y-z&=8\\ 3x-y+2z&=13\end{aligned}\right\}$

4. A person goes on a bicycle from A to B and back in a certain time at 12 miles an hour. If he had gone 10 miles an hour from A to B and returned at 16 miles an hour, he would have taken $6\frac{1}{4}$ minutes less for the double journey. Find the distance from A to B.

5. Factorise (i) $(x^2-3x)(x^2-3x-2)-24$.
(ii) $81x^4+64$.

PAPER 6

1. If A and B are two expressions, H and L their H.C.F. and L.C.M. respectively, prove that $A.B=HL$.

2. Simplify $\dfrac{1}{x+1}+\dfrac{2}{x^2+1}+\dfrac{4}{x^4+1}+\dfrac{8}{x^8-1}$.

3. Solve the equations $\left.\begin{aligned}\frac{1}{2x}-\frac{1}{3y}&=\frac{1}{12}\\ \frac{1}{6xy}&=\frac{1}{72}.\end{aligned}\right\}$

4. Divide Rs. 1,440 into two parts such that if they are invested at $3\frac{1}{2}\%$ and 5% respectively, they may together yield the same income as if the whole were invested at $4\frac{1}{2}\%$.

5. Find the value of $x^2-y^2+6x+10y-16$ when $x+y=15$ and $x-y=3$.

PAPER 7

1. The H.C.F. of two expressions is $x+2y$ and their L.C.M. is $6x^3+7x^2y-16xy^2-12y^3$, one expression is $2x^2+xy-6y^2$; find the other.

2. Simplify $\dfrac{3}{2x+3-\dfrac{3}{1-\dfrac{x}{x+6}}}$.

3. What is meant by an indeterminate equation? Illustrate it by an example.

4. A person buys 50 lbs. of tea and 60 lbs. of coffee for £10 5*s*. By selling the coffee at a loss of 5% and the tea at a gain of 10%, he makes a profit of 8*s*. 6*d*. What was the cost price of tea and coffee per lb.

5. Collect the co-efficient of x^2 from the product of $(2x^2+5x-1)$ and $(3x^2-2x+4)$ without actual multiplication.

PAPER 8

1. Find the H.C.F. and L.C.M. of x^5+x^3, x^6-x^2, $x^4-3x^3+3x^2-3x+2$.

2. Simplify $x+1-\dfrac{x}{x+2-\dfrac{x+1}{x+\dfrac{1}{x+2}}}$.

3. Solve the equations $\left.\begin{array}{r}2x+5y=33\\4x-3y=1\end{array}\right\}$ by all the three methods.

4. A rectangular garden is 36 yds. long and 25 yds. broad. A road of uniform width is constructed inside all round it. The area of the remaining garden is 672 sq. yds. Find the width of the road.

5. Factorise $2a^3-7a^2+4a-3$.

PAPER 9

1. Find the L. C. M. of $3x^3+17ax^2-62a^2x+14a^3$ and $7x^3+52ax^2-46a^2x+8a^3$.

2. Simplify $\left\{1+\frac{2b^2}{a(a-3b)}\right\}\left\{1+\frac{b}{2b-a}\right\}$

$$-\left(\frac{a^2}{b^2}+\frac{b}{a}\right)\left(\frac{a^2-ab}{a^2-ab+b^2}-1\right).$$

3. Solve the equations $\left.\begin{aligned}\frac{xy}{x+y}&=\frac{6}{5}\\ \frac{yz}{y+z}&=\frac{12}{7}\\ \frac{zx}{z+x}&=\frac{4}{3}.\end{aligned}\right\}$

4. A number of persons contribute a certain sum towards a charitable fund. If there had been 12 fewer, each would have contributed 13 as. 4 p. more; if there had been 12 more, each would have contributed 8 as. less. Find the number of persons and the contribution of each.

5. Collect the co-efficient of x^3 from the product of $(3x^4+x^3-5x^2+2x-1)(5x^3+4x^2-x+3)$ without actual multiplication.

PAPER 10

1. The H.C.F. of two expressions is $a-1$, their L.C.M. is $a^5+4a^4+6a^3+a^2-6a-6$; one expression is a^3+a^2-2, find the other.

2. Simplify $\left(\dfrac{1}{a}+\dfrac{\dfrac{1}{ab}}{\dfrac{1}{a}+\dfrac{1}{b}}\right)\left(\dfrac{1}{a}+\dfrac{\dfrac{1}{ab}}{\dfrac{1}{a}-\dfrac{1}{b}}\right)\div\dfrac{\dfrac{1}{a^2}+\dfrac{1}{b^2}}{\dfrac{1}{a^2}-\dfrac{1}{b^2}}$.

3. Solve the equations $\left.\begin{aligned} x+y&=9 \\ xy&=20. \end{aligned}\right\}$

4. In a cyclic quadrilateral $ABCD$, $A=(2x+15)$ degrees, $B=(3x-25)$ degrees, $C=(3y-10)$ degrees and $D=(2y+10)$ degrees. Find x and y.

5. Factorise $a^3p^2+2a^3q^2+b^3q^2-3a^3pq-3b^3pq+2b^3q^2$.

PART II

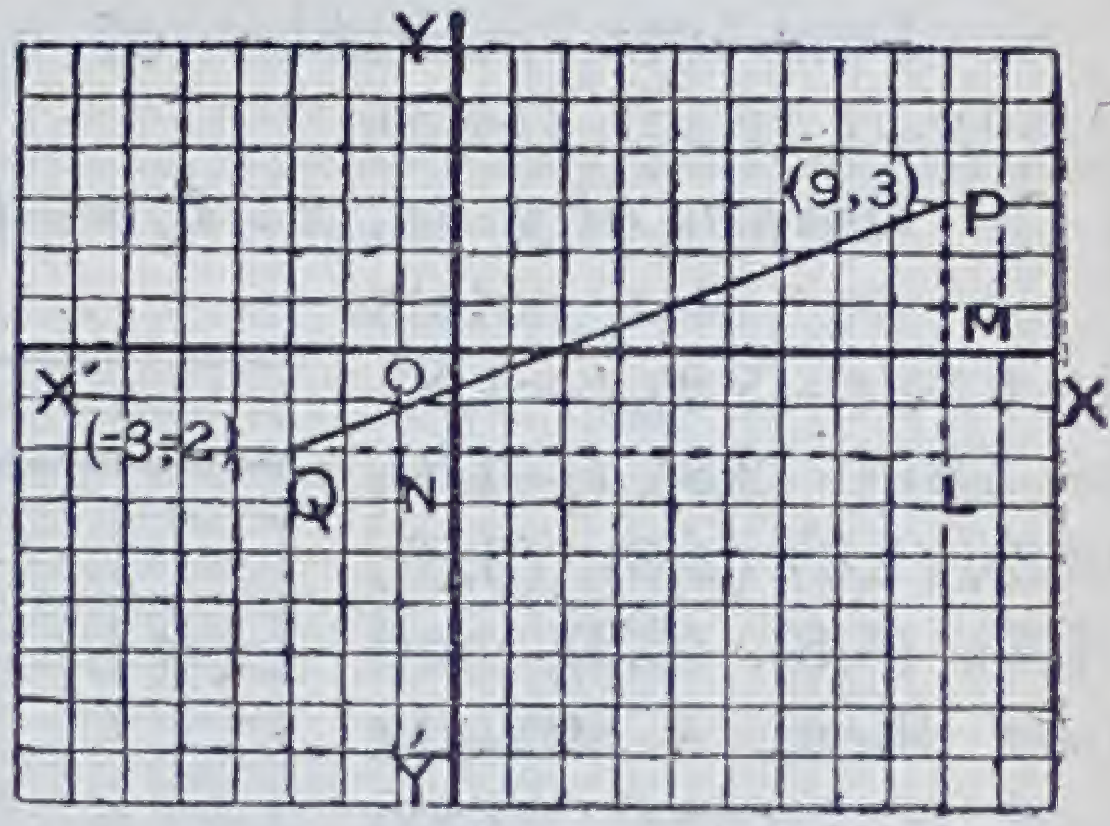
CHAPTER XI

LINEAR AND STATISTICAL GRAPHS

1. Length of Straight Line.

Example 1. The co-ordinates of P and Q are (9, 3), (−3, −2) respectively; find the length of PQ.

Draw $QNL \parallel x$-axis and $PML \perp QNL$.



$PQ^2 = QL^2 + LP^2$

$= (12)^2 + (5)^2$

$= 144 + 25 = 169$

$\therefore PQ = \sqrt{169} = 13$ units.

As one unit $= \cdot 1''$,

$\therefore \quad PQ = 1\cdot 3''$.

Otherwise.

Since $QL = QN + NL$

$= -NQ + NL = NL - NQ$

$= 9 - (-3)$.

$LP = LM + MP = -ML + MP$

$= MP - ML = 3 - (-2)$ and

$PQ^2 = QL^2 + LP^2$,

$PQ^2 = \{9 - (-3)\}^2 + \{3 - (-2)\}^2$

$\therefore \quad PQ = \sqrt{\{9-(-3)\}^2 + \{3-(-2)\}^2}$.

Generalising this process, we get the length of the line joining the points (x_1, y_1) and $(x_2, y_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.

Cor. The distance of the point (x, y) from the origin $(0,0) = \sqrt{x^2 + y^2}$.

EXERCISE 54.

1. Calculate the distance of each of the following points from the origin:

(6, 8), (10, 24), (8, 15), (7, 24).

2. Calculate the distance between the points of each pair :

(i) (3,7), (6,3) ; (ii) (4,8), (12,2) ; (iii) (3,2) (15,7) ; (iv) (2,3), (11,18).

3. Prove that the straight line joining the origin to the point (8,18) subtends a right angle at the point (13,5).

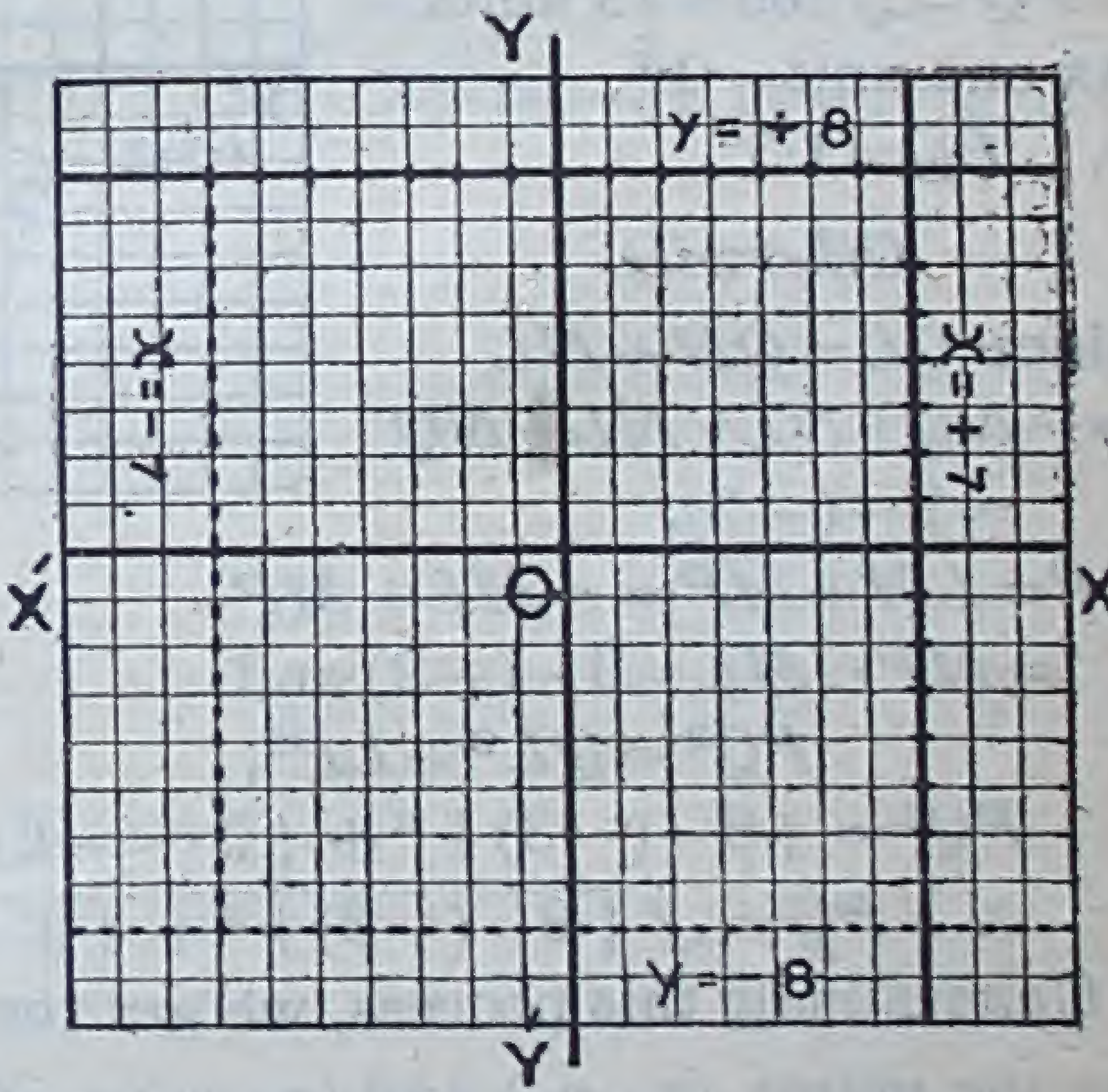
4. Prove that the triangle whose vertices are the points (3,1), (11,7), (0,5), is a right-angled triangle.

5. Shew that the points $(-2,2)$, (6,2) (2,5) are the vertices of an isosceles triangle. Calculate the length of equal sides. Verify by measurement.

6. Find the perimeter of a triangle whose vertices are (0,3), (5,15), (14,3).

7. ***Graphs of $y=a$, $x=b$, $y=0$ and $x=0$.***

If we plot points $(-6,8)$, $(-5,8)$, $(-4,8)$, $(-3,8)$, $(-2,8)$, $(-1,8)$, (0,8), (1,8), (2,8), (3,8), (4,8), (5,8), (6,8), and join them, a straight line is formed which is parallel to x-axis and is such that every point on it has its $y=+8$ and this is true for no other points. (See figure.)



The straight line is the **locus** of the points whose $y=+8$, whatever may be the x-co-ordinate, and the equation $y=+8$ represents this straight line.

Similarly, $y=-8$, represents a straight line parallel to x-axis and is such that every point on it has its $y=-8$.

Hence $\boldsymbol{y=a}$, where a is constant, represents a straight line parallel to x-axis, and $y=0$ is the x-axis itself.

Again the straight line parallel to the y-axis and at the distance of $+7$ units is such that every point on it has $x=+7$ and this is true for no other points. (See figure.)

Hence $x=+7$ represents the straight line.

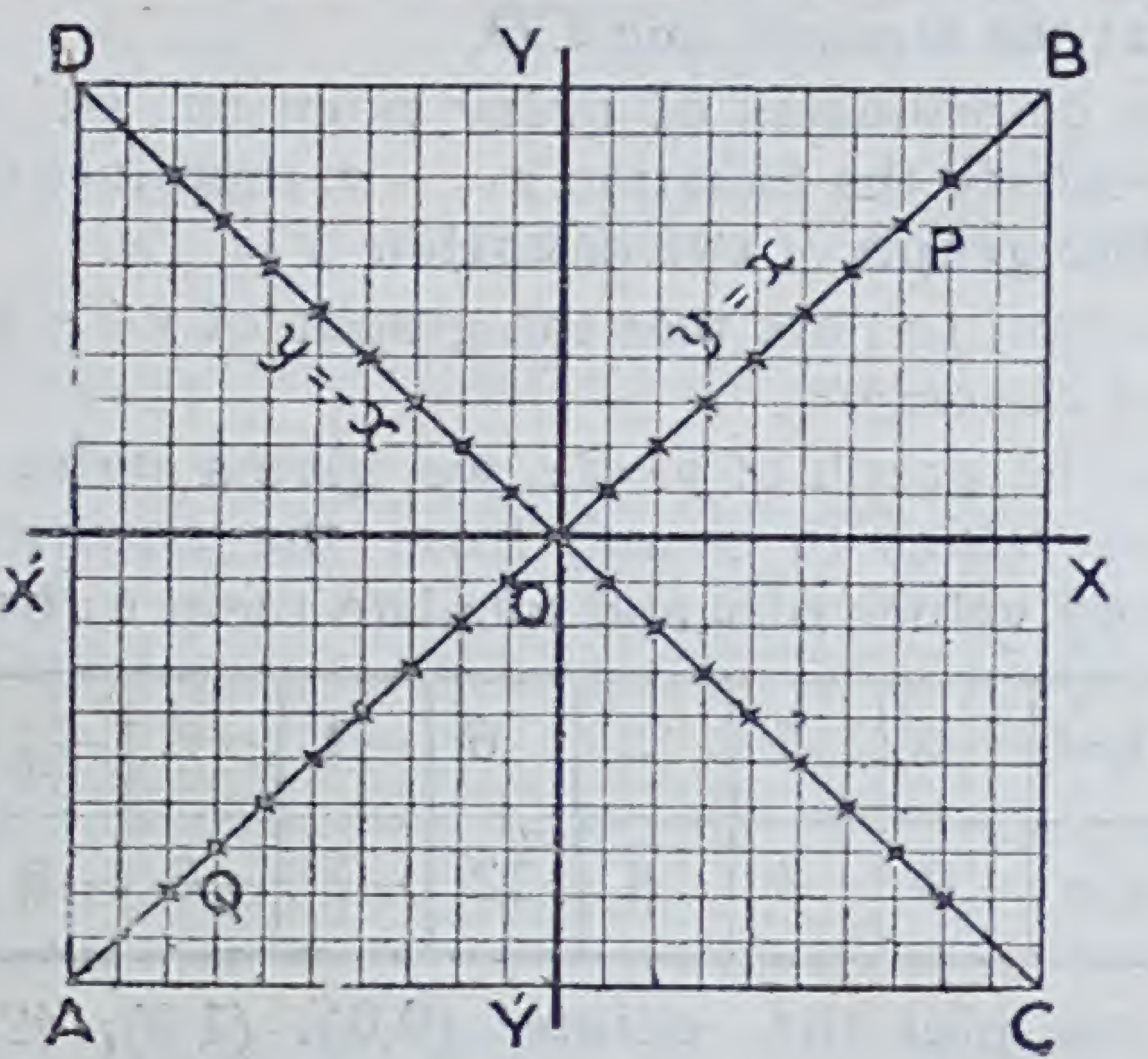
Similarly, $x=-7$ represents a straight line parallel to the y-axis and at the distance of -7 units from it.

Hence $x=b$, where b is constant, represents a straight line parallel to the y-axis, and $x=O$ is the y-axis itself.

3. Graphs of (i) $y=x$, (ii) $y=-x$, (iii) $y=2x$, (iv) $y=-2x$.

If we tabulate the values of x and the corresponding values of y from the equation $y=x$, we have

$x=$	-5	-4	-3	-2	-1	0	1	2	3	4	5
$y=$	-5	-4	-3	-2	-1	0	1	2	3	4	5



and if we plot the points $(-5,-5)$, $(-4,-4)$, $(-3,-3)$, $(-2,-2)$, $(-1,-1)$, $(0,0)$ $(1,1)$, $(2,2)$, $(3,3)$, $(4,4)$, $(5,5)$, etc., we find that all these points appear to lie on the straight line AB. (See figure.)

Now take *any* point P on the straight line AB. Its co-ordinates (7,7) satisfy the equation $y=x$. Also take *any* point Q $(-8,-8)$ whose co-ordinates satisfy the equation $y=x$. We find that Q lies on the straight line AB.

Since the co-ordinates of every point on the straight line AB satisfy the equation $y=x$ and every solution of $y=x$ gives us the co-ordinates of a point on the straight line AB, the straight line AB is the **locus or the graph** of **the equation** $y=x$.

Similarly, by tabulating the values of x and y from the equation $y=-x$, as shown below:

$x=$	5	4	3	2	1	0	-1	-2	-3	-4	-5
$y=$	-5	-4	-3	-2	-1	0	1	2	3	4	5

and plotting the points $(5,-5)$, $(4,-4)$, $(3,-3)$, $(2,-2)$, $(1,-1)$, $(0,0)$, $(-1,1)$, $(-2,2)$, $(-3,3)$, $(-4,4)$, $(-5,5)$ and joining them, we get the straight line CD.

Since the co-ordinates of every point on DC and of no other point satisfy the equation $y=-x$, therefore the straight line CD is **the graph** of **the equation** $y=-x$.

It is important to note that the graphs of $y=x$ and $y=-x$ *pass through the origin.*

In finding the graph of $y=2x$, we give a series of positive and negative values to x and from the equation find the corresponding values of y and tabulate them as below:

$x=$	0	1	2	6	-1	-3	-4
$y=$	0	2	4	12	-2	-6	-8

Now if we plot the points (0,0), (1,2), (2,4), (6,12), $(-1,-2)$, $(-3,-6)$, $(-4,-8)$, we find that all these points lie on the straight line POQ.

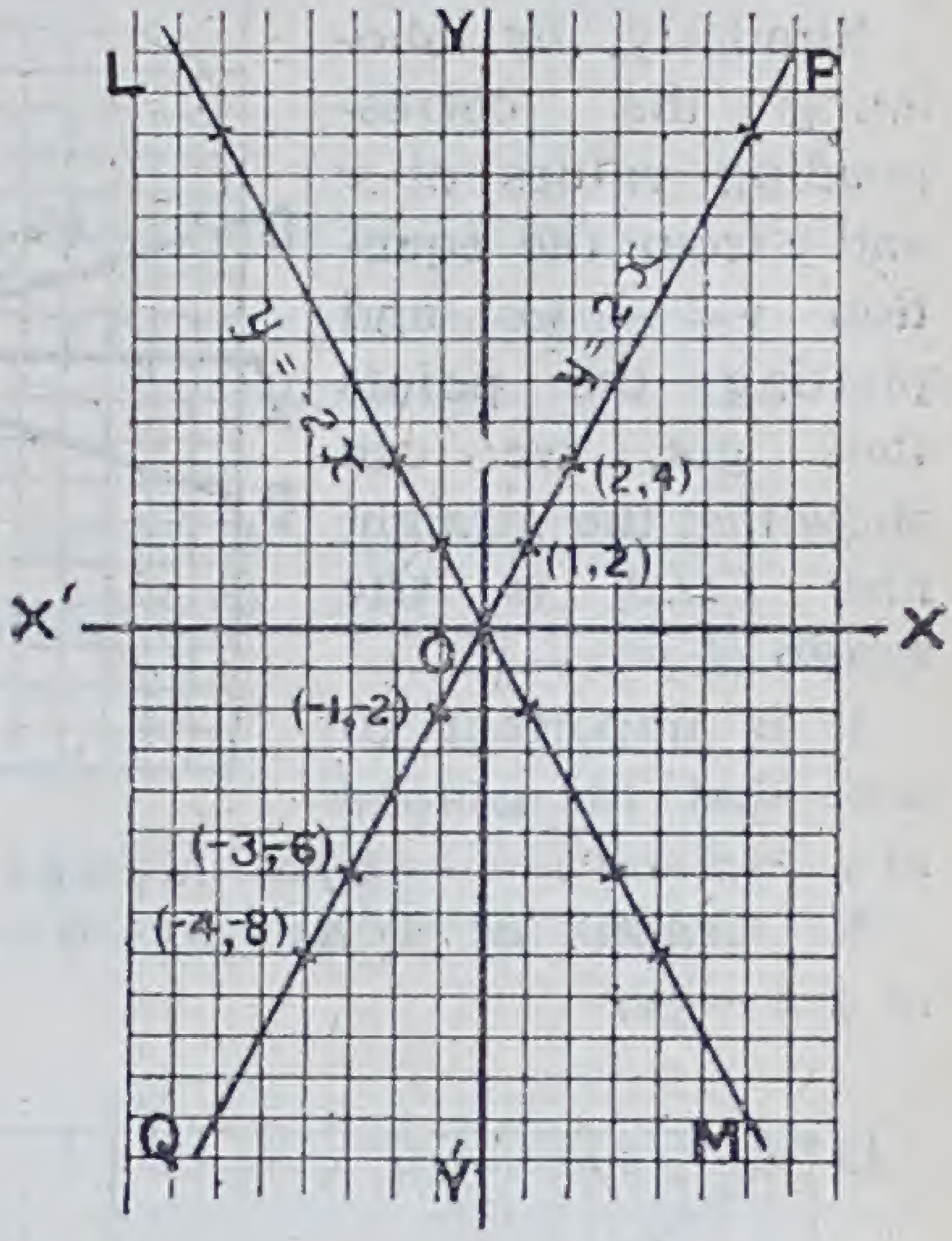
It can be easily verified that the co-ordinates of any point on the straight line POQ satisfy the equation $y=2x$

and that the co-ordinates of no other point satisfy the equation.

Thus the straight line POQ is **the graph** of $y=2x$.

For the graph of $y=-2x$, we proceed exactly in the same way and find that the straight line LOM is the required graph.

It is important to note that the graphs of $y=2x$ and $y=-2x$ *pass through the origin*.



4. Graphs of

(i) $3y-x=0$,
or $y=\frac{1}{3}x$;

(ii) $3y+x=0$,
or $y=-\frac{1}{3}x$;

Tabulating the corresponding values of x and y from the equation $y=\frac{1}{3}x$, we have

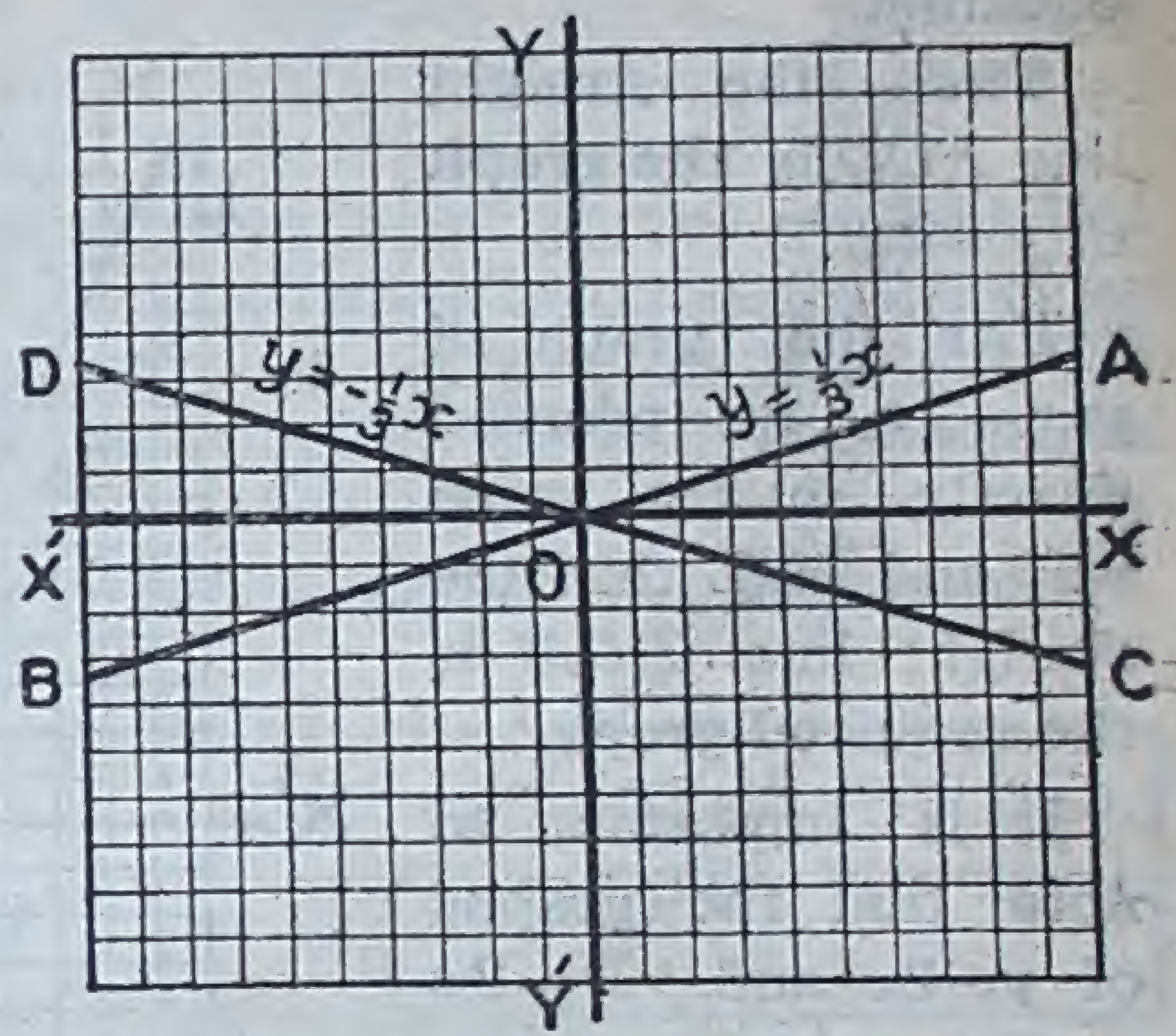
$x=$	0	1	2	3	-2	-3
$y=$	0	$\frac{1}{3}$	$\frac{2}{3}$	1	$-\frac{2}{3}$	-1

As these values involve fractions with denominator 3, it is therefore necessary for accurate drawing to take $\cdot 3''$ as the unit instead of $\cdot 1''$.

Now we plot points (0,0), $(1,\frac{1}{3})$, etc., according to this scale and observe that all these points lie on the straight line AOB.

As it can be easily verified that the co-ordinates of any point on the straight line AOB satisfy the equation $y=\frac{1}{3}x$ and

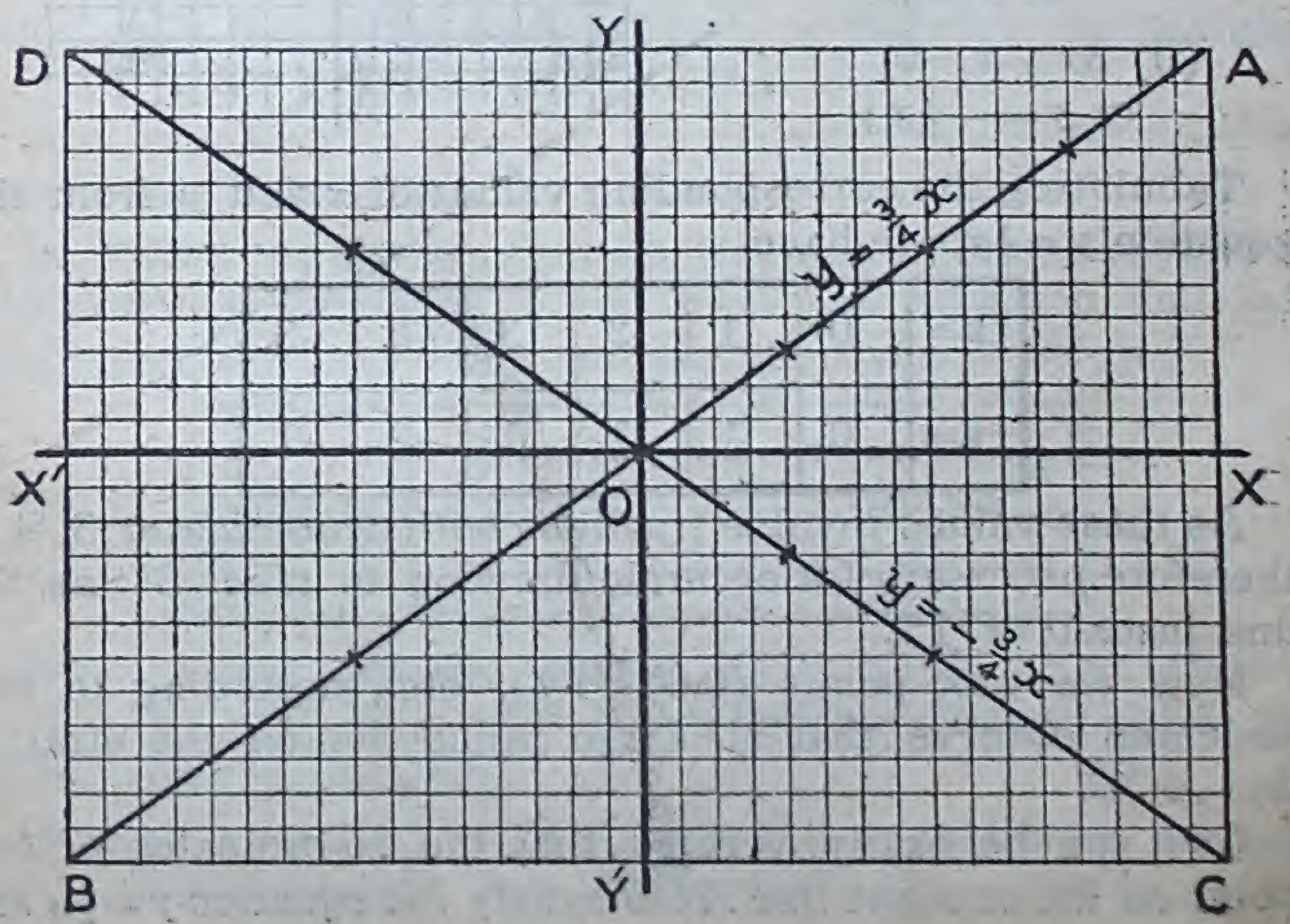
that the co-ordinates of no other point satisfy the equation the straight line AOB is **the graph** of $\mathbf{y=\frac{1}{3}x}$.



Similarly, by tabulating the corresponding values of x and y from the equation $y=-\frac{1}{3}x$ and plotting the points thus got, we can show that the straight line COD is **the graph** of $\mathbf{y=-\frac{1}{3}x}$.

It is important to note that the graphs of $y=\frac{1}{3}x$ and $y=-\frac{1}{3}x$ *pass through the origin.*

5. **Graphs of (i)** $\mathbf{4y-3x=0}$ **or** $\mathbf{y=\frac{3}{4}x}$; **(ii)** $\mathbf{4y+3x=0}$, **or** $\mathbf{y=-\frac{3}{4}x}$.



Tabulating the corresponding values of x and y from the equation $y=\frac{3}{4}x$, we have

$x=$	0	1	2	3	4	-2	-4
$y=$	0	$\frac{3}{4}$	$\frac{3}{2}$	$\frac{9}{4}$	3	$-\frac{3}{2}$	-3

As these values involve fractions with denominators 2 and 4, it is therefore necessary for accurate drawing to take $\cdot 4''$ as the unit instead of $\cdot 1''$.

Now we plot points (0, 0), $(1, \frac{3}{4})$, etc., according to this scale and observe that all these points lie on the straight line AOB.

As it can be easily verified that the co-ordinates of any point on the straight line AOB and the co-ordinates of no other point satisfy the equation $y=\frac{3}{4}x$, therefore the straight line AOB is **the graph** of $y=\frac{3}{4}x$.

Similarly, by tabulating the corresponding values of x and y from the equation $y=-\frac{3}{4}x$ and plotting the points thus got, we can show that the straight line COD is **the graph** of $y=-\frac{3}{4}x$.

It is important to note that the graphs of $y=\frac{3}{4}x$ and $y=-\frac{3}{4}x$ *pass through the origin.*

The following points are worth noting :

(i) The equations $y-x=0$, $y+x=0$, $y-2x=0$, $y+2x=0$, $3y-x=0$. $3y+x=0$, $4y-3x=0$, $4y+3x=0$, $y=x$, $y=-x$, $y=2x$, $y=-2x$, $y=\frac{1}{3}x$, $y=-\frac{1}{3}x$, $y=\frac{3}{4}x$, $y=-\frac{3}{4}x$, considered above, can be put into the form $\boldsymbol{ax+by=0}$.

Since $ax+by=0$ can again be put into the form $y=\left(-\frac{a}{b}\right)x$ or $y=mx$, therefore each of the above equations can be put into the form $\boldsymbol{y=mx}$.

(ii) In each case the graph is a straight line passing through the origin.

(iii) In each case m is the ratio between the ordinate and abscissa of a point on the graph and thus indicates the slope of the graph towards x-axis.

(iv) If m is positive, the graph lies in the first and third quadrants, if m is negative, the graph lies in the second and fourth quadrants.

EXERCISE 55.

1. If a point moves so that its ordinate is always -5, find the path traced out by it.

2. If a point moves so that its abscissa is always -8, find its locus.

3. Find the locus of a point which moves so that the ratio of its ordinate to its abscissa is $3:2$.

4. Find the locus of a point which moves so that the ratio of its ordinate to its abscissa is $-3:2$.

Draw the graphs of the following equations :

5. (i) $x=-7$, (ii) $x=12$, (iii) $x=0$, (iv) $x=-9$.
6. (i) $y=-6$, (ii) $y=11$, (iii) $y=0$, (iv) $y=-10$.
7. (i) $y=-\frac{1}{4}x$, (ii) $y=\frac{1}{5}x$, (iii) $y=-\frac{1}{6}x$.
8. (i) $y=\frac{3}{8}x$, (ii) $y=-\frac{3}{8}x$.
9. (i) $3x+2y=0$, (ii) $3x-2y=0$.
10. (i) $4x+5y=0$, (ii) $4x-5y=0$.
11. (i) $3x-5y=0$, (ii) $3x+5y=0$.
 (iii) $3y-5x=0$, (iv) $3y+5x=0$.

6. Graphs of (i) $y=2x$, (ii) $y=2x+6$, (iii) $y=2x-6$.

The corresponding values of x and y satisfying

(i) $y=2x$ are

$x=$	0	1	2	3	4	-3	-4
$y=$	0	2	4	6	8	-6	-8

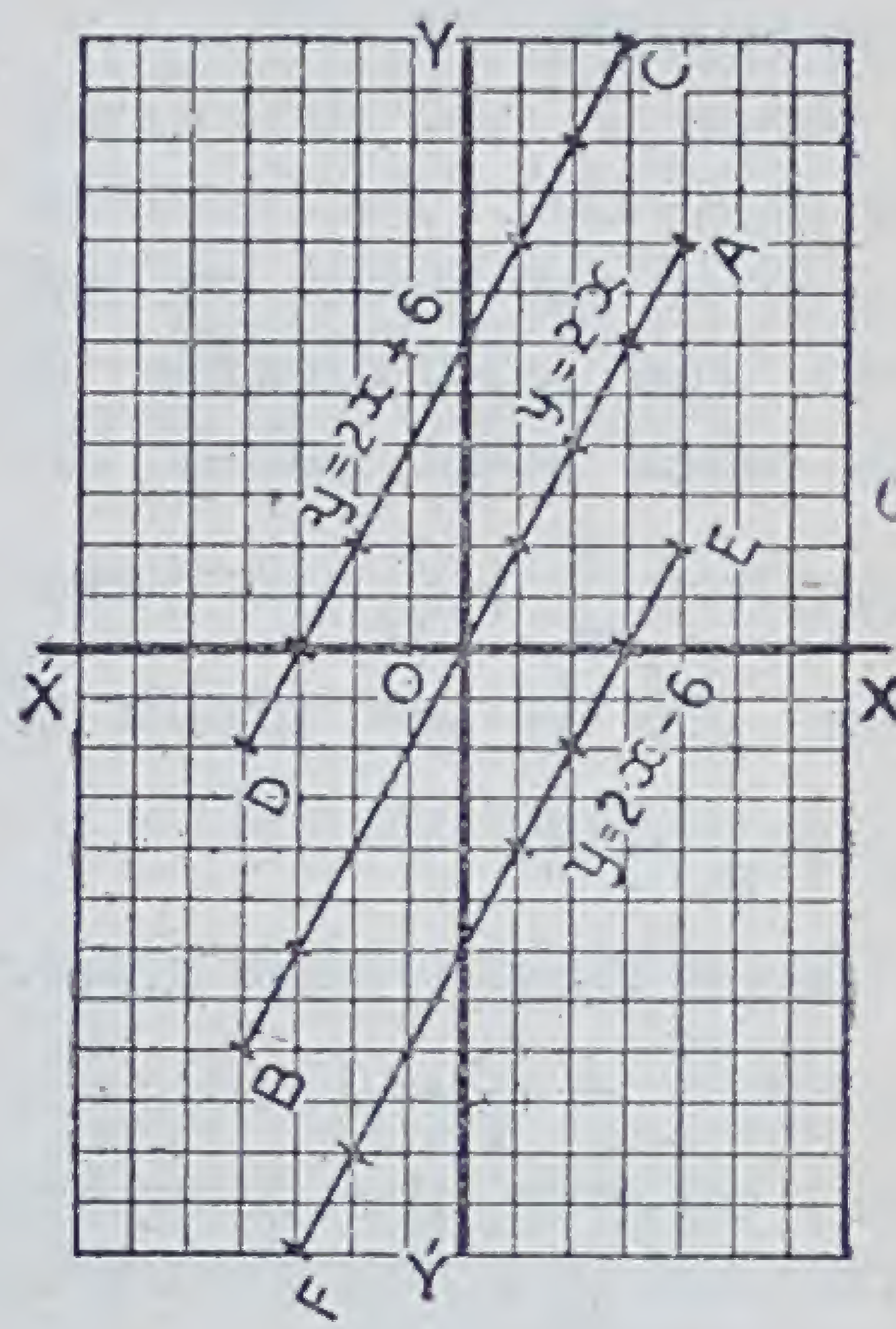
(ii) $y=2x+6$ are

$x=$	0	1	2	3	-2	-4
$y=$	6	8	10	12	2	-2

(iii) $y=2x-6$ are

$x=$	0	1	2	3	4	-2	-3
$y=$	-6	-4	-2	0	2	-10	-12

Now we plot points from table (i) and find that AOB is the graph of $y=2x$.



We plot points from table (ii) and find that CD is the graph of $y=2x+6$.

We plot points from table (iii) and find that EF is the graph of $y=2x-6$.

It will be observed that in each case the graph is a straight line and all these straight lines are *parallel.*

Moreover the intercept made by $y=2x+6$ on y-axis is $+6$, the intercept made by $y=2x$ on y-axis is 0 and the intercept made by $y=2x-6$ on y-axis is -6, or the intercept made by the graph of an equation of the form $\boldsymbol{y=mx+c}$ on y-axis is $\boldsymbol{c}$.

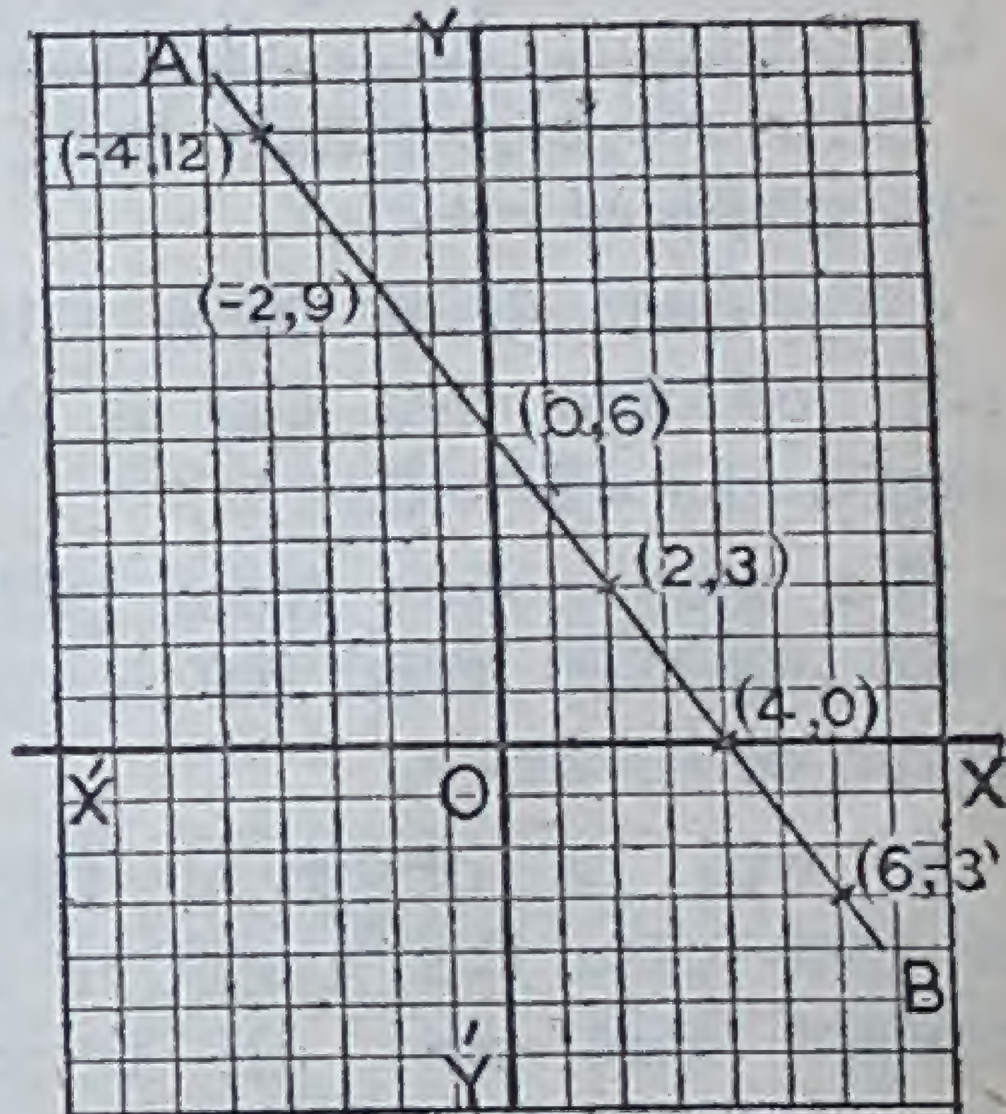
It is important to note that any equation of the form $ax+by+c=0$ can be put into the form $y=\left(-\frac{a}{b}\right)x+\left(-\frac{c}{b}\right)$, or $\boldsymbol{y=mx+c}$.

7. Graph of $\frac{x}{4}+\frac{y}{6}=1$.

Tabulating the corresponding values of x and y from this equation, we have

$x=$	0	1	2	3	4	6	-2	-4
$y=$	6	$\frac{9}{2}$	3	$\frac{3}{2}$	0	-3	9	12

Plot the points (0, 6), (2, 3), (4, 0), (6, -3), (-2, 9), (-4, 12) with integral co-ordinates and join them.



All these points lie on the straight line AB.

Since it can be easily verified that the co-ordinates of any point on AB and the co-ordinates of no other point satisfy the equation $\frac{x}{4}+\frac{y}{6}=1$, therefore AB is the graph of $\frac{x}{4}+\frac{y}{6}=1$.

Draw similarly the graphs of the following equations :

(i) $\frac{x}{3}+\frac{y}{4}=1$, (ii) $\frac{x}{4}+\frac{y}{5}=1$, (iii) $\frac{x}{5}+\frac{y}{6}=1$,

and observe the intercepts made by these lines on the axes and compare their length with the denominators of x and y in each case.

It is important to note that the straight line $\frac{x}{4}+\frac{y}{6}=1$ cuts x-axis at (4, 0) and y-axis at (0, 6) or the intercept made by it on x-axis is 4 and is the same as the denominator of x in the equation, and the intercept made by it on y-axis is 6 and is the same as the denominator of y in the equation.

Similarly, the intercepts of $\frac{x}{3}+\frac{y}{4}=1$ on the x-axis and y-axis are 3 and 4 respectively.

In *general*, if an equation is of the form $\frac{x}{a}+\frac{y}{b}=1$, its graph is a straight line which makes with x-axis an intercept$=a$ and with y-axis an intercept$=b$.

From the above examples it is clear that **the graph of an equation of the first degree in x and y, whatever be its form, is always a straight line.**

It is for this reason that an equation of the first degree in x and y is generally called a **linear equation.**

Since a straight line is determined by any two points on it, therefore for drawing the graph of a linear equation, theoretically it is sufficient to plot two points on it, but for practical purposes and *accuracy* it is necessary to plot at least three points on it *sufficiently apart.* Integral values are more helpful than fractional ones. In most cases the values we get by putting $x=0$ and $y=0$ are very convenient.

Summary :

(i) (0,0) represents the origin.

(ii) $y=0$ represents x-axis.

(iii) $x=0$ represents y-axis.

(iv) $x=a$ represents a straight line parallel to y-axis and at the distance of a from it.

(v) $y=b$ represents a straight line parallel to x-axis and at the distance of b from it.

(vi) $y=mx$ represents a straight line passing through the origin with m as its inclination towards x-axis.

(vii) $y=mx+c$ represents a straight line with m as its inclination towards x-axis and c as its intercept on y-axis.

(viii) Equations $y=mx+c_1$, $y=mx+c_2$, etc., which differ in the absolute term only, represent straight lines parallel to $y=mx$.

(ix) $\frac{x}{a}+\frac{y}{b}=1$ represents a straight line having intercepts a and b on x-axis and y-axis respectively.

8. Graph of a Linear Function of x.

An expression which contains a variable x and which has a definite value for every value of x is called **a function of x.**

Thus $3x^2+2x-5$, $3x+5$ are functions of x.

Such expressions as $6x$, $7x+3$, $\frac{3x-5}{4}$ which contain no power of x higher than the first are called **linear functions of x.**

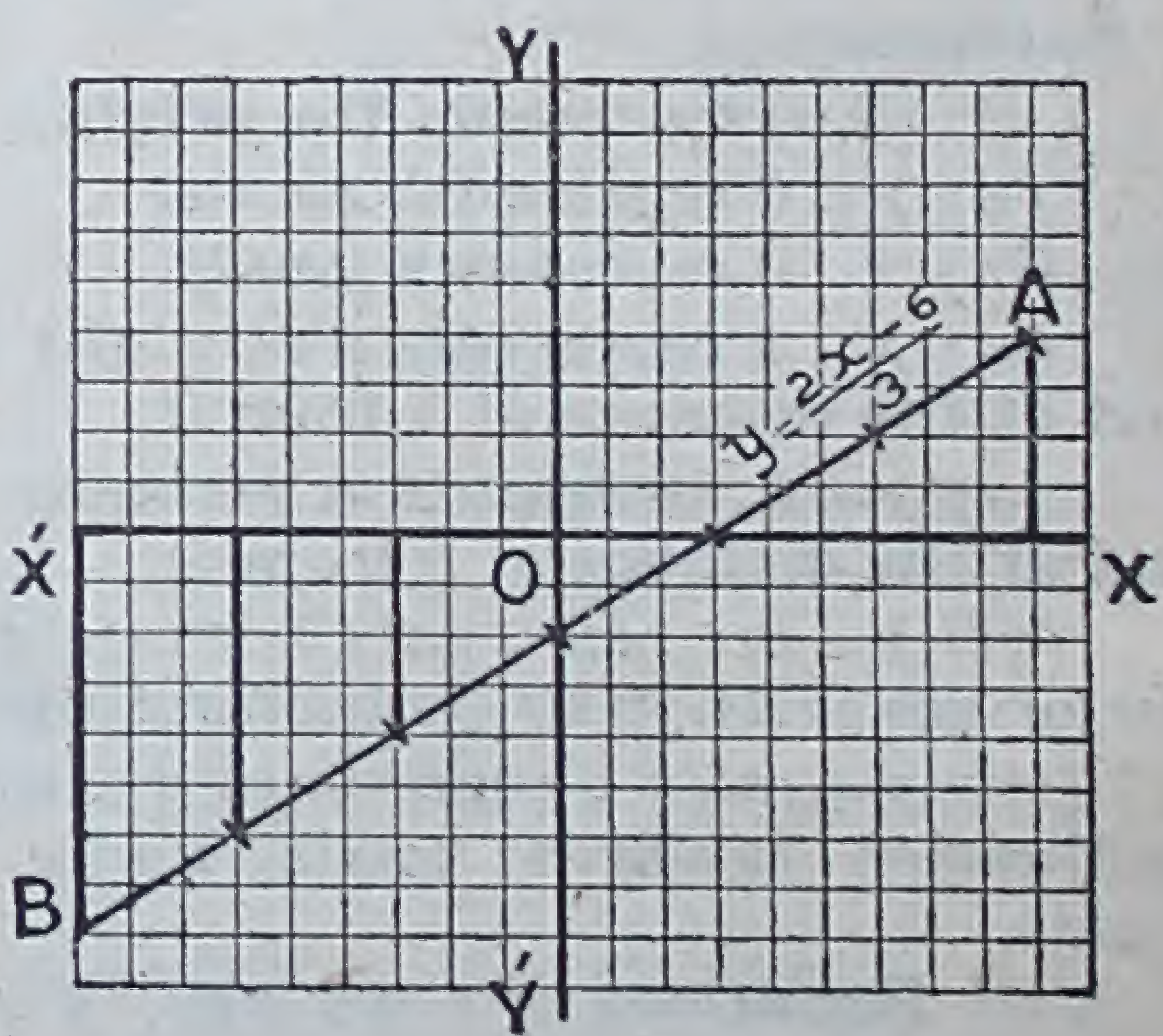
Suppose we have to draw the graph of the linear function $\frac{2x-6}{3}$.

Here we put $y=\frac{2x-6}{3}$ and by giving different values to x find the corresponding values of y.

As this is an equation of the first degree, its graph must be a straight line.

Now we choose 3 convenient points on this straight line as given below :

$x=$	0	3	6
$y=$	-2	0	2



Plot points $(0, -2)$ $(3, 0)$, $(6, 2)$ and join them. The straight line AB is the required graph.

From this graph we can read the value of the function of x, for different values of x, *e.g.*, when

(i) $x=9$, $\frac{2x-6}{3}=4$; (ii) $x=-3$, $\frac{2x-6}{3}=-4$;

(iii) $x=-6$, $\frac{2x-6}{3}=-6$; (iv) $x=-9$, $\frac{2x-6}{3}=-8$.

9. Equation of a straight line passing through two points.

Let the given points be (9, 4) and (6, 2) and the required equation of the straight line be $y = mx + c$ ... (i)

As (9, 4) lies on (i), $\therefore\ 4 = 9m + c$ ... (ii)
and as (6, 2) ,, ,, ,, $\therefore\ 2 = 6m + c$ ... (iii)
From (ii) and (iii) $m = \frac{2}{3}$ and $c = -2$,
$\therefore$ the required equation is $y = \frac{2}{3}x - 2$ or $2x - 3y = 6$.

EXERCISE 56.

[Choose a suitable scale wherever necessary, use accurate squared paper and a pencil with a sharp end.]

1. In the same diagram draw the graphs of:
(i) $y = 4x + 5$, (ii) $y = -4x + 5$, (iii) $y = 4x - 5$, (iv) $y = -4x - 5$.

2. In the same diagram draw the graphs of:
(i) $3y = x + 6$, (ii) $3y + x = 6$, (iii) $x - 3y = 6$, (iv) $x + 3y + 6 = 0$.

3. In the same diagram draw the graphs of:
(i) $y = 5x$, (ii) $y = 5x + 4$, (iii) $y = 5x - 4$.

4. In the same diagram draw the graphs of:
(i) $y + 2x = 0$, (ii) $y + 2x = 5$, (iii) $y + 2x + 5 = 0$.

Reduce each of the following equations to the form ;

(i) $y = mx + c$, (ii) $\dfrac{x}{a} + \dfrac{y}{b} = 1$, (iii) $ax + by + c = 0$,

5. $x = 2y + 5$. 6. $x + \frac{1}{2}y = 4$. 7. $3x - 4y = 10$.

8. $2y = \frac{1}{4}(x - 3)$. 9. $\dfrac{x}{3} - \dfrac{y}{4} + 1 = 0$. 10. $\dfrac{4}{5}x + \dfrac{5}{4}y = 1$.

Reduce each of the following equations to the form $y = mx + c$:

11. $y = 3$. 12. $x = 5$. 13. $y = 0$. 14. $x = 0$.

Draw the graphs of the following equations:

15. $x+3=0$.
16. $y+7=0$.
17. $3y+5x=0$.
18. $5y-3x=0$.
19. $5y+3x=0$.
20. $8y+6x=5$.
21. $8x-6y=5$.
22. $x-4y=9$.
23. $x-4y+9=0$.
24. $4x=2y+6$.
25. $x=11y-4$.
26. $x+3y=\frac{9}{2}$.
27. $\frac{x}{5}+\frac{y}{3}=1$.
28. $\frac{x}{5}-\frac{y}{3}=1$.
29. $\frac{x}{7}+\frac{y}{9}=1$.
30. $\frac{x}{7}-\frac{y}{9}=1$.
31. $\frac{x-y}{3}=\frac{3}{4}$.
32. $\frac{y}{3}=\frac{7x-6}{5}$.
33. $\frac{y}{3}=4\frac{1}{2}-x$.

34. Draw the graphs of the following equations in the same diagram and find the area included by them:

(i) $y-3=0$, (ii) $x-2=0$, (iii) $4x+5y=60$.

35. Draw the graph of the equation $3x+4y=8$. In this graph read the value of x when y is -10. [Punjab, 1918.]

36. Plot the graph of $y=\dfrac{3x+13}{2}$ and from the graph read off the value of y when $x=3$. What are the intercepts on the axes of x and y?

Show that the points $(-1, 5)$ and $(1, 8)$ lie on the graph.

Draw the graphs of the following functions of x:

37. $(4-3x)$.
38. $8(3-x)$.
39. $\frac{1-x}{2}$.
40. $\frac{3x-4}{5}$.
41. $\frac{5x-3}{6}$.
42. $\frac{7x+3}{6}$.

Find the equations of the straight lines passing through the following pairs of points:

43. $(4, 5), (6, 8)$.
44. $(-3, 5), (-5, 3)$.
45. $(0, -4), (5, 0)$.
46. $(0, 0), (-6, 8)$.

47. For what value of m and c will the points $(3, 5)$, $(-3, 8)$ lie on the straight line $y=mx+c$?

48. Plot points $(3, -3)$ and $(-3, 3)$ and write down the equation of the straight line joining them. [Punjab, 1918.]

49. Plot points (12, 4), (−3, −5) and find if the straight line joining them passes through the point (8, 2). [Punjab, 1919.]

50. Plot points $A(5, 7)$, $B(7, 10)$, $C(2, 2)$ and $D(-1, -4)$ and find the equations of the straight lines AB and CD. Find the point of their intersection. [Bombay, 1912.]

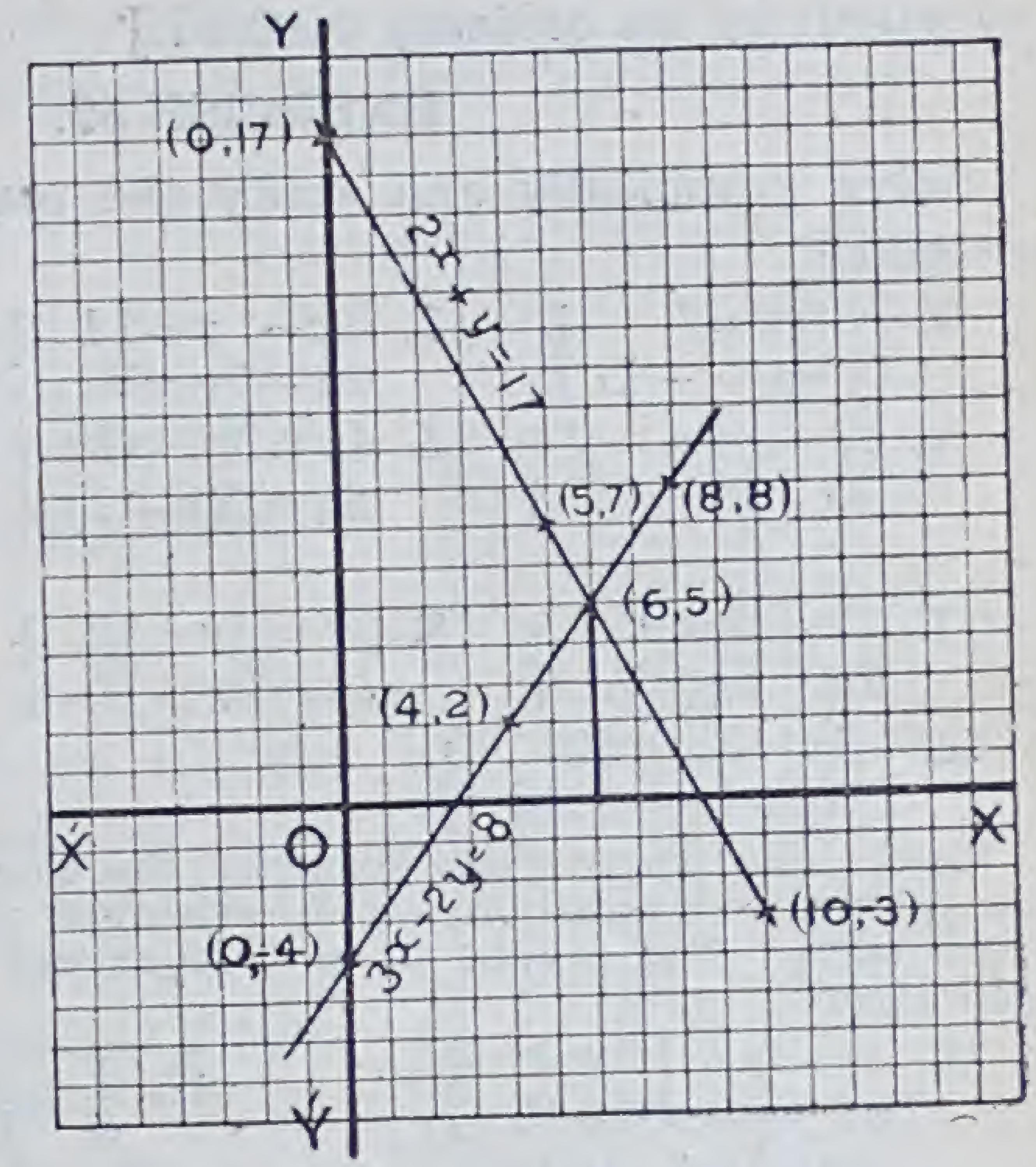
10. Graphic Solution of Simultaneous Equations.

Example. Solve graphically the equations:

$$2x + y = 17 \quad \ldots \quad \text{(i)}$$
$$3x - 2y = 8 \quad \ldots \quad \text{(ii)}$$

As both the equations are of the first degree in x and y, their graphs must be straight lines.

Now plot 3 points on each straight line sufficiently apart, as given below.



From equation $2x+y=17$, we have

$x=$	0	5	10
$y=$	17	7	−3

From equation $3x-2y=8$, we have

$x=$	0	4	8
$y=$	−4	2	8

We join the points (0, 17,) (5, 7) and (10, −3) for the graph of $2x+y=17$ and the points (0, 4), (4, 2) and (8, 8) for the graph of $3x-2y=8$, and write down the corresponding equations on the graphs for the sake of distinction.

As these graphs intersect at the point (6, 5), the co-ordinates (6, 5) satisfy both the equations.

Hence $x=6$ and $y=5$ is the solution of the given equations.

[The students may verify the result by solving the equations by the ordinary method.]

EXERCISE 57.

Solve graphically and verify the result by the ordinary method:

1. $\left.\begin{aligned} x+y&=7 \\ x-y&=-1. \end{aligned}\right\}$ 2. $\left.\begin{aligned} 3x-y&=11 \\ 2x+3y&=22. \end{aligned}\right\}$ 3. $\left.\begin{aligned} 2x+3y&=17 \\ 3x+2y&=18. \end{aligned}\right\}$

4. $\left.\begin{aligned} 2x-3y&=-7 \\ 3x-2y&=2. \end{aligned}\right\}$ 5. $\left.\begin{aligned} 4y&=3(x+2) \\ 5x&=2(y+2). \end{aligned}\right\}$ 6. $\left.\begin{aligned} 2x&=3y \\ \frac{x}{2}+\frac{y}{6}&=1\tfrac{5}{6}. \end{aligned}\right\}$

7. $\left.\begin{aligned} x+5&=0 \\ 2x-5y&=0. \end{aligned}\right\}$ 8. $\left.\begin{aligned} 2y&=3x-16 \\ \tfrac{1}{2}x&=3y+16. \end{aligned}\right\}$ 9. $\left.\begin{aligned} 3x+4y&=24 \\ 4x-3y&=7. \end{aligned}\right\}$

10. $\left.\begin{aligned} 5x-3y&=29 \\ 3x+5y+3&=0. \end{aligned}\right\}$

NOTE. In the above two examples, the co-efficients of x and y in the simultaneous equations are interchanged with a single change in their signs. The graphs of such equations are perpendicular to each other.

11. $\left.\begin{aligned} x-2y+11&=0 \\ 2x-3y+18&=0. \end{aligned}\right\}$ [Punjab, 1924]

12. $\left.\begin{aligned} \frac{x}{5}+\frac{y}{10}&=1 \\ 3x-2y&=1. \end{aligned}\right\}$

Show that the following straight lines are concurrent, also find the co-ordinates of the common points:

13. $x+y=11$, $2x-y-4=0$, $x+2y=17$.

14. $2x+y=13$, $3x-2y-2=0$, $x+11=3y$.

15. $4x-y=5$, $5x+2y=16$, $x+3y=11$.

16. Find the co-ordinates of the vertices and also the lengths of the sides of the triangle formed by the straight lines,

$$x-y=0,\ x+5=0,\ x+y=4.$$

11. Application of Graphs—Statistical Graphs.

Example. If sugar is selling @ 4 seers a rupee, we can have 8 seers of sugar for Rs. 2, 16 seers for Rs. 4, 20 seers for Rs. 5, and so on.

Here the ratio between the quantity of sugar in seers and the price in rupees$=4:1$.

If the price in rupees be represented by x and the quantity in seers by y, then $y=4x$.

Tabulating the values of x and y from this equation we have

$x=$	0	1	2	4	5
$y=$	0	4	8	16	20

and plotting these points, we have the annexed graph.

Now we can use this graph to *read off* the price of any number of srs. of sugar and *vice versa*, *e.g.*, the price of 14 srs. is Rs. 3-8 as., for when $y=14$, $x=3\frac{1}{2}$, and Rs. $4\frac{1}{2}$ is the price of 18 srs., for when $y=18$, $x=4\frac{1}{2}$.

Thus the graph can be used as a *ready reckoner.* The limit up to which it can be used as such, depends on the size of the graph. By increasing our unit and using larger paper, we can obtain a wider range of prices and quantities and also greater accuracy.

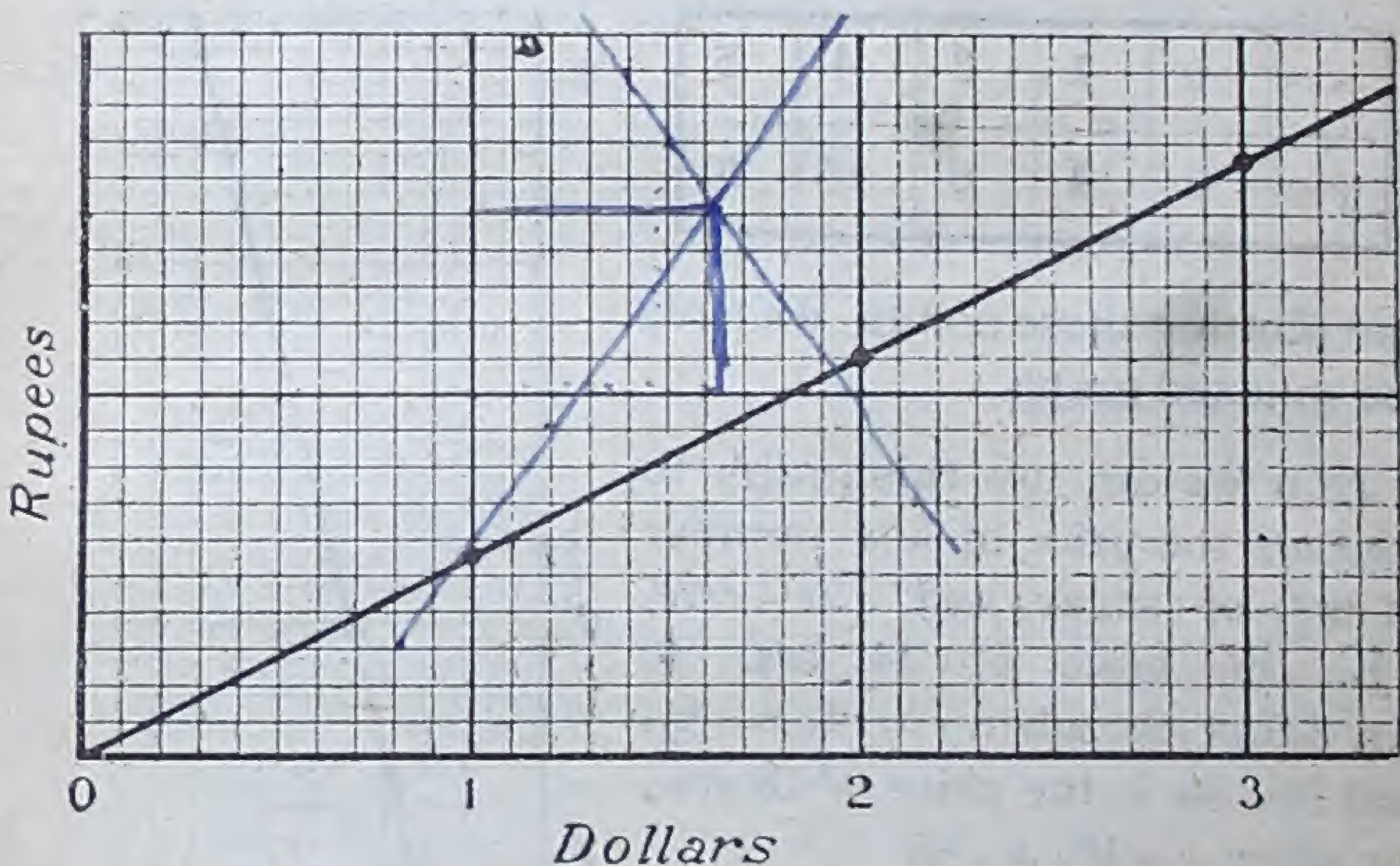
12. Conversion Graph. $\$1 = \text{Rs. } 2{\cdot}75$ (approx.), therefore if y represents the number of rupees and x the number of corresponding dollars, we have

$$y = 2{\cdot}75\ x$$

Tabulating the values of x and y from this equation, we have

$x=$	0	1	2	3
$y=$	0	2·75	5·5	8·25

Taking 1″ for \$1 along the x-axis and ·2″ for Re. 1 along the y-axis, we have the following graph:



By means of our graph we can now convert rupees into dollars and dollars into rupees with some accuracy. Thus $\$\,1{\cdot}5 = \text{Rs. } 4{\cdot}2$ (approx.) and $\text{Rs. } 6 = \$\,2{\cdot}2$ (approx). By drawing a larger graph, we can increase its range of application. Such a graph is called a **conversion graph.**

To illustrate still further the use of conversion graphs we construct here a graph representing the relation between the degrees of Centigrade and Fahrenheit scales.

Since 32°F=0°C and 100°C corresponds to 212°F and there are (212° – 32°) or 180°F for 100°C, $\therefore \frac{9}{5}$°F=1°C.

Hence the relation between the number of degrees in the two scales is given by the equation

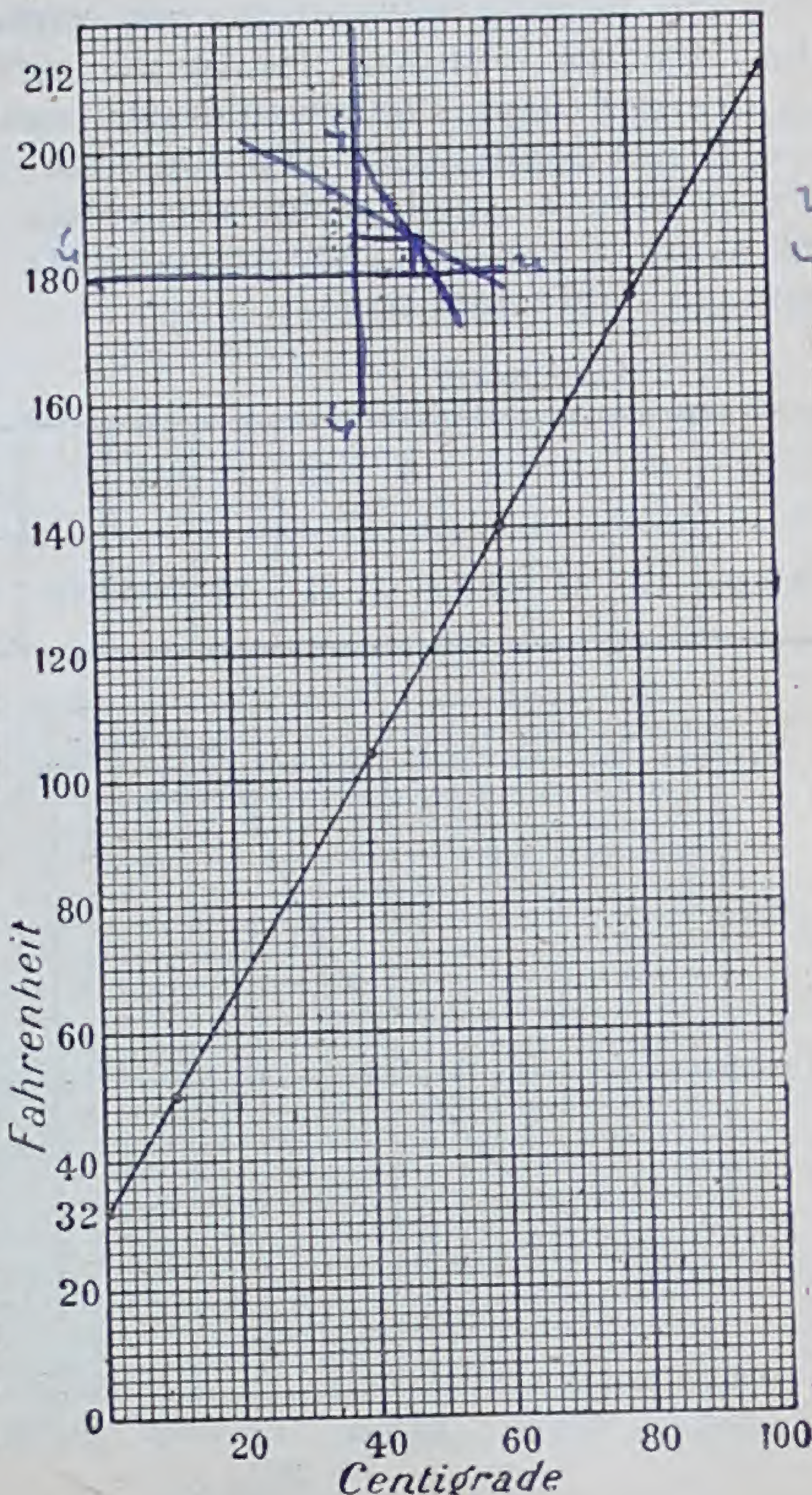
$$y=\tfrac{9}{5}x+32,$$

where x stands for the number of degrees in the Centigrade scale and y for the number of degrees in the Fahrenheit scale.

Tabulating the values of x and y from the above equation, we have

$x=$	0	10	40	60	80
$y=$	32	50	104	140	176

Taking 1 cm. to represent 20°C along x-axis and 20°F along y-axis and plotting the points, we get the annexed graph. From this graph read the value of 50°C in Fahrenheit scale and that of 120°F in Centigrade scale.

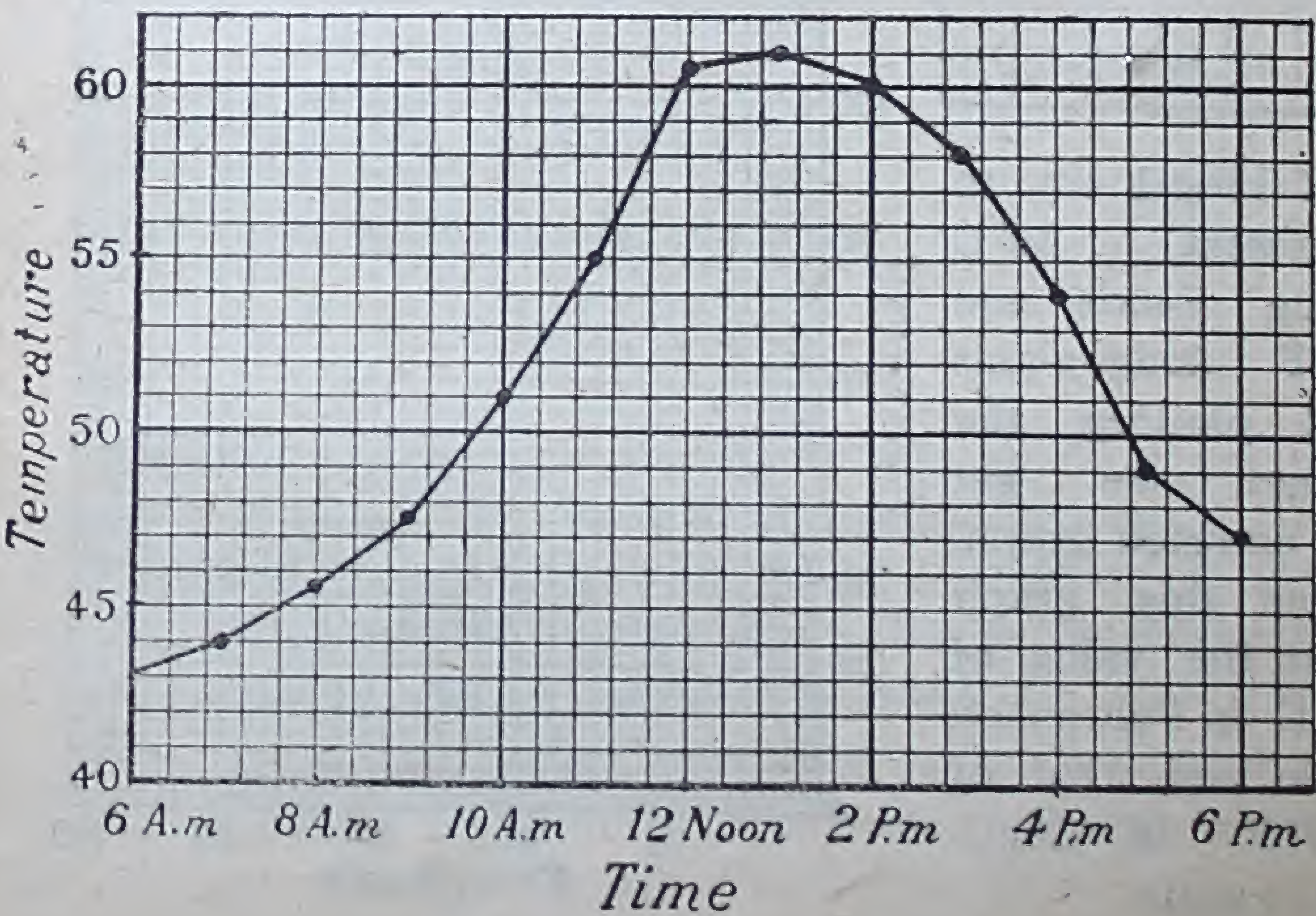


x = 4
y = 3

13. Discontinuous and Continuous Graphs.

Sometimes the corresponding values of two *variables* cannot be determined from a given relation or an equation but are obtained by actual observation or experiments, *e.g.*, the temperature at different intervals of time throughout a given period. In such cases we draw the graph representing the relation between the two variables with the help of the data thus collected and by joining the successive points by *straight lines*. The graph drawn consists of *separate lines* and is not continuous. To illustrate this point we have drawn here a graph which represents the relation between the hours of a day and the temperature of a place at such hours in Fahrenheit scale.

	Forenoon							Afternoon					
Time	6	7	8	9	10	11	12	1	2	3	4	5	6
Temp.	43	44	45·5	47·5	51	55	60·5	61	60	58	54	49	47



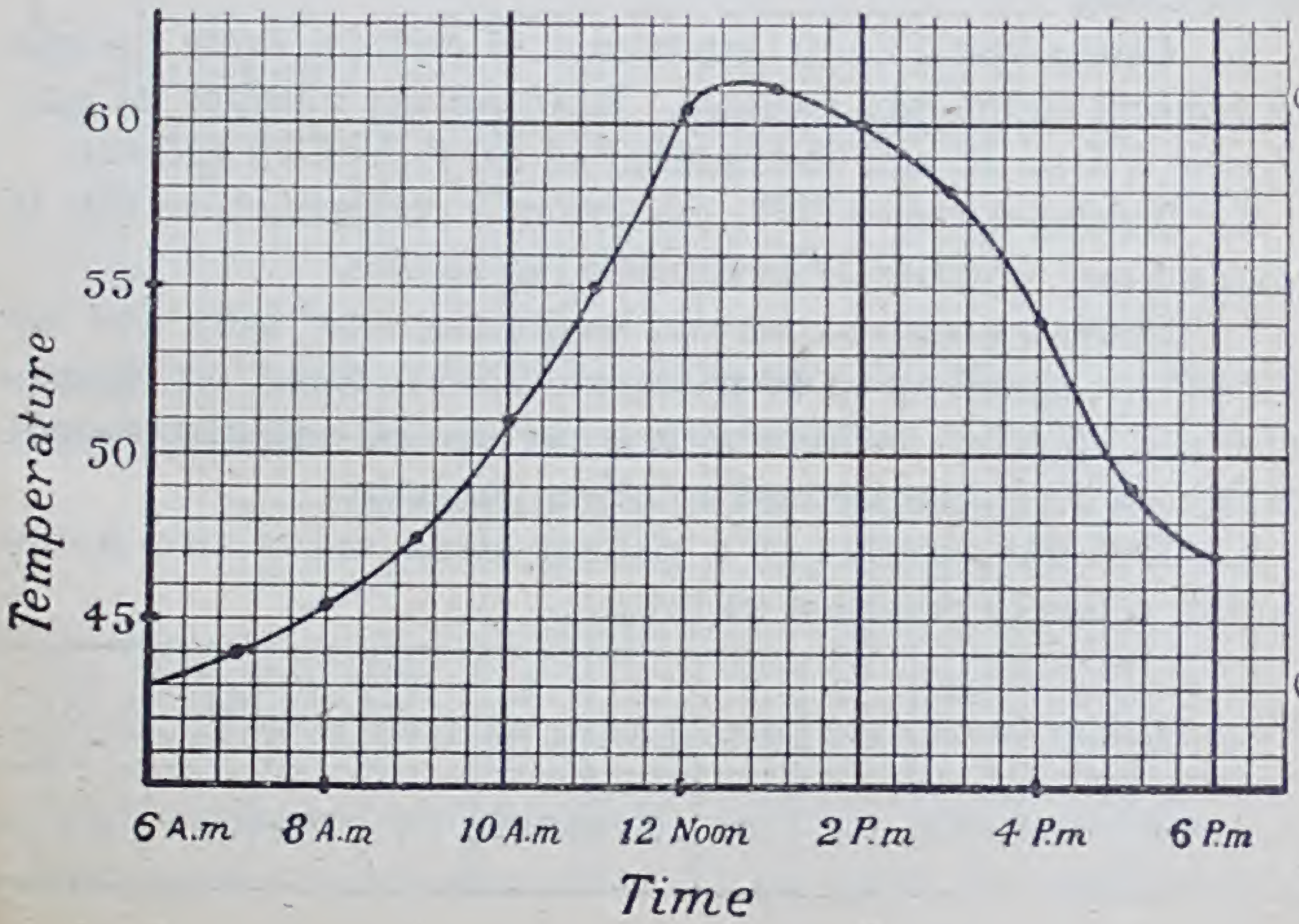
In this case we have taken 6 A.M. and 40° as the *origin*, 25″ for 1 hour along the x-axis and ·1″ for 1° along the y-axis. The successive points are joined by separate st. lines and we get a discontinuous graph.

Now we can read the approximate temperature at any particular time :

e.g., at 10-30 A.M. temp. is 53° (approx.)
and at 2-30 P.M. temp. is 58·8° (approx.)

At 1 P.M. the temperature appears to be the *maximum*, and between 12 noon and 2 P.M. the temperature is fairly steady. Between 10 A.M. and 12 noon there is a rapid increase, and between 3 P.M. and 5 P.M. there is a rapid fall.

In all experimental work, since the observations are made at intervals, the graphs made from the data thus collected would be discontinuous. Really in majority of cases the variation *is not so abrupt but gradual and continuous*, therefore the proper graph in such cases should be continuous, as shown below :



This graph gives us better representation of the variation of temperature during the day and also gives us more accurate reading, *e.g.*, the maximum temperature is not exactly at 1 P.M. but between 12 noon and 1 P.M.

EXERCISE 58.

1. If rice cost 1 anna 6 pies per seer, construct a graph to read off the price of any number of seers. Frame an equation connecting the price of rice in annas and the number of seers.

2. If 40 articles cost Re. 1 4 as., construct a graph to read off the price of articles correct up to 6 pies.

3. Taking £1 = Rs. $13\frac{1}{2}$, construct a conversion graph to read off the number of shillings for a given number of rupees.

4. Taking 1 kilometre = ·62 mile (approx.), draw a graph showing the number of miles in any number of kilometres up to 100. Read off approximately the number of kilometres in 45 miles.

5. Given that 1 cu. ft. contains 6·25 gallons, draw a graph to convert cu. ft. into gallons. Read off the number of gallons in 8·5 cu. ft. and the number of cu. ft. in 66·5 gallons.

6. Taking 1 cm. = ·39″, construct a conversion graph to read off any number of centimetres in inches.

7. Draw a graph to convert '*miles per hour*' into '*feet per second*' for speed up to 40 miles an hour. Read off speeds of 8 and 25 miles an hour as feet per second and also speeds of 24 and 35 ft. per second as miles per hour.

8. The temperature of a patient taken every two hours during a day is recorded as below:

Time	6 a.m.	8 a.m.	10 a.m.	12 noon	2 p.m.	4 p.m.	6 p.m.	8 p.m.	10 p.m.	12 mid-night	2 a.m.	4 a.m.
Temp.	99·2	100	100·4	101	103	102·4	101·6	100	99·8	99·6	99·4	99·2

Draw a continuous graph and read off the temperature at 3 P.M. and the time when the temperature was approximately 100·5°.

9. The average monthly temperature of a town is as follows:

Month	Jan.	Feb.	Mar.	Ap.	May	Jun.	July	Aug.	Sep.	Oct.	Nov.	Dec.
Temp.	63°	72°	88°	95°	102°	106°	104°	100°	96°	90°	75°	65°

Draw a continuous graph and read off approximately the hottest fortnight of the year.

10. The length of the longest day in different latitudes is given below in hours:

Latitude	0°	10°	20°	30°	40°	50°	60°	70°	80°	90°
Length of the longest day	12	12·6	13·3	14	14·8	16·1	18·5	24	24	24

Draw a continuous graph. Estimate the length of the longest day in latitudes 16°, 38°, 65°

11. The following table gives us the heights of places above the sea-level in thousands of feet and the corresponding heights of the barometer in inches:

Height above sea-level	0	5	10	15	20	25
Height of barometer	30·1	25	20·6	17·1	14·3	11

Draw a continuous graph and read off the height of the barometer at 3,000 ft. and 8,000 ft. above the sea-level.

12. Draw the graph illustrating the annual premium for Life Assurance for Rs. 1000 according to the ages at the time of the first Payment from the following table :

Age in years	22	24	26	28	30	32	34	36
Premium in Rs. & as.	25	26-2	27-6	28-12	30-6	32-2	34	36-2

Read off the premium for starting at the ages (i) 25, (ii) 33.

13. The temperature of water in a boiler is 20°C. It is continually heated for half an hour and its temperature is recorded at intervals of 5 minutes and is tabulated below:

Intervals in minutes	5	10	15	20	25	30
Temperature	24°	28·6°	32·8°	36·6°	40°	43°

Show graphically the relation between time and temperature of the water and read off the temperature after 23 minutes.

14. The formula $s=16t^2$ gives us the relation between the space (s) covered by a body falling freely under gravity in time (t). If in the following table s be given in feet and t in seconds,

t	1	2	3	4	5	6
s	16	64	144	256	400	576

draw a continuous graph for the relation between the distance and time, and read off

(i) the height of a tower from the top of which a stone takes 4·5 seconds to fall,

(ii) the depth of water-level of a well which a stone takes 2·25 seconds to reach.

Verify the results by calculation.

15. The average height of boys and girls at different ages is given below in centimetres in two different tables :

Table for boys

Age in years	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
Ht. in centi-metres	68	80	89	93	95	100	107	116	122	124	126	128	132	140	149	156	160	164

Table for girls

Age in years	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
Ht. in centi-metres	68	79	84	88	95	105	111	116	117	120	124	128	140	148	150	154	156	158

Draw *carefully* both these graphs on the same paper with the same axes. Represent one by dotted lines and the other by a continuous line.

Note the following points :

(1) The age up to which both the curves are concurrent.

(2) The ages at which the two curves intersect.

(3) The ages when the springs of boys and girls are almost parallel.

(4) The ages when the plateaus of boys and girls are almost parallel.

(5) The ages when the boys are in springs and the girls in plateaus.

Note that the *plateaus are the periods when the system can be more energetic and the springs are the periods when the system is more liable to fatigue.*

Taking this as a law, point out periods :

(i) when the boys can be made to do more work,

(ii) when the girls can be made to do more work,

(iii) when the boys should have a lighter course,

(iv) when the girls should have a lighter course,

(v) at what ages should we hold the public examinations for boys and girls ?

16. The average weights of boys and girls are given in lbs. up to the age of 15 years, in the following tables :

Table for boys

Age	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Weight	22	36	38	42	45	50	53	58	63	69	71	74	80	95	106

Table for girls

Age	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Weight	22	32	35	40	44	47	50	54	58	64	71	80	90	109	109

Draw both these graphs on the same paper with the same axes and compare the curves.

CHAPTER XII

RATIO

1. If A and B be two quantities of the *same kind* which when measured with one and the same unit are represented by the numbers a and b, the **ratio** of A to B is the quotient of a by b.

This ratio is written as a/b, $a \div b$ or $a : b$ and is read as 'the ratio of a to b.'

Thus, for example, if A and B be lengths of $3\frac{1}{4}$ ft. and $5\frac{1}{2}$ ft. respectively, the ratio of A to B is $3\frac{1}{4}/5\frac{1}{2}$ which when simplified $= \frac{13}{22}$ or $13 : 22$.

Similarly, the ratio of Rs. $3\frac{1}{4}$ to Rs. $5\frac{1}{2}$ is $13 : 22$.

In the first example the quantities compared are *two lengths* and in the second *two sums of money*; the ratio in both cases is the same.

Thus, a ratio is simply an abstract number and is independent of the concrete units employed in the quantities compared.

In the ratio $a : b$, a and b are its **terms.** The first term a is called the **antecedent** and the second term b is called the **consequent.**

The ratio $b : a$ is known as the **inverse ratio** of $a : b$.

As a ratio is represented by a vulgar fraction, *its value remains unaltered if its terms are multiplied or divided by the same number*; for instance, $\frac{a}{b} = \frac{ma}{mb}$ and $\frac{na}{nb} = \frac{a}{b}$.

A ratio is said to be a ratio of *greater inequality*, of *equality* or of *less inequality*, according as the antecedent is greater than, equal to, or less than the consequent.

Thus $4 : 3$ or $\frac{4}{3}$ is a ratio of greater inequality
$3 : 3$ or $\frac{3}{3}$,, ,, ,, equality,
$3 : 4$ or $\frac{3}{4}$,, ,, ,, less inequality.

Two or more ratios are said to be **compounded,** when they are multiplied together. Thus the ratio compounded of $a:b$ and $c:d$ is $ac:bd$.

When the ratio $a:b$ is compounded with itself, the resulting ratio $a^2:b^2$ is called the **duplicate** ratio of $a:b$.

Similarly, $a^3:b^3$ is called the **triplicate** ratio of $a:b$. When $a:b$ is the original ratio, $\sqrt{a}:\sqrt{b}$ is called the **sub-duplicate** ratio aud $\sqrt[3]{a}:\sqrt[3]{b}$ is called the **sub-triplicate** ratio of $a:b$.

2. Theorem. *A ratio of greater inequality is diminished and a ratio of less inequality is increased, by adding the same positive quantity to both the terms.*

Let $\dfrac{a}{b}$ be a ratio, a and b being positive.

Let x, a positive quantity, be added to both its terms.

The new ratio thus formed $=\dfrac{a+x}{b+x}$.

$\dfrac{a}{b}>=<\dfrac{a+x}{b+x}$ according as $\left(\dfrac{a}{b}-\dfrac{a+x}{b+x}\right)$ is positive, zero or negative.

Now $\dfrac{a}{b}-\dfrac{a+x}{b+x}=\dfrac{ax-bx}{b(b+x)}=\dfrac{x(a-b)}{b(b+x)}$.

Since x, b, $(b+x)$ are all positive, the above is positive, zero or negative according as $a-b$ is positive, zero or negative; or according as $a>=<b$.

$$\therefore \qquad \frac{a}{b}>=<\frac{a+x}{b+x} \text{ according as } a>=<b.$$

It is left to the student to prove the corresponding theorem:

A ratio of greater inequality is increased and a ratio of less inequality is diminished, by subtracting the same positive quantity from both the terms.

Example 1. Find the ratio compounded of the three ratios $4a:5b$, $b^2:2ac$ and $3c^2:2ab$.

The required ratio $=\dfrac{4a\times b^2\times 3c^2}{5b\times 2ac\times 2ab}=3c:5a$.

Example 2. If $4+x : 5+x$ be the duplicate ratio of $2 : 3$, find the value of x.

$$\frac{4+x}{5+x}=\frac{2\times 2}{3\times 3}=\frac{4}{9}$$

$\therefore$ $36+9x=20+4x$

$\therefore$ $5x=-16$

$\therefore$ $x=-3\tfrac{1}{5}$.

Example 3. What must be added to the terms of the ratio $4 : 7$ to make it equal to $5 : 6$?

Let x be the required number.

Then by the question $\dfrac{4+x}{7+x}=\dfrac{5}{6}$

or $6(4+x)=5(7+x)$

$\therefore$ $24+6x=35+5x$

$\therefore$ $x=11$.

Example 4. If $x : y=4 : 7$, find the value of $\dfrac{2x+y}{x+3y}$

Dividing the numerator and the denominator by y

we have $$\frac{2x+y}{x+3y}=\frac{2\left(\frac{x}{y}\right)+1}{\frac{x}{y}+3}$$

Substituting the value of $\dfrac{x}{y}$, we get

$$\frac{2x+y}{x+3y}=\frac{2\times\frac{4}{7}+1}{\frac{4}{7}+3}$$

$$=\frac{1\frac{8}{7}}{3\frac{4}{7}}=\frac{15}{25}=\frac{3}{5}.$$

Example 5. If $3x+4y=5x-2y$, find the ratio $x : y$.

Since $3x+4y=5x-2y$

$\therefore$ $2x=6y$

or $x=3y$

$\therefore$ $\dfrac{x}{y}=3=\dfrac{3}{1}$

$\therefore$ $x : y=3 : 1$.

EXERCISE 59.

[Do the first five questions *mentally*.]

1. Write down the ratio compounded of the ratios of :

(i) $3:4,\ 5:6,\ 2:3$;

(ii) $a:b,\ b:c,\ c:d$;

(iii) $(x+1):(x-1),\ (x^2-x+1):(x^2+x+1)$.

2. Write down the duplicate ratio of :

(i) $3:7$. (ii) $ab:cd$, (iii) $(a+x):(a-x)$,

(iv) $(x+1):(x-1)$.

3. Write down the triplicate ratio of :

(i) $4:5$, (ii) $mn:pq$, (iii) $(a-b):(a+b)$, (iv) $x^2:2x$.

4. Write down the sub-duplicate ratio of :

(i) $16:25$, (ii) $x^6:y^8$, (iii) $x^2+4ax+4a^2:x^2-4ax+4a^2$.

5. Write down the sub-triplicate ratio of :

(i) $64:27$, (ii) $x^9:y^6$, (iii) $125a^6b^{12}:343x^3y^6$.

6. Compare the ratios :

(i) $2:3$ and $5:6$, (ii) $14:11$ and $8:5$.

7. If $4x+3:7x-1$ be the duplicate ratio of $3:4$, find the value of x.

8. If $m:n$ be the duplicate ratio of $2m-x:n-2x$, shew that $x^2=mn$.

9. If $(a+x):(a-x)$ be the duplicate ratio of $(a+b):(a-b)$, then $(x-b):(a-x)=b(a+b):a(a-b)$.

10. If $p:q$ be the triplicate ratio of $p+x:q+x$, shew that $x^3-3pqx-pq(p+q)=0$.

11. What must be added to the terms of $5:9$ to make it equal to $7:8$?

12. What must be added to the terms of $a:b$ so that the resulting ratio may be in the duplicate ratio of $a:b$?

13. What must be added to the terms of $a:b$ so that the resulting ratio may be in the triplicate ratio of $a:b$?

14. Two numbers are in the ratio of 7 : 8 and if four be added to each, they will be in the ratio of 8 : 9. Find the numbers.

[*Hint.* Let $7x$ and $8x$ be the required numbers.]

15. Two numbers are in the ratio of 3 : 4 and if five be subtracted from each, they will be in the ratio of 2 : 3. Find the numbers.

16. Find two numbers in the ratio of 9 : 7 whose difference is 11.

17. Two numbers are in the ratio of 3 : 5 and the sum of their squares is 1666; find them.

18. If x be added to the antecedent of the ratio $a : b$, what quantity must be added to the consequent so that the value of the ratio may remain unaltered?

19. If $x : y = 5 : 6$, find the value of $\dfrac{5x-3y}{6x+3y}$.

20. If $x : y = 3 : 2$, find the value of $\dfrac{4x^2-xy+2y^2}{3x^2+xy-y^2}$.

21. If $\dfrac{7x-4y}{3x+y} = \dfrac{5}{13}$, find the value of $\dfrac{x}{y}$.

22. If $\dfrac{2x^2-3y^2}{x^2+y^2} = \dfrac{2}{41}$, find $\dfrac{x}{y}$.

23. If $(8x-2y)^2 = (3x+4y)^2$, find $x : y$.

24. Which is greater, $\dfrac{y+x}{y-x}$ or $\dfrac{y^2+x^2}{y^2-x^2}$, when x and y are both positive and $y > x$?

25. Which is greater, $\dfrac{x^3+y^3}{x^2+y^2}$ or $\dfrac{x^2+y^2}{x+y}$ when x and y are both positive?

26. Find the least integer which must be added to each term of the ratio 3 : 5 so that the resulting ratio may be greater than 2 : 3.

27. Find two numbers whose difference is 9 and whose sub-duplicate ratio is 1 : 2.

28. The ages of two persons are in the ratio of 5 : 4; 9 years ago they were in the ratio 4 : 3. Find their ages.

29. The bases of two triangles are in the ratio of 3 : 4 and their heights in the ratio of 6 : 5. Find the ratio of their areas.

30. The ratio of the radii of two circles is $p : q$ and the ratio of their areas is $p-x : q-x$. Express x in terms of p and q.

*31. At a height of h feet we can see to a distance of $\sqrt{\frac{3h}{2}}$ miles. An aeroplane ascends from 2,000 ft. to 5,000 ft.: in what ratio does the pilot increase his range of vision?

*32. The soldiers in two armies when they met in a battle were in the ratio of 10 : 3. Their respective losses were as 20 : 3 and the survivors as 40 : 13; if the number of survivors in the larger army be 24,000, find the original number of soldiers in each army.

Particular Cases of Ratio

*3. Meaning of $\frac{0}{n}$, $\frac{m}{\infty}$, $\frac{m}{0}$, $\frac{0}{0}$.

(i) The fraction $\frac{m}{n}$ stands for the share of each person when a quantity m is divided among n persons.

Similarly, $\frac{0}{n}$ stands for the share of each person when 0 is divided among n persons. As in this case the quantity to be divided is zero, the share of each must be zero.

Or $\frac{0}{n}=0$ ... ... (i)

Cor. When $n=1, \frac{0}{1}=0$.

(ii) If the numerator of a fraction is constant (say m), its value decreases as its denominator increases, *e.g.*, $\frac{m}{3}<\frac{m}{2}$, $\frac{m}{4}<\frac{m}{3}$ $\frac{m}{5}<\frac{m}{4}$ ...and when the denominator is *very*

very large the value of the fraction is *very very small*, and when the denominator is *infinitely large* or ∞, its value is *infinitely small* or

$$\frac{m}{\infty}=0. \qquad \ldots \qquad \ldots \qquad \ldots \qquad \text{(ii)}$$

Cor. When $m=1, \frac{1}{\infty}=0.$

(iii) If the numerator of a fraction is constant (say m), its value increases as its denominator decreases, *e.g.*, $\frac{m}{\frac{1}{3}}>\frac{m}{\frac{1}{2}}$ and $\frac{m}{\frac{1}{4}}>\frac{m}{\frac{1}{3}}$ ...and when the denominator is *very very small, the* value of the fraction is *very very large*, and when the denominator is *infinitely small* or 0, its value is *infinitely large*, or

$$\frac{m}{0}=\infty. \qquad \ldots \qquad \ldots \qquad \ldots \qquad \text{(iii)}$$

Cor. When $m=1, \frac{1}{0}=\infty.$

(iv) $\frac{x^2-1}{x-1}=\frac{1-1}{1-1}=\frac{0}{0}$ when $x=1$ (By substitution).

But $\frac{x^2-1}{x-1}\equiv x+1=1+1=2$ when $x=1$ (By identity).

$$\therefore \quad \frac{0}{0}=2. \qquad \ldots \qquad \ldots \qquad \ldots \qquad \text{(i)}$$

Again, $\frac{x^3-1}{x-1}=\frac{1-1}{1-1}=\frac{0}{0}$ when $x=1$ (By substitution).

But $\frac{x^3-1}{x-1}\equiv x^2+x+1=1+1+1=3$ when $x=1$ (By identity).

$$\therefore \quad \frac{0}{0}=3 \qquad \ldots \qquad \ldots \qquad \ldots \qquad \text{(ii)}$$

Again, $\frac{x^2-1}{x^3-1}=\frac{1-1}{1-1}=\frac{0}{0}$ when $x=1$ (By substitution).

But $\frac{x^2-1}{x^3-1}\equiv\frac{x+1}{x^2+x+1}=\frac{1+1}{1+1+1}=\frac{2}{3}$ when $x=1$ (By identity).

$$\therefore \quad \frac{0}{0}=\frac{2}{3}. \qquad \ldots \qquad \ldots \qquad \ldots \qquad \text{(iii)}$$

Again, $\dfrac{x^3-1}{x^4-1}=\dfrac{1-1}{1-1}=\dfrac{0}{0}$ when $x=1$ (By substitution).

But $\dfrac{x^3-1}{x^4-1}\equiv\dfrac{x^2+x+1}{x^3+x^2+x+1}=\dfrac{1+1+1}{1+1+1+1}=\dfrac{3}{4}$ when $x=1$.
(By identity).

$$\therefore \quad \frac{0}{0}=\frac{3}{4} \quad \ldots \quad \ldots \quad \ldots \quad \text{(iv)}$$

Thus, $\dfrac{0}{0}$ has several *different* values. By taking suitable identities and proceeding as above, we can prove that $\dfrac{0}{0}$ can have *any* value or is *indeterminate.*

Example 6. Find the value of $\dfrac{x^2+ax-2a^2}{x^2+4ax-5a^2}$ when $x=a$.

If we substitute a for x in the given expression, it is reduced to the indeterminate form $\dfrac{0}{0}$.

In order to avoid the indeterminate form, we proceed as follows:

$$\text{The expression}=\frac{(x+2a)(x-a)}{(x+5a)(x-a)}\equiv\frac{x+2a}{x+5a}$$
$$=\frac{a+2a}{a+5a}$$
$$=\frac{3a}{6a}=\tfrac{1}{2}.$$

Find the value of:

*33. $\dfrac{2x^2+5x-7}{3x^2-8x+5}$ when $x=1$.

*34. $\dfrac{x^2+3x-40}{x^2+4x-32}$ when $x=-8$.

*35. $\dfrac{x^3+4x^2-8x+3}{2x^3-5x^2+2x+1}$ when $x=1$.

*36. $\dfrac{2x^3-7x^2a+6a^2x-a^3}{3x^4-3x^3a+4a^3x-4a^4}$ when $x=a$.

*37. $\dfrac{3x^3-27ax^2+78a^2x-72a^3}{2x^3+10ax^2-4a^2x-48a^3}$ when $x=2a$.

*38. $\dfrac{x^3-5ax^2+3a^2x+9a^3}{x^3-7ax^2+15a^2x-9a^3}$ when $x=3a$.

CHAPTER XIII

PROPORTION

1. If $\frac{a}{b}=\frac{c}{d}$, the numbers a, b, c, d are said to be **in proportion** (or **proportionals**) and it is read as 'a to b equals c to d' or 'a is to b as c is to d'.

Of this proportion a, d are called the **extremes,** b, c the **means** and d the **fourth proportional** to a, b and c.

Two quantities are said to be *proportional* to two others when the ratio of the first two is equal to the ratio of the second two, and two quantities are said to be *inversely proportional* to two others when the *inverse* ratio of the first two is equal to the ratio of the other two.

If a, b, c, d, e...be such that $a : b=b : c=c : d=d : e=$&c., a, b, c, d, e are said to be in **continued proportion.**

When a, b, c, are in continued proportion, b is said to be the **mean proportional** between a and c and c is said to be the **third proportional** to a and b.

2. If $a : b=c : d$, then $ad=bc$.

Since $\frac{a}{b}=\frac{c}{d}$, multiplying both sides by bd, we get

$$ad=bc.$$

Thus, if *four quantities are proportional, the product of the extremes is equal to the product of the means.*

Conversely, if $ad=bc$, we can easily prove by dividing both sides by bd that $\frac{a}{b}=\frac{c}{d}$, or $a : b=c : d$.

Cor. If $a : b=b : c$, then $ac=b^2$, *i.e., if three quantities are in continued proportion, the product of the extremes is equal to the square of the means.*

3. If $a : b = b : c$, then $a : c = a^2 : b^2$.

For $\frac{a}{b} = \frac{b}{c}$

$\therefore \quad \frac{a}{b} \times \frac{b}{c} = \frac{a}{b} \times \frac{a}{b}$

$\therefore \quad \frac{a}{c} = \frac{a^2}{b^2}$, or $a : c = a^2 : b^2$.

Thus, if *three quantities are in continued proportion, the first is to the third in the duplicate ratio of the first to the second.*

4. If $a : b = b : c = c : d$, then $a : d = a^3 : b^3$.

For $\frac{a}{b} = \frac{b}{c} = \frac{c}{d}$

$\therefore \quad \frac{a}{b} \times \frac{b}{c} \times \frac{c}{d} = \frac{a}{b} \times \frac{a}{b} \times \frac{a}{b}$

$\therefore \quad \frac{a}{d} = \frac{a^3}{b^3}$, or $a : d = a^3 : b^3$.

Thus, if *four quantities are in continued proportion, the first is to the fourth in the triplicate ratio of the first to the second.*

Example 1. Find the fourth proportional to 2, 3, 4.

Let x be the fourth proportional required.

Then $\frac{2}{3} = \frac{4}{x}$, which gives $x = 6$

$\therefore$ the 4th proportional is 6.

Example 2. Find the mean proportional to 9, 25.

Let x be the mean proportional required.

Then $\frac{9}{x} = \frac{x}{25}$, $\quad \therefore \quad x^2 = 9 \times 25$, $\quad \therefore \quad x = 15$.

$\therefore$ the mean proportional is 15.

Example 3. Find the third proportional to 2, 8.

Let x be the third proportional required.

Then $\frac{2}{8} = \frac{8}{x}$, $\quad \therefore \quad 2x = 64$, $\quad \therefore \quad x = 32$.

$\therefore$ the 3rd proportional is 32.

EXERCISE 60.

Find the fourth proportional to:

1. 8, 12, 14. 2. 9, 12, 15. 3. ·0024, 2·4, ·04. 4. $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$.

5. $3a, 6a^2, 2ab^2$.

Find the mean proportional to :

6. 9, 16. 7. 49, 25. 8. 9, 64. 9. 121, 144. 10. $4\frac{1}{2}$, 2.

11. x^2, y^2. 12. $(a+b)(a-b)^3$ and $(a+b)^3(a-b)$.

Find the third proportional to :

13. 18, 12. 14. 20, 16. 15. 15, 6. 16. $x^2-y^2, (x+y)^2$.

17. If $1+x, 3+x, 6+x$ are in continued proportion, find x.

Example 4. What must be added to each of the numbers 7, 11 and 19 so that the resulting numbers may be in continued proportion ?

Let x be the required number.

Then $\dfrac{7+x}{11+x}=\dfrac{11+x}{19+x}$.

Multiplying cross-wise, we have

$$(7+x)(19+x)=(11+x)^2$$

$$\therefore \quad 133+26x+x^2=121+22x+x^2$$

$$\therefore \quad 4x=-12, \text{ or } x=-3.$$

$\therefore$ The number to be added is -3.

18. What must be added to each of the numbers 3, 7, 13 and 22 so that the results may be in proportion ?

19. What must be subtracted from each of the numbers 27, 41, 30 and 46 so that the results may be in proportion ?

20. Shew that the fourth proportional to $x^2+7x+12$, $5(x+3)$ and $6(x+4)$ is independent of x.

21. Find two numbers such that their mean proportional is 12 and the third proportional is 324.

22. The mean proportional between 45 and a certain number is three times the mean proportional between 5 and 22. Find the number.

23. If y is the mean proportional between x and z, shew that $xy+yz$ is the mean proportional between x^2+y^2 and y^2+z^2.

24. Two numbers consisting of the same two digits are in the ratio of 4 : 7. Find the numbers.

25. A certain kind of brass is made up of copper, zinc, lead and tin. The ratio of copper to zinc is 1 : 2, that of zinc to lead is 3 : 4 and that of lead to tin is 4 : 5. Find the quantity of each metal in 540 lbs. of brass.

***Example 5.** A cask is filled with wine and water in the ratio of 4 : 3. If 12 gallons of the mixture be drawn off and the cask be filled with water, their ratio becomes 3 : 4. How many gallons can the cask hold ?

Let x be the required number of gallons.

When 12 gallons are drawn off, the number of gallons in it $=x-12$. Out of every 7 gallons of mixture, 4 are wine and 3 are water, or $\frac{4}{7}$ of the mixture is wine and $\frac{3}{7}$ is water.

Hence $\frac{4}{7}(x-12)$ gallons of wine are left and $\frac{3}{7}(x-12)$ gallons of water.

When the cask is again filled up with water, the quantity of water $=\frac{3}{7}(x-12)+12$.

By the question, we have

$$\frac{\frac{4}{7}(x-12)}{\frac{3}{7}(x-12)+12}=\frac{3}{4}.$$

Multiplying cross-wise, we have

$$16(x-12)=9(x-12)+252$$

$$\therefore \quad 7x=336$$

$$\therefore \quad x=48.$$

$\therefore$ the cask can hold 48 gallons.

*26. Two vessels contain a mixture of wine and water in the ratios of 8 : 5 and 4 : 1. In what ratio must liquids be drawn from each to give a mixture of wine and water in the ratio of 3 : 1 ?

***27** Two vessels contain mixtures of wine and water in the ratios of 7 : 3 and 3 : 2. In what ratio must liquids be drawn from each to give a mixture of wine and water in the ratio of 9 : 5?

5. **Propositions.** When $a : b = c : d$, the following relations always hold good. The student is advised not only to reproduce them, but to learn each by heart.

I. If $a : b = c : d$, then $b : a = d : c$.

For if $\frac{a}{b} = \frac{c}{d}$, then $1 \div \frac{a}{b} = 1 \div \frac{c}{d}$

$$\text{or} \quad \frac{b}{a} = \frac{d}{c}$$

$$\text{or} \quad b : a = d : c.$$

This proposition is known as **Invertendo.**

II. If $a : b = c : d$, then $a : c = b : d$.

For if $\frac{a}{b} = \frac{c}{d}$, then $ad = bc$.

Dividing both sides by cd, we get

$$\frac{a}{c} = \frac{b}{d}$$

$$\text{or} \quad a : c = b : d.$$

This proposition is known as **Alternando** and is frequently used in the solution of fractional equations.

III. If $a : b = c : d$, then $(a + b) : b = (c + d) : d$.

For if $\frac{a}{b} = \frac{c}{d}$, then $\frac{a}{b} + 1 = \frac{c}{d} + 1$

$$\therefore \quad \frac{a+b}{b} = \frac{c+d}{d}$$

$$\text{or} \quad (a+b) : b = (c+d) : d.$$

This proposition is known as **Componendo.**

IV. If $a : b = c : d$, then $(a - b) : b = (c - d) : d$.

For if $\frac{a}{b} = \frac{c}{d}$, then $\frac{a}{b} - 1 = \frac{c}{d} - 1$.

$$\therefore \qquad \frac{a-b}{b}=\frac{c-d}{d}$$

or $\qquad (a-b) : b=(c-d) : d.$

This proposition is known as **Dividendo.**

V. If $a : b=c : d$, then $(a+b) : (a-b)=(c+d) : (c-d)$.

By III, we have $\qquad \dfrac{a+b}{b}=\dfrac{c+d}{d}.$

By IV, we have $\qquad \dfrac{a-b}{b}=\dfrac{c-d}{d}.$

By division, we have $\dfrac{a+b}{a-b}=\dfrac{c+d}{c-d}.$

Or $\quad (a+b) : (a-b)=(c+d) : (c-d).$

This proposition is known as **Componendo and Dividendo** and is frequently used in the solution of fractional equations.

Example 1. If $\dfrac{4a+5b}{4a-5b}=\dfrac{4c+5d}{4c-5d}$, then $\dfrac{a}{b}=\dfrac{c}{d}$.

By componendo and dividendo, we have

$$\frac{(4a+5b)+(4a-5b)}{(4a+5b)-(4a-5b)}=\frac{(4c+5d)+(4c-5d)}{(4c+5d)-(4c-5d)}$$

$$\therefore \qquad \frac{8a}{10b}=\frac{8c}{10d}.$$

Multiplying both sides by $\frac{10}{8}$, we have

$$\frac{a}{b}=\frac{c}{d}$$

or $\qquad a : b=c : d.$

Example 2. If $a : b=c : d$, then $\dfrac{2a+3b}{2a-3b}=\dfrac{2c+3d}{2c-3d}$.

First Method.

Let $\dfrac{a}{b}=\dfrac{c}{d}=k$; then $a=bk$ and $c=dk$.

$$\frac{2a+3b}{2a-3b}=\frac{2bk+3b}{2bk-3b}=\frac{b(2k+3)}{b(2k-3)}=\frac{2k+3}{2k-3}$$

and $\quad \dfrac{2c+3d}{2c-3d}=\dfrac{2dk+3d}{2dk-3d}=\dfrac{d(2k+3)}{d(2k-3)}=\dfrac{2k+3}{2k-3}$

$\therefore \quad \dfrac{2a+3b}{2a-3b}=\dfrac{2c+3d}{2c-3d}.$

Second Method.

Since $\dfrac{a}{b}=\dfrac{c}{d}$, $\therefore \quad \dfrac{a}{c}=\dfrac{b}{d}$ (Alternando).

or $\quad \dfrac{2a}{2c}=\dfrac{3b}{3d}$

$\therefore \quad \dfrac{2a}{3b}=\dfrac{2c}{3d}$ (Alternando).

$\therefore \quad \dfrac{2a+3b}{2a-3b}=\dfrac{2c+3d}{2c-3d}$ (Componendo and dividendo.)

[The second method is more elegant, but the first is surer.]

Example 3. If $3a+4b+3c+4d : 2a+3b+2c+3d$
$=3a+4b-6c-8d : 2a+3b-4c-6d,$

then $\quad a : b=c : d.$

Let $\quad \dfrac{3a+4b+3c+4d}{2a+3b+2c+3d}=\dfrac{3a+4b-6c-8d}{2a+3b-4c-6d}=k.$

Then $\quad 3a+4b+3c+4d=(2a+3b+2c+3d)k \quad \ldots \quad$ (i)

and $\quad 3a+4b-6c-8d=(2a+3b-4c-6d)k \quad \ldots \quad$ (ii)

From (i) and (ii), we have

$a(3-2k)+b(4-3k)+c(3-2k)+d(4-3k)=0 \quad \ldots \quad$ (iii)

$a(3-2k)+b(4-3k)-2c(3-2k)-2d(4-3k)=0 \quad \ldots \quad$ (iv)

From (iii) and (iv), by factors and transposition, we have

$(a+c)(3-2k)=(b+d)(3k-4) \quad \ldots \quad \ldots \quad$ (v)

$(a-2c)(3-2k)=(b-2d)(3k-4) \quad \ldots \quad \ldots \quad$ (vi)

From (v) and (vi), we have

$\dfrac{a+c}{b+d}=\dfrac{3k-4}{3-2k} \quad \ldots \quad \ldots \quad$ (vii)

$\dfrac{a-2c}{b-2d}=\dfrac{3k-4}{3-2k} \quad \ldots \quad \ldots \quad$ (viii

From (vii) and (viii), we have

$$\frac{a+c}{b+d}=\frac{a-2c}{b-2d}.$$

Multiplying cross-wise, we have

$$ab-2ad+bc-2cd=ab+ad-2bc-2cd$$

$\therefore$ $-3ad=-3bc$

or $ad=bc$

or $\dfrac{a}{b}=\dfrac{c}{d}$

or $a:b=c:d.$

EXERCISE 61.

If $a:b=c:d$, prove that

1. $a+3c : b+3d=3a+c : 3b+d.$
2. $3a-2c : 3b-2d=5a-4c : 5b-4d.$
3. $ma+nc : mb+nd=pa+qc : pb+qd.$
4. $ma-nc : mb-nd=pa-qc : pb-qd.$
5. $a+b : ma-nb=c+d : mc-nd.$
6. $a^2+b^2 : a^2-b^2=c^2+d^2 : c^2-d^2.$
7. $\dfrac{ma+nb}{mc+nd}=\dfrac{b^2c}{d^2a}.$
8. $pa^2+qc^2 : pb^2+qd^2=ma^2-nc^2 : mb^2-nd^2.$
9. $a^2+ab+b^2 : a^2-ab+b^2=c^2+cd+d^2 : c^2-cd+d^2.$
10. $a^3+c^3 : b^3+d^3=\left(\dfrac{a+c}{b+d}\right)^3.$
11. $\dfrac{la^2+mab+nb^2}{pa^2+qab+rb^2}=\dfrac{lc^2+mcd+nd^2}{pc^2+qcd+rd^2}.$
12. $\dfrac{(a-b)(a-c)}{a}=(a+d)-(b+c).$

[*Hint*. Simplify the left-hand side and substitute ad for bc.]

13. $4(a+b)(c+d)=bd\left\{\dfrac{a+b}{b}+\dfrac{c+d}{d}\right\}^2.$

[*Hint*. Apply componendo, then $\dfrac{a+b}{b}+\dfrac{c+d}{d}=\dfrac{2(a+b)}{b}=\dfrac{2(c+d)}{d}$;
hence $\left(\dfrac{a+b}{b}+\dfrac{c+d}{d}\right)^2=\dfrac{2(a+b)}{b}\times\dfrac{2(c+d)}{d}$.]

14. $(5a+6c)(5b-6d)=(5a-6c)(5b+6d)$.

15. If $\dfrac{2a+3b}{2a-3b}=\dfrac{2c+3d}{2c-3d}$, then $a:b=c:d$.

16. If $\dfrac{a^2+c^2}{b^2+d^2}=\dfrac{ac}{bd}$, then $a:b=c:d$.

17. If $\left(\dfrac{a-c}{b-d}\right)^2=\dfrac{a^2}{b^2}$, then $a:b=c:d$

18. If $\left(\dfrac{a+c}{b+d}\right)^3=\dfrac{a(a+c)^2}{b(b+d)^2}$, then $a:b=c:d$.

19. If $\dfrac{ac+bd}{ac-bd}=\dfrac{a^2+b^2}{a^2-b^2}$, then $a:b=c:d$.

20. If $(a+b+c+d)(a-b-c+d)=(a-b+c-d)(a+b-c-d)$, then $a:b=c:d$.

21. If $(2a+3b+5c+4d)(2a-3b-5c+4d)=(2a+3b-5c-4d)(2a-3b+5c-4d)$, then $ad:bc=15:8$.

22. If $\dfrac{3a+b-9c-3d}{a+2b-3c-6d}=\dfrac{3a+b+6c+2d}{a+2b+2c+4d}$, then $a:b=c:d$.

23. If $(a^2+b^2)(x^2+y^2)=(ax+by)^2$, then $a:x=b:y$.

24. If $2b=a+c$ and $\dfrac{2}{c}=\dfrac{1}{b}+\dfrac{1}{d}$, shew that $a:b=c:d$.

Example 4. If $x=\dfrac{2ab}{a+b}$, find the value of $\dfrac{x+a}{x-a}+\dfrac{x+b}{x-b}$.

Since $x=\dfrac{2ab}{a+b}$, $\quad\therefore\quad \dfrac{x}{a}=\dfrac{2b}{a+b}$ ... ... (i)

and $\dfrac{x}{b}=\dfrac{2a}{a+b}$ ... ... (ii)

Applying componendo and dividendo in (i) and (ii), we get $\dfrac{x+a}{x-a}=\dfrac{a+3b}{-a+b}$ ... ... (iii)

and $\dfrac{x+b}{x-b}=\dfrac{3a+b}{a-b}$ ... . (iv)

Adding (iii) and (iv), we get

$$\frac{x+a}{x-a}+\frac{x+b}{x-b}=\frac{a+3b}{-a+b}+\frac{3a+b}{a-b}$$

$$=\frac{-a-3b+3a+b}{a-b}$$

$$=\frac{2(a-b)}{a-b}=2.$$

25. If $x=\frac{4ab}{a+b}$, find the value of $\frac{x+2a}{x-2a}+\frac{x+2b}{x-2b}$.

26. If $x=\frac{6ab}{a+b}$, find the value of $\frac{x+3a}{x-3a}+\frac{x+3b}{x-3b}$.

Example 5. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, then $\frac{a^3+c^3}{b^3+d^3}=\frac{e^3}{f^3}$.

Let $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=k$, then $a=bk$, $c=dk$ and $e=fk$.

$$\therefore \quad \frac{a^3+c^3}{b^3+d^3}=\frac{b^3k^3+d^3k^3}{b^3+d^3}=\frac{k^3(b^3+d^3)}{b^3+d^3}=k^3.$$

But $\frac{e}{f}=k$ $\qquad \therefore \quad \frac{e^3}{f^3}=k^3$

$\frac{a^3+c^3}{b^3+d^3}=\frac{e^3}{f^3}$, for each is equal to the same quantity.

If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, prove that

27. $\frac{a+c}{b+d}=\frac{c-e}{d-f}$.

28. $\frac{b-3f}{a-3e}=\frac{d-b}{c-a}$.

29. $\frac{c^3+e^3}{d^3+f^3}=\frac{a^3-a^2c}{b^3-b^2d}$.

30. $\frac{bd-f^2}{b^2+d^2}=\frac{ac-e^2}{a^2+c^2}$.

31. $\frac{a^3+c^3+e^3}{b^3+d^3+f^3}=\frac{ace}{bdf}$.

Example 6. If $\frac{x^2+y^2}{xy}=\frac{p^2+q^2}{pq}$, then $\frac{x+y}{x-y}=\frac{p+q}{p-q}$.

Since $\frac{x^2+y^2}{xy}=\frac{p^2+q^2}{pq}$,

$\therefore \quad \frac{x^2+y^2}{2xy}=\frac{p^2+q^2}{2pq}$ [Dividing by 2.]

$\therefore \quad \frac{x^2+y^2+2xy}{x^2+y^2-2xy}=\frac{p^2+q^2+2pq}{p^2+q^2-2pq}$

[Componendo and dividendo.]

$\therefore \quad \frac{(x+y)^2}{(x-y)^2}=\frac{(p+q)^2}{(p-q)^2}$

$\therefore \quad \frac{x+y}{x-y}=\frac{p+q}{p-q}$. [Sq. root of both sides.]

32. If $\dfrac{x^2+y^2}{p^2+q^2}=\dfrac{xy}{pq}$, then $\dfrac{(x^2+y^2)^2}{(x^2-y^2)^2}=\dfrac{(p^2+q^2)^2}{(p^2-q^2)^2}$.

33. If $\dfrac{x+y}{p+q}=\dfrac{y}{q}$, then $\dfrac{x(x+2y)}{y^2}=\dfrac{p(p+2q)}{q^2}$.

34. If $\dfrac{x+y+p+q}{x-y+p-q}=\dfrac{x+y-p-q}{x-y-p+q}$, then $\dfrac{x+p}{y+q}=\dfrac{x-p}{y-q}$

35. If $\dfrac{x+y-p+q}{x-y+p+q}=\dfrac{y+p+x+q}{y-p-x+q}$, then $\dfrac{x+q}{y-p}=\dfrac{y+q}{p+x}$.

36. If $\dfrac{x^3+3xy^2}{3x^2y+y^3}=\dfrac{p^3+3pq^2}{3p^2q+q^3}$, then $\dfrac{x+y}{x-y}=\dfrac{p+q}{p-q}$.

Example 7. If $a:b=b:c$, then $\dfrac{a^2+ab+b^2}{b^2+bc+c^2}=\dfrac{a^2}{b^2}$.

Since $\dfrac{a}{b}=\dfrac{b}{c}$ $\quad\therefore\ b^2=ac.$

$$\therefore \quad \frac{a^2+ab+b^2}{b^2+bc+c^2}=\frac{a^2+ab+ac}{ac+bc+c^2}=\frac{a(a+b+c)}{c(a+b+c)}=\frac{a}{c}$$

$$=\frac{a}{b}\times\frac{b}{c}=\frac{a}{b}\times\frac{a}{b}=\frac{a^2}{b^2}.$$

NOTE. Putting each of the ratios $=k$, we get $b=ck$ and $a=bk=ck^2$. Using these results, we can do this example by the ordinary **k-method.**

If $\dfrac{a}{b}=\dfrac{b}{c}=\dfrac{c}{d}$, prove that

37. $\dfrac{a^2+b^2+c^2}{b^2+c^2+d^2}=\dfrac{ab+bc+c^2}{bc+cd+d^2}$.

38. $\dfrac{a^2+b^2+c^2}{b^2+c^2+d^2}=\dfrac{a^2+bc}{b^2+cd}$.

39. $\dfrac{a^3+b^3+c^3}{b^3+c^3+d^3}=\dfrac{a}{d}$.

40. $\left(\dfrac{a-b}{b-c}\right)^3=\dfrac{a}{d}$.

41. $a-2b+c=\dfrac{(a-b)^2}{a}=\dfrac{(b-c)^2}{c}$.

42. $\dfrac{(a+b+c)^2}{a^2+b^2+c^2}=\dfrac{a+b+c}{a-b+c}$.

43. $(b-c)^2=(a-b)(c-d)$.

44. $(a+b+c)(a-b+c)=a^2+b^2+c^2$.

45. $(a-d)^2=(b-c)^2+(c-a)^2+(d-b)^2$.

46. $(a^2+b^2+c^2)(b^2+c^2+d^2)=(ab+bc+cd)^2$.

47. a^2+b^2, b^2+c^2 and c^2+d^2 are in continued proportion.

48. If a, b, c are in continued proportion, find the simplest value of $\dfrac{abc(a+b+c)^3}{(ab+ac+bc)^3}$.

49. If a, b, c are in continued proportion and if $a(b-c)=2b$, prove that $a-c=2\left(\dfrac{a+b}{a}\right)$.

6. **Theorem.** If $\dfrac{a}{b}=\dfrac{c}{d}=\dfrac{e}{f}=\ldots$then each of the ratios $=\dfrac{pa+qc+re+}{pb+qd+rf+}\ldots$, where $p, q, r\ldots$ are any quantities whatever.

Let $$\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\ldots=k.$$

Then $$a=bk,\ c=dk,\ e=fk\ldots$$

$\therefore$ $$pa=pbk,\ qc=qdk,\ re=rfk\ldots$$

By addition we have

$$pa+qc+re+\ldots=k(pb+qd+rf+\ldots)$$

$\therefore$ $$\frac{pa+qc+re+}{pb+qd+rf+\ldots}=k=\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\ldots$$

Cor. 1. Putting $p=q=r=\ldots=1$, we have

$$\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\ldots=\frac{a+c+e+\ldots}{b+d+f+\ldots}.$$

Thus, *when several fractions are equal, each of them is equal to the sum of all the numerators divided by the sum of all the denominators.*

Cor. 2. Putting $p=1$ and $q=-1$, we have

$$\frac{a}{b}=\frac{c}{d}=\ldots=\frac{a-c}{b-d}.$$

Thus, *when two fractions are equal, each is equal to the first numerator minus the second numerator divided by the first denominator minus the second denominator.*

EXERCISE 62.

If $a : b = c : d = e : f$, then

1. $\dfrac{a^2+c^2+e^2}{b^2+d^2+f^2}=\dfrac{a^2}{b^2}$.

2. $\dfrac{a^3-2c^3+3e^3}{b^3-2d^3+3f^3}=\dfrac{a^3}{b^3}$.

3. $\dfrac{3a^2-4c^2+5e^2}{3b^2-4d^2+5f^2}=\dfrac{ac}{bd}$.

4. $\dfrac{3a+4c+2e}{3b+4d+2f}=\dfrac{2a-3c-4e}{2b-3d-4f}$.

5. $\dfrac{a^3+c^3+e^3}{b^3+d^3+f^3}=\dfrac{a^2c+c^2e+e^2a}{b^2d+d^2f+f^2b}$.

6. $(a^2+c^2+e^2)(b^2+d^2+f^2)=(ab+cd+ef)^2$.

7 If $\dfrac{4x+5}{6x+7}=\dfrac{4x+6}{6x+4}$, then each $=-\dfrac{1}{3}$.

8. If $\dfrac{1-4x}{1+6x}=\dfrac{3+2x}{2-3x}$, then each $=\dfrac{7}{5}$.

9. If $\dfrac{x^2+3x+2}{x^2+7x+12}=\dfrac{x+3}{x+7}$, then each $=\dfrac{1}{6}$.

10. If $\left(\dfrac{2x+3}{2x+5}\right)^2=\dfrac{x+3}{x+5}$, then $\dfrac{9}{x+3}=\dfrac{25}{x+5}$.

Fill up the blanks in the following:

11. $\dfrac{x}{8}=\dfrac{y}{3}=\dfrac{x+y}{\quad}=\dfrac{x-y}{\quad}=\dfrac{15x-12y}{\quad}$.

12. $\dfrac{x}{7}=\dfrac{y}{3}=\dfrac{\quad}{10}=\dfrac{\quad}{4}=\dfrac{\quad}{36}$.

13. If $\dfrac{x}{a}=\dfrac{y}{b}=\dfrac{z}{c}$, shew that $\left(\dfrac{ax+by+cz}{a^2+b^2+c^2}\right)^3=\dfrac{xyz}{abc}$.

14. If a, b, c, d are in proportion, prove that $ab+cd$ is the mean proportional between a^2+c^2 and b^2+d^2; and if a, b, c, d are in continued proportion, prove that $b+c$ is the mean proportional between $a+b$ and $c+d$.

7. A few important Examples.

Example 1. If $\frac{x}{b+c-a}=\frac{y}{c+a-b}=\frac{z}{a+b-c}$, find the value of $(b-c)x+(c-a)y+(a-b)z$.

Let $\frac{x}{b+c-a}=\frac{y}{c+a-b}=\frac{z}{a+b-c}=k.$

Then
$$x=(b+c-a)k$$
$$y=(c+a-b)k$$
$$z=(a+b-c)k$$

$\therefore$ the given expression

$=(b-c)(b+c-a)k+(c-a)(c+a-b)k+(a-b)(a+b-c)k$

$=k\{b^2-c^2-a(b-c)+c^2-a^2-b(c-a)+a^2-b^2-c(a-b)\}$

$=k\times 0=0.$

Example 2. If $\frac{a}{b+c}=\frac{b}{c+a}=\frac{c}{a+b}$, then each $=\frac{1}{2}$ or -1.

Each ratio $=\dfrac{\text{sum of the numerators}}{\text{sum of the denominators}}$

$=\dfrac{a+b+c}{2a+2b+2c}$

$=\dfrac{a+b+c}{2(a+b+c)}$

$=\frac{1}{2}$, if $a+b+c$ *is not equal to* 0.

When $a+b+c=0$, then $b+c=-a$

$\therefore$ each ratio $=\dfrac{a}{b+c}=\dfrac{a}{-a}=-1.$

Example 3. If $\frac{x}{b+c}=\frac{y}{c+a}=\frac{z}{a+b}$,

then $$\frac{a}{y+z-x}=\frac{b}{z+x-y}=\frac{c}{x+y-z}.$$

Let $$\frac{x}{b+c}=\frac{y}{c+a}=\frac{z}{a+b}=k.$$

Then $x=(b+c)k$, $y=(c+a)k$ and $z=(a+b)k$.

$\therefore$ $y+z-x=(c+a)k+(a+b)k-(b+c)k$

$=k(c+a+a+b-b-c)$

$=2ka$

$\therefore$ $\dfrac{a}{y+z-x}=\dfrac{1}{2k}.$

Similarly, $\dfrac{b}{z+x-y}=\dfrac{1}{2k}$ and $\dfrac{c}{x+y-z}=\dfrac{1}{2k}$

$$\therefore \quad \frac{a}{y+z-x}=\frac{b}{z+x-y}=\frac{c}{x+y-z}.$$

Example 4. If $\dfrac{ay-bx}{c}=\dfrac{cx-az}{b}=\dfrac{bz-cy}{a}$,

shew that $\dfrac{x}{a}=\dfrac{y}{b}=\dfrac{z}{c}$.

Let each of the given ratios $=k$.

Then $ay-bx=ck$, $cx-az=bk$, $bz-cy=ak$

$\therefore \quad c(ay-bx)=c^2k$, $b(cx-az)=b^2k$, $a(bz-cy)=a^2k$.

Adding these results, we get

$$k(a^2+b^2+c^2)=0$$

$$\therefore \quad k=0$$

$$\therefore \quad ay-bx=0 \text{ or } ay=bx, \quad \therefore \quad \frac{x}{a}=\frac{y}{b},$$

also $\quad cx-az=0$ or $cx=az$, $\quad \therefore \quad \dfrac{x}{a}=\dfrac{z}{c}$.

$$\therefore \quad \frac{x}{a}=\frac{y}{b}=\frac{z}{c}.$$

EXERCISE 63.

1. If $\dfrac{x}{a-b}=\dfrac{y}{b-c}=\dfrac{z}{c-a}$, prove that $x+y+z=0$ and $cx+ay+bz=0$.

2. If $\dfrac{x}{bc(b-c)}=\dfrac{y}{ca(c-a)}=\dfrac{z}{ab(a-b)}$, prove that $a(b+c)x+b(c+a)y+c(a+b)z=0$.

3. If $\dfrac{a}{b+c-a}=\dfrac{b}{c+a-b}=\dfrac{c}{a+b-c}$, prove that each ratio $=1$.

4. If $\dfrac{a}{b+c}=\dfrac{b}{c+a}=\dfrac{c}{a+b}$ and $a+b+c$ be not equal to zero, then $a=b=c$.

*5. Assuming that $\dfrac{a+b-c}{a+b}=\dfrac{b+c-a}{b+c}=\dfrac{c+a-b}{c+a}$ and $a+b+c$ is not zero, shew that $a=b=c$.

*6. If $\dfrac{a+b}{b+c}=\dfrac{c+d}{d+a}$, prove that either $a=c$ or $a+b+c+d$ $=0$.

*7. If $\dfrac{x}{a+2b+2c}=\dfrac{y}{b+2c+2a}=\dfrac{z}{c+2a+2b}$, shew that each of these ratios $=\dfrac{x+y-z}{a+b+3c}$.

*8. If $\dfrac{x}{ax+by+cz}=\dfrac{y}{bx+cy+az}=\dfrac{z}{cx+ay+bz}$, shew that each of these ratios $=\dfrac{1}{a+b+c}$, provided $x+y+z$ is not equal to zero.

*9. If $(a+b+c)x=(b+c-a)y=(c+a-b)z=(a+b-c)w$ then $\dfrac{1}{y}+\dfrac{1}{z}+\dfrac{1}{w}=\dfrac{1}{x}$.

*10. If $a+c=2b$ and $\dfrac{1}{b}+\dfrac{1}{d}=\dfrac{2}{c}$, shew that $a:b=c:d$.

CHAPTER XIV

FRACTIONAL AND LITERAL EQUATIONS

1. Equations involving fractions which contain the unknown in the denominators.

Example 1. Solve $\frac{1}{x}-\frac{3}{2x}+\frac{5}{3x}-\frac{7}{4x}=\frac{5}{6x}-\frac{1}{24x}-1.$

Multiplying both sides by $24x$, the L.C.M. of the denominators, we have

$$24-36+40-42=20-1-24x.$$

By transposition, we get

$24x=20-1-24+36-40+42$

or $24x=33$

$\therefore \quad x=1\frac{3}{8}.$

Example 2. Solve $\frac{2}{3x-4}-\frac{3}{4x-5}=0.$

Multiplying both sides by $(3x-4)(4x-5)$, the L.C.M. of the denominators, we have

$$2(4x-5)-3(3x-4)=0$$

$$8x-10-9x+12=0$$

or $$-x=-2$$

or $$x=2.$$

Otherwise

By transposition, we have

$$\frac{2}{3x-4}=\frac{3}{4x-5}.$$

Multiplying both sides by $(3x-4)(4x-5)$, the L.C.M. of the denominators, or *multiplying cross-wise*, we have

$$2(4x-5)=3(3x-4)$$

or $$8x-10=9x-12$$

or $$x=2.$$

EXERCISE 64.

Solve the equations:

1. $\frac{1}{2x}+\frac{1}{3x}-2=\frac{4}{3x}-3.$ 2. $\frac{1}{x}+\frac{1}{2x}+\frac{1}{3x}+\frac{1}{4x}=6.$

3. $\frac{\cdot 4}{x}=\cdot 25-\frac{\cdot 65}{x}.$ 4. $\frac{3}{2x}+\frac{4}{5x}+11=\frac{1}{x}+\frac{5}{3x}.$

5. $\frac{a}{bx}-\frac{b}{ax}=a^2-b^2.$ 6. $\frac{1}{5x-11}=\frac{2}{3x-1}.$

7. $\frac{5}{3x+4}=\frac{4}{5(x-3)}.$ 8. $\frac{3}{2x+3}=\frac{4}{3x+2}.$

9. $\frac{5}{3-4x}=\frac{6}{6-5x}.$ 10. $\frac{2}{3x-1}+\frac{3}{2x-1}=0.$

11. $\frac{3}{8x+5}=\frac{5}{6x-47}=0.$ 12. $\frac{a}{a-x}+\frac{b}{b-x}=0.$

13. $\frac{a}{bx+a}+\frac{b}{ax+b}=0.$ 14. $\frac{3}{6x^2+2x+5}=\frac{4}{8x^2+3x+2}.$

15. $\frac{a}{ax^2+bx+1}=\frac{b}{bx^2+ax+1}.$

16. $\frac{2}{4x^3+2x^2+x+1}=\frac{3}{6x^3+3x^2+2x+1}.$

Example 3. Solve $\frac{1}{(x-1)(x-2)}+\frac{1}{(x-1)(x-3)}$

$$=\frac{1}{(x-2)(x-3)}+\frac{1}{(x-3)(x-4)}.$$

By transposition, we have

$$\frac{1}{(x-1)(x-2)}-\frac{1}{(x-2)(x-3)}=\frac{1}{(x-3)(x-4)}-\frac{1}{(x-1)(x-3)}$$

$$\therefore \quad \frac{x-3-x+1}{(x-1)(x-2)(x-3)}=\frac{x-1-x+4}{(x-1)(x-3)(x-4)}$$

$$\therefore \quad \frac{-2}{(x-1)(x-2)(x-3)}=\frac{3}{(x-1)(x-3)(x-4)}$$

Multiplying both sides by $(x-1)(x-3)$, we have

$$\frac{-2}{x-2}=\frac{3}{x-4}.$$

Multiplying cross-wise, we have

$$-2x+8=3x-6$$

$$\therefore \quad 5x=14$$

$$\text{or} \quad x=2\tfrac{4}{5}.$$

NOTE. In the above process, the first step of transposition is taken to remove x from the numerators on both sides, when they are simplified.

Example 4. Solve $\dfrac{10}{2x+1}+\dfrac{3}{3x-2}=\dfrac{2}{2x-3}+\dfrac{15}{3x+2}$

Here, we arrange the terms in such a way as to remove x from the numerator on both sides, when they are simplified.

By transposition, we have

$$\frac{10}{2x+1}-\frac{15}{3x+2}=\frac{2}{2x-3}-\frac{3}{3x-2}$$

$$\therefore \quad \frac{30x+20-30x-15}{(2x+1)(3x+2)}=\frac{6x-4-6x+9}{(2x-3)(3x-2)}$$

$$\therefore \quad \frac{5}{(2x+1)(3x+2)}=\frac{5}{(2x-3)(3x-2)}$$

$$\therefore \quad \frac{1}{(2x+1)(3x+2)}=\frac{1}{(2x-3)(3x-2)}.$$

Multiplying cross-wise, we have

$$(2x-3)(3x-2)=(2x+1)(3x+2)$$

$$\therefore \quad 6x^2-13x+6=6x^2+7x+2$$

$$\therefore \quad 20x=4$$

$$\therefore \quad x=\tfrac{1}{5}.$$

Solve the equations:

17. $\dfrac{1}{x+1}-\dfrac{1}{x+3}=\dfrac{1}{x+2}-\dfrac{1}{x+4}.$

18. $\dfrac{1}{x+2}+\dfrac{1}{x+10}=\dfrac{1}{x+4}+\dfrac{1}{x+8}.$

19. $\dfrac{1}{x}-\dfrac{1}{x-1}+\dfrac{1}{x+3}-\dfrac{1}{x+4}=0.$

20. $\dfrac{1}{x-6}-\dfrac{1}{x-3}=\dfrac{1}{x-5}-\dfrac{1}{x-2}.$

21. $\dfrac{1}{x-10}-\dfrac{1}{x-5}=\dfrac{1}{x-7}-\dfrac{1}{x-2}.$

22. $\dfrac{1}{x^2+7x+12}+\dfrac{1}{x^2+8x+15}=\dfrac{1}{x^2+9x+18}+\dfrac{1}{x^2+9x+20}.$

23. $\dfrac{1}{(x+a)^2-b^2}+\dfrac{1}{(x+b)^2-a^2}=\dfrac{1}{x^2-(a+b)^2}+\dfrac{1}{x^2-(a-b)^2}.$

24. $\dfrac{2}{2x-1}+\dfrac{3}{3x+1}=\dfrac{2}{2x+1}+\dfrac{3}{3x-2}.$

25. $\dfrac{9}{x-4}+\dfrac{3}{x-8}=\dfrac{4}{x-9}+\dfrac{8}{x-3}.$

26. $\dfrac{3}{x-3}-\dfrac{4}{x+9}-\dfrac{5}{x-27}+\dfrac{6}{x-15}=0.$

27. $\dfrac{6}{5x+7}-\dfrac{4}{5x+13}=\dfrac{9}{5x+13}-\dfrac{7}{5x+19}.$

28. $\dfrac{1}{2x-1}+\dfrac{8}{4x-1}-\dfrac{6}{3x-1}-\dfrac{3}{6x-1}=0.$

Example 5. Solve $\dfrac{1}{x+5}+\dfrac{10}{2x-5}=\dfrac{18}{3x-5}.$

Method. Decompose $\dfrac{18}{3x-5}$ into two such fractions that when one of them is combined with $\dfrac{1}{x+5}$ and the other with $\dfrac{10}{2x-5}$ and then simplified, x disappears from the numerators on both sides.

Thus, $\dfrac{1}{x+5}+\dfrac{10}{2x-5}=\dfrac{3+15}{3x-5}=\dfrac{3}{3x-5}+\dfrac{15}{3x-5}.$

Combining $\dfrac{3}{3x-5}$ with $\dfrac{1}{x+5}$ and $\dfrac{15}{3x-5}$ with $\dfrac{10}{2x-5}$,

we have $\dfrac{1}{x+5}-\dfrac{3}{3x-5}=\dfrac{15}{3x-5}-\dfrac{10}{2x-5}$

$$\therefore \quad \frac{3x-5-3x-15}{(x+5)(3x-5)}=\frac{30x-75-30x+50}{(3x-5)(2x-5)}$$

$$\therefore \quad \frac{-20}{(x+5)(3x-5)}=\frac{-25}{(3x-5)(2x-5)}.$$

Multiplying both sides by $\frac{3x-5}{-5}$ we have

$$\therefore \quad \frac{4}{x+5}=\frac{5}{2x-5}.$$

Multiplying cross-wise we have

$$8x-20=5x+25$$

$$\therefore \quad 3x=45$$

$$\therefore \quad x=15.$$

Example 6. Solve $\frac{6}{2x+3}-\frac{5}{5x+2}=\frac{2}{x+6}$.

Method. Decompose $\frac{2}{x+6}$ into two such fractions that when one is combined with $\frac{6}{2x+3}$ and the other with $\frac{5}{5x+2}$ and then simplified, x disappears from the numerators on both sides. Thus,

$$\frac{6}{2x+3}-\frac{5}{5x+2}=\frac{3-1}{x+6}=\frac{3}{x+6}-\frac{1}{x+6}.$$

Combining $\frac{3}{x+6}$ with $\frac{6}{2x+3}$ and $\frac{1}{x+6}$ with $\frac{5}{5x+2}$, we have

$$\frac{6}{2x+3}-\frac{3}{x+6}=\frac{5}{5x+2}-\frac{1}{x+6}$$

$$\therefore \quad \frac{6x+36-6x-9}{(2x+3)(x+6)}=\frac{5x+30-5x-2}{(5x+2)(x+6)}$$

$$\therefore \quad \frac{27}{(2x+3)(x+6)}=\frac{28}{(5x+2)(x+6)}.$$

Multiplying both sides by $x+6$, we have

$$\frac{27}{2x+3}=\frac{28}{5x+2}.$$

Multiplying cross-wise, we have

$$27(5x+2)=28(2x+3)$$

or

$$135x+54=56x+84$$

$$\therefore \quad 79x=30$$

$$\therefore \quad x=\tfrac{30}{79}.$$

Solve the equations :

29. $\dfrac{1}{x+1}+\dfrac{2}{x+2}=\dfrac{3}{x+3}.$

30. $\dfrac{3}{x+1}+\dfrac{4}{x+2}=\dfrac{7}{x+3}.$

31. $\dfrac{2}{2x-5}+\dfrac{1}{x-3}=\dfrac{6}{3x-1}.$

32. $\dfrac{7}{x+2}+\dfrac{3}{x+4}=\dfrac{10}{x-5}.$

33. $\dfrac{1}{x+a}+\dfrac{1}{x+b}=\dfrac{2}{x}.$

34. $\dfrac{a}{x-1}+\dfrac{b}{x-2}=\dfrac{a+b}{x}.$

35. $\dfrac{a}{x-a}+\dfrac{b}{x-b}=\dfrac{a+b}{x-c}.$

36. $\dfrac{a}{x+a}-\dfrac{b}{x+b}=\dfrac{a-b}{x+c}.$

37. $\dfrac{3}{4x+1}+\dfrac{4}{4x+5}=\dfrac{7}{4x+3}.$

38. $\dfrac{6}{3x+15}-\dfrac{1}{x-5}=\dfrac{2}{2x-5}.$

39. $\dfrac{4}{5x+1}+\dfrac{7}{10x+1}=\dfrac{3}{2x+1}.$

40. $\dfrac{6}{5-6x}+\dfrac{13}{6x+19}=\dfrac{7}{6x+7}.$

41. $\dfrac{6}{2x-3}+\dfrac{2}{1-2x}=\dfrac{2}{x-1}.$

42. $\dfrac{10}{5x-9}+\dfrac{14}{2x+9}=\dfrac{9}{x+8}.$

43. $\dfrac{7}{7x+1}+\dfrac{6}{4x+1}=\dfrac{15}{6x+5}.$

44. $\dfrac{b-c}{x+a}+\dfrac{a-b}{x+b}=\dfrac{a-c}{x+c}.$

45. $\dfrac{1}{x+2}+\dfrac{3}{x+3}+\dfrac{5}{x+5}=\dfrac{9}{x+4}.$

[*Hint.* Decompose $\dfrac{9}{x+4}$ into three parts with numerators 1, 3, 5, and then associate these parts with the fractions on the left-hand side so that x may disappear from the numerators after simplification.]

46. $\dfrac{1}{x-6a}+\dfrac{2}{x+3a}+\dfrac{3}{x-2a}=\dfrac{6}{x-a}.$

47. $\dfrac{3}{3x+1}+\dfrac{4}{4x+1}+\dfrac{48}{8x+1}=\dfrac{48}{6x+1}.$

2. Equations involving fractions which contain the unknown both in the numerators and the denominators.

Example 1. Solve $\dfrac{4x+1}{5x+2}=\dfrac{2}{3}.$

Multiplying cross-wise, we have

$$3(4x+1)=2(5x+2)$$

$$\therefore\quad 12x+3=10x+4$$

$$\therefore\quad 2x=1 \text{ and } x=\tfrac{1}{2}.$$

Example 2. Solve $\frac{4x+5}{6x+7}=\frac{2x+3}{3x+2}$.

Multiplying cross-wise, we have

$$(4x+5)(3x+2)=(2x+3)(6x+7)$$

$$\therefore \quad 12x^2+23x+10=12x^2+32x+21$$

$$\therefore \quad -9x=11$$

$$\therefore \quad x=-1\tfrac{2}{9}.$$

EXERCISE 65.

Solve the equations:

1. $\frac{5x+2}{3x-4}=3\frac{2}{5}$.
2. $\frac{7x-3}{2x+5}=1\frac{2}{9}$.
3. $\frac{4x+5}{6x-7}=1\frac{6}{11}$.
4. $\frac{2-3x}{1-4x}+\frac{1}{2}=0$.
5. $\frac{11-5x}{7-4x}=2$.
6. $\frac{ax+b}{bx+a}=\frac{c}{d}$.
7. $\frac{2x+1}{3x+2}=\frac{2x+5}{3x-4}$.
8. $\frac{3x+2}{3x+1}=\frac{4x-3}{4x+1}$.
9. $\frac{3-5x}{7-3x}=\frac{5x+2}{3x+4}$.
10. $\frac{7-12x}{5-2x}=\frac{6x-5}{x-3}$.
11. $\frac{x^2+x+1}{x^2+3x+3}=\frac{x+1}{x+3}$.
12. $\frac{ax+b^2}{ax+c^2}=\frac{x+a}{x+b}$.

Example 3. Solve $\frac{x^2+2x-3}{x^2-9}=3$.

The left-hand side $=\frac{(x+3)(x-1)}{(x+3)(x-3)}=\frac{x-1}{x-3}$

$$\therefore \quad \frac{x-1}{x-3}=3.$$

Multiplying cross-wise, we have

$$3x-9=x-1$$

$$\therefore \quad x=4.$$

Solve the equations:

13. $\frac{x^2+x-6}{x^2-4}=1\frac{1}{5}$.
14. $\frac{x^2+x-12}{x^2-16}=2$.

15. $\dfrac{x^2-7x+12}{x^2-9}=\frac{2}{9}$. 16. $\dfrac{x^2-x-20}{x^2+x-12}=\frac{1}{2}$.

17. $\dfrac{x^2+4x-21}{x^2+3x-28}=1\frac{1}{4}$. 18. $\dfrac{x^2+x-30}{x^2+3x-18}=\frac{1}{2}$.

Example 4. Solve $\dfrac{4x+3}{9}+\dfrac{29-7x}{12-5x}=\dfrac{8x+19}{18}$.

Transposing the terms with numerical denominators 1 one side, we have

$$\frac{29-7x}{12-5x}=\frac{8x+19}{18}-\frac{4x+3}{9}$$
$$=\frac{8x+19-8x-6}{18}$$
$$=\frac{13}{18}.$$

Multiplying cross-wise, we have

$$18(29-7x)=13(12-5x)$$
$$\therefore \quad 522-126x=156-65x$$
$$61x=366 \text{ and } x=6.$$

Solve the equations :

19. $\dfrac{x-3}{4}+\dfrac{x+1}{12x+7}=\dfrac{3x-11}{12}$.

20. $\dfrac{2x-1}{5}+\dfrac{3x+1}{15x+8}=\dfrac{8x-7}{20}$.

21. $\dfrac{x+4}{3}+\dfrac{3x+2}{2x+5}=\dfrac{3x+16}{9}$. 22. $\dfrac{x-a}{b}+\dfrac{2x+a}{x+b}=\dfrac{ax+b^2}{ab}$.

23. $\dfrac{x}{2}+\dfrac{8x-12}{10x-4}=\dfrac{18x+20}{36}$.

24. $\dfrac{91x-21}{56}+\dfrac{24x-93}{35x-138}=\dfrac{13x+9}{8}$.

Example 5. Solve $\dfrac{4x-13}{2x-5}+\dfrac{9x-17}{3x-7}=5$.

Decomposing the terms on the left-hand side into integral and fractional parts, we have

$$\frac{4x-10-3}{2x-5}+\frac{9x-21+4}{3x-7}=5$$

or $$\frac{2(2x-5)-3}{2x-5}+\frac{3(3x-7)+4}{3x-7}=5$$

or $$2-\frac{3}{2x-5}+3+\frac{4}{3x-7}=5$$

$\therefore$ $$-\frac{3}{2x-5}+\frac{4}{3x-7}=0$$

By transposition, we have

$$\frac{4}{3x-7}=\frac{3}{2x-5}.$$

Multiplying cross-wise, we have

$$8x-20=9x-21$$

$\therefore$ $$x=1.$$

Example 6. Solve $\dfrac{6x-1}{3x+4}+\dfrac{25x-40}{5x-6}=\dfrac{7x+9}{x+2}$.

Decomposing all the terms into integral and fractional parts, we have

$$\frac{2(3x+4)-9}{3x+4}+\frac{5(5x-6)-10}{5x-6}-\frac{7(x+2)-5}{x+2}$$

or $$2-\frac{9}{3x+4}+5-\frac{10}{5x-6}=7-\frac{5}{x+2}$$

$$\frac{9}{3x+4}+\frac{10}{5x-6}=\frac{5}{x+2}$$

$$=\frac{3+2}{x+2}$$

$$=\frac{3}{x+2}+\frac{2}{x+2}.$$

By transposition, we have

$$\frac{9}{3x+4}-\frac{3}{x+2}=\frac{2}{x+2}-\frac{10}{5x-6}$$

$\therefore$ $$\frac{9x+18-9x-12}{(3x+4)(x+2)}=\frac{10x-12-10x-20}{(x+2)(5x-6)}$$

$\therefore$ $$\frac{6}{(3x+4)(x+2)}=\frac{-32}{(x+2)(5x-6)}.$$

Multiplying both sides by $\dfrac{x+2}{2}$, we have

$$\frac{3}{3x+4}=\frac{-16}{5x-6}.$$

Multiplying cross-wise, we have

$$15x-18=-48x-64$$

$$\therefore \quad 63x=-46$$

$$\therefore \quad x=-\frac{46}{63}.$$

Solve the equations:

25. $\dfrac{3x+1}{x-1}+\dfrac{2x+3}{x+1}=5.$

26. $\dfrac{8x-7}{2x+1}+\dfrac{6x+1}{3x-2}=6.$

27. $\dfrac{5-2x}{4-x}-\dfrac{3-x}{1-x}=1.$

28. $\dfrac{5-4x}{3-2x}+\dfrac{2-3x}{1-x}=5.$

29. $\dfrac{x^2+5x+6}{x+1}+\dfrac{x^2+3x+5}{x+2}=2x+5.$

[*Hint.* Decompose these fractions by division.]

30. $\dfrac{2x^2-9x-8}{2x+1}+\dfrac{x^2-7x+15}{x-3}=2x-9.$

31. $\dfrac{4x+3}{4x-3}+\dfrac{15x-7}{5x-4}=\dfrac{8x+1}{2x-1}.$

32. $\dfrac{6x+1}{2x-1}+\dfrac{6x+7}{3x-1}=\dfrac{25x+20}{5x-1}.$

33. $\dfrac{x+2}{x+1}+\dfrac{x+3}{x+2}=\dfrac{2x+8}{x+3}.$

34. $\dfrac{15x+11}{3x+4}+\dfrac{12x+5}{4x+3}=\dfrac{8x+44}{x+6}.$

35. $\dfrac{6x-5}{2x-3}-\dfrac{6x-14}{3x-1}=\dfrac{4x+23}{4x-1}.$

36. $\dfrac{42x-37}{6x-1}+\dfrac{20x+13}{5x+12}=\dfrac{11x+76}{x+8}.$

37. $\dfrac{2x+7}{x+2}+\dfrac{3x+13}{x+3}=\dfrac{5x+27}{x+4}.$

38. $\dfrac{4x-7}{4x+5}+\dfrac{15x+11}{5x+7}=\dfrac{12x+1}{3x+4}.$

39. $\dfrac{x^2+4}{x-1}+\dfrac{x^2-x+2}{x-2}=\dfrac{2x^2-4x+3}{x-3}.$

40. $\dfrac{2x^2+x+5}{2x-1}+\dfrac{3x^2-2x+20}{3x+1}=\dfrac{2x^2+6x+10}{x+3}.$

41. $\dfrac{x}{2x-a}+\dfrac{x}{2x-b}=1.$

[*Hint.* Multiply both sides by 2 and proceed as before.]

42. $\dfrac{3x+5}{x+1}=\dfrac{4x+8}{3x+3}+\dfrac{10x+1}{6x+3}.$

***Example 7.** Solve $\dfrac{2x+7}{2x+3}+\dfrac{6x+11}{6x-1}=\dfrac{3x+7}{3x+1}+\dfrac{x+3}{x+1}.$

Decomposing each term into integral and fractional parts, we have

$$\frac{(2x+3)+4}{2x+3}+\frac{(6x-1)+12}{6x-1}=\frac{(3x+1)+6}{3x+1}+\frac{(x+1)+2}{x+1}$$

$$\therefore \left(1+\frac{4}{2x+3}\right)+\left(1+\frac{12}{6x-1}\right)=\left(1+\frac{6}{3x+1}\right)+\left(1+\frac{2}{x+1}\right);$$

$$\therefore \quad \frac{4}{2x+3}+\frac{12}{6x-1}=\frac{6}{3x+1}+\frac{2}{x+1}.$$

Taking two terms on one side and two on the other, in such a way as to remove x from the numerators on both sides when they are simplified, we have

$$\frac{4}{2x+3}-\frac{6}{3x+1}=\frac{2}{x+1}-\frac{12}{6x-1}$$

$$\therefore \quad \frac{12x+4-12x-18}{(2x+3)(3x+1)}=\frac{12x-2-12x-12}{(x+1)(6x-1)}$$

$$\therefore \quad \frac{-14}{(2x+3)(3x+1)}=\frac{-14}{(x+1)(6x-1)}.$$

Dividing both sides by -14 and then multiplying cross-wise, we have

$$\begin{aligned} (x+1)(6x-1)&=(2x+3)(3x+1)\\ \therefore \quad 6x^2+5x-1&=6x^2+11x+3\\ \therefore \quad 6x&=-4\\ \therefore \quad x&=-\tfrac{2}{3}. \end{aligned}$$

Solve the equations:

*43. $\dfrac{x+1}{x+2}-\dfrac{x+2}{x+3}=\dfrac{x+4}{x+5}-\dfrac{x+5}{x+6}.$

*44. $\dfrac{2x-11}{x-2}+\dfrac{x+4}{x-3}=\dfrac{x-5}{x+2}+\dfrac{2x+9}{x+1}.$

*45. $\dfrac{7x-55}{x-8}+\dfrac{2x-17}{x-9}=\dfrac{6x-71}{x-12}+\dfrac{3x-14}{x-5}$.

*46. $\dfrac{x-8}{x-10}-\dfrac{x-5}{x-7}=\dfrac{x-7}{x-9}-\dfrac{x-4}{x-6}$.

*47. $\dfrac{2x-3}{x-4}-\dfrac{2x+1}{x-2}=5\left\{\dfrac{x-2}{x-1}-\dfrac{x-4}{x-3}\right\}$.

*48. $\dfrac{4(x+1)}{2x-3}+\dfrac{2x+1}{x+3}=\dfrac{4(x+2)}{2x-1}+\dfrac{2x+3}{x+4}$.

*49. $\dfrac{x}{x-2}+\dfrac{9-x}{7-x}=\dfrac{x+1}{x-1}+\dfrac{8-x}{6-x}$.

*50. $\dfrac{2x-3}{x-2}+\dfrac{3x-20}{x-7}=\dfrac{x-3}{x-4}+\dfrac{4x-19}{x-5}$.

*51. $\dfrac{2x+11}{x+5}-\dfrac{9x-9}{3x-4}=\dfrac{4x+13}{x+3}-\dfrac{15x-47}{3x-10}$.

*52. $\dfrac{x^2+4x+5}{x+2}+\dfrac{x^2+10x+26}{x+5}=\dfrac{x^2+10x+25}{x+6}+\dfrac{x^2+4x+4}{x+1}$.

*53. $\dfrac{x^2-4x-20}{x-7}+\dfrac{x^2+7x+11}{x+5}=\dfrac{x^2+3x+3}{x+2}+\dfrac{x^2-15}{x-4}$.

*3. **Solution of fractional equations by the application of the fundamental principles of proportion.**

***Example 1.** Solve $\dfrac{4x+5}{6x+7}=\dfrac{2x+3}{3x+2}$.

The equations of this type can be easily solved by multiplying cross-wise, but the method illustrated below is considered to be neater.

Multiplying the numerator and the denominator of the term on the right-hand side by 2 and rewriting the equation,

$$\frac{4x+5}{6x+7}=\frac{4x+6}{6x+4}$$

$$\therefore \text{ each }=\frac{(4x+5)-(4x+6)}{(6x+7)-(6x+4)}=-\tfrac{1}{3}$$

$$\therefore \quad \frac{4x+6}{6x+4} \text{ or } \frac{2x+3}{3x+2}=-\tfrac{1}{3}$$

or $\quad 6x+9=-3x-2$

or $\quad 9x=-11$

$\therefore \quad x=-1\tfrac{2}{9}$.

***Example 2.** Solve $\frac{1-4x}{1+6x}=\frac{3+2x}{2-3x}$.

Multiplying the numerator and the denominator of the term on the right-hand side by 2, and rewriting the equation,

$$\frac{1-4x}{1+6x}=\frac{6+4x}{4-6x}$$

$$\therefore \quad \text{each}=\frac{(1-4x)+(6+4x)}{(1+6x)+(4-6x)}=\frac{7}{5}$$

$$\therefore \quad \frac{1-4x}{1+6x}=\frac{7}{5}$$

or $$7+42x=5-20x$$

$$\therefore \quad 62x=-2$$

$$\therefore \quad x=-\frac{1}{31}.$$

EXERCISE 66.

Solve the equations:

*1. $\frac{6x+1}{4x+3}=\frac{3x-5}{2x-7}$. *2. $\frac{9x-5}{6x-1}=\frac{3x-2}{2x-3}$.

*3. $\frac{8x-7}{12x+5}=\frac{2x-3}{3x+4}$. *4. $\frac{3-4x}{1+6x}=\frac{1+2x}{2-3x}$.

*5. $\frac{2+3x}{3-5x}=\frac{1-9x}{4+15x}$. *6. $\frac{1+2x}{2-3x}=\frac{5-6x}{1+9x}$.

*7. $\frac{x+p}{x+q}=\frac{px+q^2}{px+r^2}$. *8. $\frac{3+bx}{2+abx}=\frac{1-x}{1-ax}$.

***Example 3.** Solve $\frac{(x+1)(x+2)}{(x+3)(x+4)}=\frac{x+3}{x+7}$

Removing the brackets, we have

$$\frac{x^2+3x+2}{x^2+7x+12}=\frac{x+3}{x+7}.$$

Multiplying the numerator and the denominator of the term on the right-hand side by x, we have

$$\frac{x^2+3x+2}{x^2+7x+12}=\frac{x^2+3x}{x^2+7x}$$

$$\therefore \quad \text{each}=\frac{(x^2+3x+2)-(x^2+3x)}{(x^2+7x+12)-(x^2+7x)}=\frac{2}{12}=\frac{1}{6}.$$

$$\therefore \quad \frac{x+3}{x+7}=\frac{1}{6}$$

$$\therefore \quad 6x+18=x+7$$

$$\therefore \quad 5x=-11$$

$$\therefore \quad x=-2\tfrac{1}{5}.$$

Otherwise

By *alternando*, we have

$$\frac{x^2+3x+2}{x+3}=\frac{x^2+7x+12}{x+7}.$$

Decomposing each fraction into integral and fractional parts, we have

$$x+\frac{2}{x+3}=x+\frac{12}{x+7}$$

$$\therefore \quad \frac{2}{x+3}=\frac{12}{x+7}$$

$$\frac{1}{x+3}=\frac{6}{x+7}$$

$$\therefore \quad 6x+18=x+7$$

$$\therefore \quad x=-2\tfrac{1}{5}.$$

Solve the following equations by both the methods illustrated above :

*9. $\dfrac{(x+2)(x+3)}{(x+4)(x+5)}=\dfrac{x+5}{x+9}$. *10. $\dfrac{(x+3)(x+4)}{(x+5)(x+6)}=\dfrac{x+7}{x+11}$.

*11. $\dfrac{x^2+6x+8}{x^2+8x+15}=\dfrac{x+6}{x+8}$. *12. $\dfrac{x^2+7x+10}{x^2+10x+24}=\dfrac{x+7}{x+10}$.

*13. $\dfrac{(x+3)(x+5)}{(x+4)(x+1)}=\dfrac{x+8}{x+5}$. *14. $\dfrac{2-5x-6x^2}{2-2x-3x^2}=\dfrac{6x+5}{3x+2}$.

***Example 4.** Solve $\left(\dfrac{2x+3}{2x+5}\right)^2=\dfrac{x+3}{x+5}$.

By *alternando*, we have

$$\frac{(2x+3)^2}{x+3}=\frac{(2x+5)^2}{x+5}$$

or

$$\frac{4x^2+12x+9}{x+3}=\frac{4x^2+20x+25}{x+5}$$

Decomposing each fraction into integral and fractional parts, we have

$$4x+\frac{9}{x+3}=4x+\frac{25}{x+5}$$

$$\therefore \qquad \frac{9}{x+3}=\frac{25}{x+5}.$$

Multiplying cross-wise, we have

$$25x+75=9x+45$$

$$\therefore \qquad 16x=-30$$

$$\therefore \qquad x=-1\tfrac{7}{8}.$$

Solve the equations:

*15. $\left(\frac{x+2}{x+3}\right)^2=\frac{x+4}{x+6}.$ *16. $\left(\frac{x+3}{x+5}\right)^2=\frac{x+6}{x+10}.$

*17. $\left(\frac{x-4}{x-5}\right)^2=\frac{x-8}{x-10}.$ *18. $\left(\frac{2x-3}{2x-5}\right)^2=\frac{x-3}{x-5}.$

*19. $\left(\frac{3x-5}{3x-2}\right)^2=\frac{3x-10}{3x-4}.$ *20. $\left(\frac{x+3}{x+5}\right)^3=\frac{x+1}{x+7}.$

*21. $\left(\frac{ax-b}{ax-c}\right)^2=\frac{ax-2b}{ax-2c}.$ *22. $\left(\frac{ax-b}{ax+c}\right)^3=\frac{ax-2b-c}{ax+2c+b}.$

***Example 5.** Solve $(x^2+3x-7)(x^2+2x-5)$
$=(x^2-3x+7)(x^2-2x+5).$

Dividing both sides of the equation by $(x^2-3x+7)(x^2+2x-5)$, we have

$$\frac{x^2+3x-7}{x^2-3x+7}=\frac{x^2-2x+5}{x^2+2x-5}.$$

By componendo and dividendo, we have

$$\frac{(x^2+3x-7)+(x^2-3x+7)}{(x^2+3x-7)-(x^2-3x+7)}=\frac{(x^2-2x+5)+(x^2+2x-5)}{(x^2-2x+5)-(x^2+2x-5)}.$$

or

$$\frac{2x^2}{6x-14}=\frac{2x^2}{-4x+10}.$$

Dividing both sides by $2x^2$ and multiplying cross-wise, we have

$$-4x+10=6x-14$$

or

$$-10x=-24$$

$$\therefore \qquad x=\tfrac{12}{5}=2\tfrac{2}{5}.$$

***Example 6.** Solve $16\left(\frac{1-x}{1+x}\right)^3=\frac{1+x}{1-x}$.

Multiplying both sides by $\frac{1-x}{1+x}$, we get

$$16\left(\frac{1-x}{1+x}\right)^4=1.$$

Dividing both sides by 16, we get

$$\left(\frac{1-x}{1+x}\right)^4=\frac{1}{16}=\left(\frac{1}{2}\right)^4$$

$$\therefore \qquad \frac{1-x}{1+x}=\tfrac{1}{2}.$$

From this, we get $\qquad x=\tfrac{1}{3}$.

Solve the equations:

*23. $\frac{x^2-2x+3}{x^2+2x-3}=\frac{x+5}{x-5}$.

*24. $\frac{x^2-3x+4}{3x-4}=\frac{x^2-2x+5}{2x-5}$.

*25. $\frac{x^2+2x-5}{x^2-2x+5}=\frac{x^2-x+3}{x^2+x-3}$.

*26. $(x^2+5x+9)(x^2+3x-7)=(x^2-3x+7)(x^2-5x-9)$.

*27. $(x^2-7x+15)(3x-5)=(x^2-3x+5)(7x-15)$.

*28. $\frac{x^2-2x+5}{3x^2+4x-1}=\frac{x^2+2x-5}{3x^2-4x+1}$.

*29. $81\left(\frac{2-x}{2+x}\right)^3=\left(\frac{2+x}{2-x}\right)$.

*30. $\frac{x^3+3x}{3x^2+1}=\frac{76}{49}$.

4. Literal Equations. The methods *generally* employed in the solution of literal equations are similar to those used in ordinary equations. However, a few typical examples are added below.

Example 1. Solve $a(x-a)+b(x-b)=2ab$.

Removing the brackets, we get

$$ax-a^2+bx-b^2=2ab.$$

By transposition, $ax+bx=a^2+b^2+2ab$

or $$x(a+b)=(a+b)^2$$

$$\therefore \quad x=\frac{(a+b)^2}{a+b}$$

$$=a+b.$$

Example 2. Solve $(x+a)^3+(x+b)^3+(x+c)^3$
$$=3(x+a)(x+b)(x+c).$$

By transposition,

$$(x+a)^3+(x+b)^3+(x+c)^3-3(x+a)(x+b)(x+c)=0$$

Factorising the left-hand side, we get

$$\tfrac{1}{2}\{(x+a)+(x+b)+(x+c)\}\ [\{(x+a)-(x+b)\}^2+\{(x+a)-(x+c)\}^2+\{(x+b)-(x+c)\}^2]=0.$$

On simplification, we get

$$\tfrac{1}{2}(3x+a+b+c)\{(a-b)^2+(a-c)^2+(b-c)^2\}=0.$$

Either the first factor $=0$ or the second $=0$,

but the second being a *constant* cannot be equal to zero

Hence $(3x+a+b+c)=0$

$$\therefore \quad x=-\frac{a+b+c}{3}.$$

Example 3. Solve $\dfrac{x+a}{b+c}+\dfrac{x+b}{c+a}+\dfrac{x+c}{a+b}+3=0.$

Decomposing 3 into $1+1+1$ and associating *one* with each fraction, we have

$$\left(\frac{x+a}{b+c}+1\right)+\left(\frac{x+b}{c+a}+1\right)+\left(\frac{x+c}{a+b}+1\right)=0$$

or $$\frac{x+a+b+c}{b+c}+\frac{x+a+b+c}{c+a}+\frac{x+a+b+c}{a+b}=0$$

or $$(x+a+b+c)\left\{\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right\}=0.$$

Since the second factor is constant,

$$\therefore \quad x+(a+b+c)=0$$

$$\therefore \quad x=-(a+b+c).$$

***Example 4.** Solve $(x-2a)^3+(x-2b)^3=2(x-a-b)^3$.

Putting p for $(x-2a)$, q for $(x-2b)$ and r for $(x-a-b)$,

we get $\quad p^3+q^3=2r^3$

or $\quad p^3-r^3=r^3-q^3$

or $\quad (p-r)(p^2+pr+r^2)=(r-q)(r^2+rq+q^2)$.

But $p-r=r-q$, for each $=b-a$ ... ... (i)

$$\therefore \quad p^2+pr+r^2=r^2+rq+q^2$$

$$\text{or} \quad p^2+pr=rq+q^2$$

$$\therefore \quad p^2-q^2=rq-pr$$

$$\text{or} \quad (p+q)(p-q)=-r(p-q)$$

$$\therefore \quad p+q=-r. \qquad \text{... ... (ii)}$$

Substituting the values of p, q and r in (ii), we get

$$(x-2a)+(x-2b)=-(x-a-b)$$

whence $\quad 3x=3a+3b$

or $\quad x=a+b$.

***Example 5.** Solve $\dfrac{mx+b}{x+a}+\dfrac{nx+a}{x+b}=m+n$.

By transposition, $\left(\dfrac{mx+b}{x+b}-m\right)=\left(n-\dfrac{nx+a}{x+a}\right)$

$$\text{or} \frac{mx+b-mx-mb}{x+b}=\frac{nx+na-nx-a}{x+a}$$

$$\text{or} \quad \frac{b-mb}{x+b}=\frac{na-a}{x+a}.$$

Multiplying cross-wise, we have

$$(x+a)(b-mb)=(x+b)(na-a)$$

or $xb-xmb+ab-abm$

$$=xna-xa+abn-ab.$$

By transposition,

$$xb-xmb-xna+xa=abn-ab-ab+abm$$

$$\text{or} \quad x(b-mb-na+a)=ab(n-2+m)$$

$$\therefore \quad x=\frac{ab(m+n-2)}{a+b-mb-na}.$$

EXERCISE 67.

Solve the equations:

1. $a^2(x-a)+b^2(x-b)=abx.$

2. $b(x-2a)+a(x-2b)=(a-b)^2.$

3. $x(x-a)+x(x-b)=2(x-a)(x-b).$

4. $(x-a)^2-(x-b)^2=0.$

5. $(x-a)^2+(x-b)^2=2(x-a-b)^2.$

6. $\dfrac{a-x}{a}+\dfrac{2a-x}{2a}=\dfrac{3a-x}{3a}.$

7. $\dfrac{a}{bx}+\dfrac{b}{ax}=a^2+b^2.$

8. $\dfrac{c-x^2}{bx}-\dfrac{b-x}{c}=\dfrac{c-x}{b}-\dfrac{b-x^2}{cx}.$

9. $\dfrac{bx}{a}+\dfrac{ax}{b}+\dfrac{c^2x}{ab}=a^2+b^2+c^2.$

10. $(x-a)^3+(x-b)^3+(x-c)^3=3(x-a)(x-b)(x-c).$

11. $(x-a)^3+(b-x)^3+(a-b)^3=0.$

12. $\dfrac{x-b}{a+1}+\dfrac{x-a}{b+1}+\dfrac{x-1}{a+b}=3.$

*13. $\dfrac{x-ab}{b+a}+\dfrac{x-bc}{b+c}+\dfrac{x-ca}{c+a}=a+b+c.$

*14. $\dfrac{bc(ax-1)}{b+c}+\dfrac{ca(bx-1)}{c+a}+\dfrac{ab(cx-1)}{a+b}=a+b+c.$

*15. $\dfrac{x-a^2}{b+c}+\dfrac{x-ab}{c+a}+\dfrac{x-ac}{a+b}=3a.$

*16. $\dfrac{x-a^3}{b^2-bc+c^2}+\dfrac{x-b^3}{c^2-ca+a^2}+\dfrac{x-c^3}{a^2-ab+b^2}=2(a+b+c).$

[*Hint.* $2(a+b+c)=(a+b)+(b+c)+(c+a)$.]

*17. $\dfrac{a+b}{x-a-b}=\dfrac{a}{x-a}+\dfrac{b}{x-b}$.

*18. $\dfrac{1}{x+a}+\dfrac{1}{x+b}=\dfrac{1}{x+a+b}+\dfrac{1}{x}$.

*19. $\dfrac{ax}{ax-1}+\dfrac{bx}{bx-1}=2$.

*20. $\dfrac{1}{ab-ax}+\dfrac{1}{bc-bx}=\dfrac{1}{ac-ax}$.

*21. $\dfrac{a}{ax+1}+\dfrac{b}{bx+1}=\dfrac{2c}{cx+1}$.

*22. $\dfrac{1}{x+a}+\dfrac{1}{x-a}=\dfrac{2}{x^2-a^2}$.

*23. $\dfrac{ax}{x-b}+\dfrac{bx}{x-a}=a+b$.

*24. $\dfrac{x+2a}{x-2a}-\dfrac{x-2b}{x+2b}=\dfrac{4(a+b)}{x}$.

*25. $\dfrac{ax^2+bx+c}{px^2+qx+r}=\dfrac{ax+b}{px+q}$.

*26. $(x+a+b-c)^3+(x+b+c-a)^3=2(x+b)^3$

*27. $(x-27)^3+(x-23)^3=2(x-25)^3$

*28. $\left(\dfrac{x+a}{x+b}\right)^3=\dfrac{x+2a-b}{x+2b-a}$.

*29. $16\left(\dfrac{a-x}{a+x}\right)^3=\dfrac{a+x}{a-x}$.

CHAPTER XV

SIMULTANEOUS EQUATIONS *(Continued.)*

1. Cross-multiplication Method.

Theorem. If $ax+by+cz=0$ ... ... (i)

and $a'x+b'y+c'z=0$ ... ... (ii)

then $\dfrac{x}{bc'-b'c}=\dfrac{y}{ca'-c'a}=\dfrac{z}{ab'-a'b}$.

Proof. Multiplying (i) by c' and (ii) by c we get

$ac'x+bc'y+cc'z=0$ ... ... (iii)

$a'cx+b'cy+cc'z=0$ ... ... (iv)

Subtracting (iv) from (iii), we have

$$(ac'-a'c)x+(bc'-b'c)y=0$$

$\therefore \quad (bc'-b'c)y=-(ac'-a'c)x=(ca'-c'a)x.$

Dividing both sides by $(bc'-b'c)(ca'-c'a)$, we have

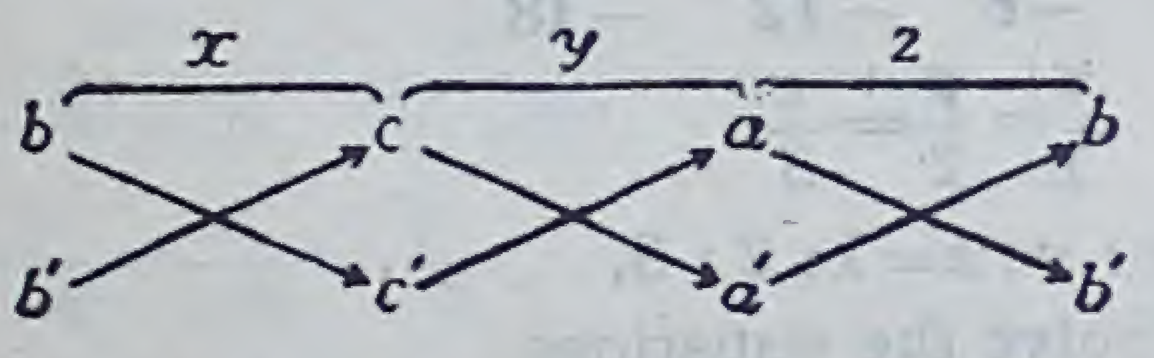
$$\frac{y}{ca'-c'a}=\frac{x}{bc'-b'c}$$

Similarly, $\dfrac{y}{ca'-c'a}=\dfrac{z}{ab'-a'b}$.

$\therefore \quad \dfrac{x}{bc'-b'c}=\dfrac{y}{ca'-c'a}=\dfrac{z}{ab'-a'b}$ ... ... (v)

This theorem is very useful. The rule of writing down the denominators under x, y and z may be thoroughly grasped.

Rule. **Write down in two rows, as given below, the co-efficients of y, z, x, y in order, from the original equations after they have been reduced to the standard**



form, and multiply the co-efficients diagonally as indicated by the arrows and put down

(i) $bc'-b'c$ as the denominator under x,

(ii) $ca'-c'a$ as the denominator under y,

(iii) $ab'-a'b$ as the denominator under z.

This is known as the rule of **Cross-multiplication.**

Cor. If $z=1$, the equations (i) and (ii) become

$$ax+by+c=0$$

and $$a'x+b'y+c'=0$$

and (v) becomes

$$\frac{x}{bc'-b'c}=\frac{y}{ca'-c'a}=\frac{1}{ab'-a'b}.$$

NOTE. The original theorem is useful in solving simultaneous equations of the first degree, involving three variables, provided two of them are of the form $ax+by+cz=0$, and the corollary gives us an elegant method of solving simultaneous equations of the first degree, involving two variables. The following examples are illustrative :

Example 1. Find $x:y:z$, if

$$4x-5y+2z=0 \quad \cdots \quad \cdots \quad \text{(i)}$$

$$2x-7y+4z=0 \quad \cdots \quad \cdots \quad \text{(ii)}$$

Arranging the co-efficients of y, z, x, y, we have

$$\begin{array}{cccc} -5 & 2 & 4 & -5 \\ -7 & 4 & 2 & -7, \end{array}$$

whence, by the rule of cross-multiplication, we obtain

$$\frac{x}{(-5)(4)-(-7)(2)}=\frac{y}{(2)(2)-(4)(4)}=\frac{z}{(4)(-7)-(2)(-5)}$$

or $$\frac{x}{-20+14}=\frac{y}{4-16}=\frac{z}{-28+10}$$

or $$\frac{x}{-6}=\frac{y}{-12}=\frac{z}{-18}$$

or $$\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$$

$\therefore$ $$x:y:z=1:2:3.$$

Example 2. Solve the equations

$$2x-3y+3z=0 \quad \cdots \quad \cdots \quad \text{(i)}$$

$$x-4y+5z=0 \quad \cdots \quad \cdots \quad \text{(ii)}$$

$$x+2y+3z=32 \quad \cdots \quad \cdots \quad \text{(iii)}$$

From (i) and (ii), by the rule of cross-multiplication,

$$\frac{x}{(-3)(5)-(-4)(3)}=\frac{y}{(3)(1)-(5)(2)}=\frac{z}{(2)(-4)-(1)(-3)}$$

or $$\frac{x}{-15+12}=\frac{y}{3-10}=\frac{z}{-8+3}$$

or $$\frac{x}{-3}=\frac{y}{-7}=\frac{z}{-5}$$

or $$\frac{x}{3}=\frac{y}{7}=\frac{z}{5}.$$

Let $$\frac{x}{3}=\frac{y}{7}=\frac{z}{5}=k.$$

$\therefore$ $$x=3k,\ y=7k,\ z=5k.$$

Substituting these values of x, y, z in (iii), we get

$$3k+14k+15k=32$$

or $$32k=32$$

or $$k=1$$

$\therefore$ $x=3,\quad y=7,\quad z=5.$

Example 3. Solve the equations

$$4x-5y+8=0 \quad \ldots \quad \ldots \quad \text{(i)}$$

$$2x-3y+6=0 \quad \ldots \quad \ldots \quad \text{(ii)}$$

From (i) and (ii), by rule of cross-multiplication,

$$\frac{x}{(-5)\times 6-(-3)\times 8}=\frac{y}{8\times 2-6\times 4}=\frac{1}{4\times(-3)-2\times(-5)}$$

or $$\frac{x}{-30+24}=\frac{y}{16-24}=\frac{1}{-12+10}$$

or $$\frac{x}{-6}=\frac{y}{-8}=\frac{1}{-2}$$

or $$\frac{x}{3}=\frac{y}{4}=1$$

$\therefore$ $x=3,\ y=4.$

EXERCISE 68.

Find $x : y : z$, if

1. $\left.\begin{aligned} x-2y+z&=0 \\ 9x-8y+3z&=0. \end{aligned}\right\}$

2. $\left.\begin{aligned} 6x-8y+7z&=0 \\ 2x-7y+11z&=0. \end{aligned}\right\}$

3. $\left.\begin{aligned} 4x-13y+8z&=0 \\ 7x+6y-9z&=0. \end{aligned}\right\}$

4. $\left.\begin{aligned} 2x-3y+4z&=0 \\ 7x+2y-6z&=0. \end{aligned}\right\}$

5. $\left.\begin{aligned} ax+by+cz&=0 \\ bx+cy+az&=0. \end{aligned}\right\}$

Solve the equations by the method of Cross-multiplication :

6. $\left.\begin{aligned} 4x-2y-z&=0 \\ 6x-y-4z&=0 \\ x+y+z&=25. \end{aligned}\right\}$

7. $\left.\begin{aligned} 5x+6y+8z&=0 \\ 3x+4y+6z&=0 \\ x+5y+16z&=3. \end{aligned}\right\}$

8. $\left.\begin{aligned} x+2y+z&=0 \\ 2x-5y-4z&=0 \\ 4x+5y+6z&=24. \end{aligned}\right\}$

9. $\left.\begin{aligned} 4(x+y)&=3(2z-y) \\ 5(x-2y)&=3(2y-3z) \\ 3(x-2)+5(y-3)+7(z-4)&=53. \end{aligned}\right\}$

10. $\left.\begin{aligned} 15x=10y&=6z \\ 3x+4y+5z&=172. \end{aligned}\right\}$

11. $\left.\begin{aligned} 4x-13y+8z&=0 \\ 7x+6y-9z&=0 \\ \frac{6}{x}+\frac{8}{y}+\frac{20}{z}&=8. \end{aligned}\right\}$

12. $\left.\begin{aligned} x-2y+z&=0 \\ 2x-5y+4z&=0 \\ 3x^2+4y^2+5z^2&=48. \end{aligned}\right\}$

13. $\left.\begin{aligned} 7x-5y+z&=0 \\ 5x+2y-3z&=0 \\ 2x^3+5y^3-z^3&=15. \end{aligned}\right\}$

14. $\left.\begin{aligned} x+y+z&=0 \\ ax+by+cz&=0 \\ \frac{x}{b-c}+\frac{y}{c-a}+\frac{z}{a-b}&=3. \end{aligned}\right\}$

15. $x-y=y-z=\frac{x+z}{6}=2.$

16. $\left.\begin{aligned} \frac{y+z}{4}=\frac{z+x}{3}&=\frac{x+y}{2}. \\ x+y+z&=27. \end{aligned}\right\}$

17. $\left.\begin{aligned} 3x+4y&=15 \\ 6y-5x&=-6. \end{aligned}\right\}$

18. $\left.\begin{aligned} 7y&=9x \\ 3y-4x+1&=0. \end{aligned}\right\}$

19. $\left.\begin{array}{l}7x+4y=5\\5x+6y=2.\end{array}\right\}$

20. $\dfrac{x+5}{y+2}=\dfrac{5x+3}{4y+1}=2.$

21. $\left.\begin{array}{l}\dfrac{x+y}{2}+\dfrac{3x-5y}{4}=2\\[2ex]\dfrac{x}{14}+\dfrac{y}{18}=1.\end{array}\right\}$

22. $\left.\begin{array}{l}3x+y-2p=0\\-5x+py-3=0.\end{array}\right\}$

23. $\left.\begin{array}{l}lx+my=p\\mx=ly.\end{array}\right\}$

24. $x-ay+b=ax-y+b=1.$

25. $\left.\begin{array}{l}\dfrac{x}{a}+\dfrac{y}{b}=c\\[2ex]\dfrac{x}{b}+\dfrac{y}{a}=d.\end{array}\right\}$

26. If $x+5y=18$ and $3x+2y=41$, find the value of $x-8y$.

27. Find the values of x and y from the equation

$$(3x-2y+8)^2+(5x+13y-3)^2=0.$$

[*Hint.* As the sum of squares $=0$, each $=0$.]

Example 4. Solve the equations

$$\frac{3}{x}+\frac{4}{y}=18 \quad \ldots \quad \ldots \quad \ldots \quad \text{(i)}$$

$$\frac{2}{x}+\frac{5}{y}=19 \quad \ldots \quad \ldots \quad \ldots \quad \text{(ii)}$$

By transposition, we get

$$\frac{3}{x}+\frac{4}{y}-18=0 \quad \ldots \quad \ldots \quad \text{(iii)}$$

$$\frac{2}{x}+\frac{5}{y}-19=0 \quad \ldots \quad \ldots \quad \text{(iv)}$$

By the cross-multiplication method, we get

$$\frac{\frac{1}{x}}{(-76)-(-90)}=\frac{\frac{1}{y}}{(-36)-(-57)}=\frac{1}{15-8}$$

or $\dfrac{\frac{1}{x}}{14}=\dfrac{\frac{1}{y}}{21}=\dfrac{1}{7}$

$\therefore \dfrac{1}{x}=\dfrac{14}{7}=2$ and $\dfrac{1}{y}=\dfrac{21}{7}=3$

$\therefore x=\dfrac{1}{2}$ and $y=\dfrac{1}{3}$.

Example 5. Solve the equations

$$5x+4y=9xy \quad \dots \quad \dots \quad \text{(i)}$$
$$3x+2y=5xy \quad \dots \quad \dots \quad \text{(ii)}$$

Dividing (i) and (ii) by xy, we get

$$\left.\begin{aligned}\frac{5}{y}+\frac{4}{x}=9\\ \frac{3}{y}+\frac{2}{x}=5\end{aligned}\right\}$$

or
$$\frac{4}{x}+\frac{5}{y}-9=0 \quad \dots \quad \dots \quad \text{(iii)}$$
$$\frac{2}{x}+\frac{3}{y}-5=0 \quad \dots \quad \dots \quad \text{(iv)}$$

By the cross-multiplication method, we get

$$\frac{\frac{1}{x}}{-25+27}=\frac{\frac{1}{y}}{-18+20}=\frac{1}{12-10}$$

or
$$\frac{\frac{1}{x}}{2}=\frac{\frac{1}{y}}{2}=\frac{1}{2}$$

$\therefore$
$$\frac{1}{x}=\frac{1}{y}=1$$

$\therefore$ $x=1$ and $y=1$.

Solve the equations by the method of cross-multiplication :

28. $\left.\begin{aligned}\frac{3}{x}+\frac{4}{y}=18\\ \frac{2}{x}+\frac{5}{y}=19.\end{aligned}\right\}$

29. $\left.\begin{aligned}\frac{3}{x}-\frac{2}{y}=-4\\ \frac{2}{x}+\frac{3}{y}=19.\end{aligned}\right\}$

30. $\left.\begin{aligned}\frac{3}{x}-\frac{4}{y}=5\\ \frac{4}{x}-\frac{5}{y}=6.\end{aligned}\right\}$

31. $\left.\begin{aligned}\frac{3}{x}+\frac{5}{y}=11\\ \frac{2}{x}-\frac{3}{y}=-5\tfrac{1}{3}.\end{aligned}\right\}$

32. $\left.\begin{aligned}\frac{1}{3x}+\frac{1}{5y}=2\\ \frac{1}{4x}+\frac{1}{6y}=\frac{19}{12}.\end{aligned}\right\}$

33. $\left.\begin{aligned}2x+\frac{3}{y}=5\\ 5x-\frac{2}{y}=3.\end{aligned}\right\}$

34. $\left.\begin{array}{l}\dfrac{2}{3x}-\dfrac{5}{4y}=\dfrac{1}{12}\\[2mm]\dfrac{3}{4x}-\dfrac{7}{6y}=\dfrac{1}{3}.\end{array}\right\}$

35. $\left.\begin{array}{l}\dfrac{1}{5x}+\dfrac{y}{9}=5\\[2mm]\dfrac{1}{3x}+\dfrac{y}{2}=14.\end{array}\right\}$

36. $\left.\begin{array}{l}\dfrac{a}{x}+\dfrac{b}{y}=m\\[2mm]\dfrac{b}{x}+\dfrac{a}{y}=n.\end{array}\right\}$

37. $\left.\begin{array}{l}\dfrac{2}{ax}+\dfrac{3}{by}=5\\[2mm]\dfrac{5}{ax}-\dfrac{2}{by}=3.\end{array}\right\}$

38. $\left.\begin{array}{l}\dfrac{8}{x+y}+\dfrac{6}{x-y}=25\\[2mm]\dfrac{4}{x+y}-\dfrac{3}{x-y}=10.\end{array}\right\}$

39. $\left.\begin{array}{l}10y+14x=9xy\\4x-6y=7xy.\end{array}\right\}$

40. $\left.\begin{array}{l}2y-x=4xy\\[2mm]\dfrac{4}{y}-\dfrac{3}{x}=9.\end{array}\right\}$

41. $\dfrac{2+5x}{3x}=\dfrac{y+4}{2}=\dfrac{2xy+9x+2}{6x}.$

2. Simultaneous equations with three variables.

Example 1. Solve the equations

$$4x+2y-3z=2 \quad \ldots \quad \text{(i)}$$
$$3x+4y-2z=10 \quad \ldots \quad \text{(ii)}$$
$$2x-5y+4z=5 \quad \ldots \quad \text{(iii)}$$

Multiplying (i) by 5 and then subtracting (ii) from it,

$$17x+6y-13z=0 \quad \ldots \quad \text{(iv)}$$

Multiplying (iii) by 2 and then subtracting it from (ii),

$$-x+14y-10z=0 \quad \ldots \quad \text{(v)}$$

By the cross-multiplication method from (iv) and (v), we get

$$\frac{x}{-60+182}=\frac{y}{13+170}=\frac{z}{238+6}$$

or $$\frac{x}{122}=\frac{y}{183}=\frac{z}{244}$$

or $$\frac{x}{2}=\frac{y}{3}=\frac{z}{4}=k \text{ (Let each be equal to } k.)$$

$\therefore$ $x=2k$, $y=3k$ and $z=4k$.

Substituting these values in (i), we get

$$8k+6k-12k=2$$

or $$k=1$$

$\therefore$ $x=2,\ y=3,\ z=4.$

Example 2. Solve the equations

$$x+5y+6z=3 \quad \dots \quad \dots \quad \text{(i)}$$
$$2x-3y+7z=22 \quad \dots \quad \dots \quad \text{(ii)}$$
$$5x+y+8z=19 \quad \dots \quad \dots \quad \text{(iii)}$$

Multiplying (i) by 2 and then subtracting (ii) from it,

$$13y+5z=-16$$

or $$13y+5z+16=0 \quad \dots \quad \dots \quad \text{(iv)}$$

Multiplying (i) by 5 and then subtracting (iii) from it,

$$24y+22z=-4$$

or $$12y+11z+2=0 \quad \dots \quad \dots \quad \text{(v)}$$

From (iv) and (v), by the method of cross-multiplication,

$$\frac{y}{10-176}=\frac{z}{192-26}=\frac{1}{143-60}$$

or $$\frac{y}{-166}=\frac{z}{166}=\frac{1}{83}$$

$\therefore$ $$y=-2,\ z=2.$$

Substituting these values in (i), we get $x=1$.

EXERCISE 69.

Solve the equations :

1. $\left.\begin{array}{l} x+y+z=1 \\ 2x+3y+z=4 \\ 4x+9y+z=16. \end{array}\right\}$

2. $\left.\begin{array}{l} 4x-3y+2z=8 \\ 3x-4y+5z=6 \\ -6x+5y+7z=-1. \end{array}\right\}$

3. $\left.\begin{array}{l} 3x-2y+4z=5 \\ 5x-7y+8z=8 \\ 11x-10y+16z=21. \end{array}\right\}$

4. $\left.\begin{array}{l} x+5y-4z=5 \\ 3x-2y+2z=14 \\ -10x+8y+z=6. \end{array}\right\}$

5. $x+2y+3z=3x+y+2z=2x+3y+z=6.$

*6. $\left.\begin{array}{l} \frac{1}{x}+\frac{1}{y}+\frac{1}{z}=6 \\ \frac{2}{x}-\frac{3}{y}+\frac{4}{z}=8 \\ \frac{3}{x}-\frac{4}{y}+\frac{5}{z}=10. \end{array}\right\}$

*7. $\left.\begin{array}{l} 3x+\frac{4}{y}+\frac{5}{z}=-13 \\ 4x+\frac{5}{y}+\frac{3}{z}=7. \\ 5x+\frac{3}{y}+\frac{4}{z}=6. \end{array}\right\}$

*8. $\left.\begin{array}{l} x-z=1 \\ 2y+z=11 \\ 2x+3y=23. \end{array}\right\}$

SECTIONAL REVISION IV

TEST PAPERS

PAPER 1

1. Explain the terms :

(i) duplicate ratio and sub-duplicate ratio,
(ii) triplicate ratio and sub-triplicate ratio.

2. Prove that a ratio of greater inequality is increased and a ratio of less inequality is diminished by subtracting the same positive quantity from both the terms.

3. If $\dfrac{6a-5b}{6a+5b}=\dfrac{6c-5d}{6c+5d}$, shew that $\dfrac{a}{b}=\dfrac{c}{d}$.

4. If $\dfrac{a}{b}=\dfrac{c}{d}=\dfrac{e}{f}$, shew that each is equal to $\dfrac{a+c+e}{b+d+f}$.

5. Solve the equations :

(i) $\dfrac{a}{ax+1}+\dfrac{b}{bx+1}=\dfrac{2c}{cx+1}$.

(ii) $9\left(\dfrac{x-a}{x+a}\right)^3=\dfrac{x-a}{x+a}$.

6. The co-ordinates of P and Q are $(-7, 5)$ and $(+8, -4)$ respectively. Find the length of PQ correct to 2 places of decimals.

7. Solve graphically the equations

$$\left.\begin{aligned} x+y&=1 \\ x-y&=5. \end{aligned}\right\}$$

PAPER 2

1. Find the compounded ratio of the following three ratios :

$3a : 4b$, $b^2 : 5ac$ and $5c^2 : 6ab$.

2. If $6+x : 8+x$ be the sub-duplicate ratio of $4 : 9$, find x.

3. If $a : b=c : d$, then $\dfrac{4a+3b}{4a-3b}=\dfrac{4c+3d}{4c-3d}$.

4. If $\frac{x}{a+b-c}=\frac{y}{b+c-a}=\frac{z}{c+a-b}$, shew that each of these fractions $=\frac{x+y+z}{a+b+c}$.

5. Solve the equations

(i) $(x^2+5x-4)(x^2+3x-2)=(x^2-5x+4)(x^2-3x+2)$.

(ii) $\frac{6x+5}{8x+3}=\frac{3x+2}{4x+1}$.

6. Find the equation of a straight line passing through (3, 0) and (–3, 4).

PAPER 3

1. What must be added to the terms of 5 : 8 to make it equal to 9 : 10?

2. If $x : y=5 : 6$, find the value of $\frac{3x+4y}{4x+3y}$.

3. Two vessels contain a mixture of wine and water in the ratio of 5 : 3 and 3 : 1. In what ratio must liquids be drawn from each to give a mixture of wine and water in the ratio of 2 : 1?

4. If $x=\frac{ab}{a+b}$, find the value of $\frac{x+a}{x-a}+\frac{x+b}{x-b}$.

5. Solve the equations:

(i) $\frac{5-6x}{3+4x}=\frac{2+3x}{5-2x}$.

(ii) $\frac{(x+2)(x+3)}{(x+4)(x+5)}=\frac{x+5}{x+9}$.

6. Find the equation of a straight line which is parallel to $5x+4y=3$ and passes through $(6, -4)$.

7. Solve the following equations by the method of cross-multiplication:

$$\left.\begin{aligned} \frac{3}{x}+\frac{2}{y}-2 &=0 \\ \frac{4}{x}+\frac{3}{y}-2\tfrac{5}{6} &=0. \end{aligned}\right\}$$

PAPER 4

1. If $7x-6y=5x+2y$, find the value of $x:y$.

2. The bases of two triangles are in the ratio of 5 : 6 and their altitudes are in the ratio of 8 : 9. Find the ratio between their areas.

3. If $a:b::c:d$, prove that $\frac{a+b}{a-b}=\frac{c+d}{c-d}$.

4. If l, m, n are in continued proportion and if $l(m-n)=2m$, prove that $l-n=2\left(\frac{l+m}{l}\right)$.

5. Solve the equations:

(i) $\left(\frac{2x-3}{2x-5}\right)^2=\frac{x-3}{x-5}$.

(ii) $(x-3)(x+4)(x+7)=(x+2)(x+1)(x+5)$.

6. Draw the graph of the function $\frac{2x-3}{6}$.

7. Solve the equations: $\left.\begin{aligned}\frac{3}{x}+\frac{2}{y}&=\frac{17}{12}\\ \frac{2}{z}-\frac{1}{y}&=\frac{2}{3}\\ \frac{1}{x}+\frac{1}{z}&=\frac{3}{4}.\end{aligned}\right\}$

PAPER 5

1. Prove that $\frac{0}{n}=0, \frac{m}{\infty}=0$ and $\frac{m}{0}=\infty$, where m and n are finite quantities.

2. What must be added to each of the numbers 12, 16, 21 so that the resulting numbers may be in continued proportion?

3. If $\frac{2a^2+3ab+4b^2}{5a^2+6ab+7b^2}=\frac{2c^2+3cd+4d^2}{5c^2+6cd+7d^2}$, prove that $a:b=c:d$.

4. If $\frac{a}{y+z-x}=\frac{b}{z+x-y}=\frac{c}{x+y-z}$, shew that

$$\frac{x}{b+c}=\frac{y}{c+a}=\frac{z}{a+b}.$$

5. Solve the equations:

(i) $81\left(\frac{a-x}{a+x}\right)^3=\frac{a-x}{a+x}$.

(ii) $\frac{x^2-5x+7}{5x-7}=\frac{x^2-3x+5}{3x-5}$.

6. Find the equations of the sides of a triangle whose vertices are $(-5, 4)$, $(6, -2)$ and $(3, 2)$.

PAPER 6

1. Prove that $\frac{0}{0}$ is indeterminate.

2. Find two numbers of the same two digits which are in the ratio of $8 : 3$.

3. If $\frac{pa+qb}{pc+qd}=\frac{b^2c}{d^2a}$, prove that $a : b=c : d$.

4. If $\frac{a}{x}=\frac{b}{y}=\frac{c}{z}$, shew that $\frac{abc}{xyz}=\frac{a^3+b^3+c^3}{x^3+y^3+z^3}$.

5. Solve the equations:

(i) $\frac{x^2-3x+4}{2x^2-5x+1}=\frac{x^2+3x-4}{2x^2+5x-1}$.

(ii) $\frac{x^3+3x}{3x^2+1}=\frac{91}{37}$.

6. Solve graphically the equations $\left.\begin{aligned} y&=2x-1 \\ 6y&=2x-3. \end{aligned}\right\}$

7. The average weights of infants for the first 12 months are given below in lbs.:

Months	1	2	3	4	5	6	7	8	9	10	11	12
Weights	8	$9\frac{1}{2}$	11	$12\frac{1}{2}$	14	$15\frac{1}{2}$	$16\frac{3}{4}$	18	19	20	21	22

Draw the graph and compare the weight of any infant you know.

PAPER 7

1. Find the value of $\frac{x^2+2x-35}{x^2+4x-21}$ when $x=-7$.

If $1+x$, $2+x$, $4+x$ are in continued proportion, find x.

3. If $a : b = c : d$, prove that $(6a-5c)(6b+5d)$
$$=(6a+5c)(6b-5d).$$

4. If $\frac{x}{a}=\frac{y}{b}$, prove that $(x^2+y^2)(a^2+b^2)=(ax+by)^2$.

5. Solve the equations :

(i) $\frac{x^2+x-30}{x^2-25}=2$.

(ii) $\frac{3x-2}{x-1}+\frac{6x-13}{2x-5}=\frac{3x-5}{x-2}+\frac{6x-7}{2x-3}$.

6. Solve graphically and verify the solution of $\left.\begin{aligned}\frac{x+y}{3}&=5\\ 2x-3y&=5.\end{aligned}\right\}$

PAPER 8

1. The soldiers in two armies when met in a battle were in the ratio of 5 : 3, their respective losses were in the ratio of 3 : 2 and their survivors as 69 : 41. If the number of survivors in the smaller army be 16,400, find the original number of soldiers.

2. Find (i) the mean proportional between 16, 49 ;
(ii) the fourth proportional to 12, 15, 18 ;
and (iii) the third proportional to 24, 30.

3. If $2p=m+n$ and $\frac{2}{p}=\frac{1}{x}+\frac{1}{y}$, shew that $m : x = n : y$.

4. If a, b, c, d are in continued proportion, shew that $ab+bc+cd$ is the mean proportional between $a^2+b^2+c^2$ and $b^2+c^2+d^2$.

5. Solve the equations:

(i) $\frac{2x+7}{x+2}+\frac{3x+13}{x+3}=\frac{5x+27}{x+4}$.

(ii) $\frac{x-8}{x-10}-\frac{x-5}{x-7}=\frac{x-7}{x-9}-\frac{x-4}{x-6}$.

6. Find the speed of a train if 90 minutes are saved in 150 miles when the speed is increased by 5 miles an hour.

7. Find the vertices of the triangle formed by the straight lines
$$\left.\begin{aligned}8x+5y+7&=0\\ 3x+7y-23&=0\\ 5x-2y-11&=0.\end{aligned}\right\}$$

CHAPTER XVI

INDICES

1. In a^5, ^{5} is the index and it indicates that a is repeated as a factor 5 times, or $a^5 = a \times a \times a \times a \times a$.

This definition of an index does not obviously involve the idea of direction. An index used hitherto is neither a positive integer, nor a negative one—it is a non-directed integer. Thus a^{+5} or a^{-5} has no meaning. Again, as an index indicates the number of times a factor is repeated, $a^{\frac{1}{5}}$ or $a^{\frac{2}{3}}$ is also meaningless.

In order to include the directed and fractional indices in the domain of Algebra, we have to extend the meaning of an index.

We have already established the following laws of indices from the ordinary meaning of an index :

$$a^m \times a^n = a^{m+n} \quad \ldots \quad \ldots \quad \ldots \quad \text{(I)}$$

$$a^m \div a^n = a^{m-n} \quad \ldots \quad \ldots \quad \ldots \quad \text{(II)}$$

$$(a^m)^n = a^{mn} \quad \ldots \quad \ldots \quad \ldots \quad \text{(III)}$$

Now the problem is to put such interpretation on a^{+m}, a^{-n}, $a^{\frac{m}{n}}$ and a^0 that they may be included in the domain of Algebra.

The extension of Algebraic domain, with a view to include such *mysterious* and *foreign elements*, is possible only if these elements would obey all the rules which constitute the Algebraic government.

Starting with the assumption that these foreign and mysterious elements do accept all the rules of Algebraic government, we lay down the following principles :

(i) that a^{+m} means exactly the same thing as a^m ;

(ii) that $a^{\frac{m}{n}}$, a^0, a^{-n} are subject to the ordinary laws of Algebra.

2. Meaning of $a^{\frac{m}{n}}$.

By the law of indices,

$$a^{\frac{2}{3}} \times a^{\frac{2}{3}} \times a^{\frac{2}{3}} = a^{\frac{2}{3}+\frac{2}{3}+\frac{2}{3}}$$

$$\therefore \quad \left(a^{\frac{2}{3}}\right)^3 = a^{\frac{2}{3}\times 3} = a^2.$$

Taking the cube root of both sides, we have

$$a^{\frac{2}{3}} = \sqrt[3]{a^2}$$

Similarly, $a^{\frac{4}{5}} = \sqrt[5]{a^4}$ and $a^{\frac{7}{11}} = \sqrt[11]{a^7}$.

For further generalisation, we proceed as below.

By the law of indices,

$$a^{\frac{m}{n}} \times a^{\frac{m}{n}} \times a^{\frac{m}{n}} \times \ldots \text{ to } n \text{ factors}$$

$$a^{\frac{m}{n}+\frac{m}{n}+\frac{m}{n}+\ldots\text{to } n \text{ terms}}$$

$$\therefore \quad \left(a^{\frac{m}{n}}\right)^n = a^{\frac{m}{n}\times n} = a^m.$$

Taking the nth root of both sides, we have

$$a^{\frac{m}{n}} = \sqrt[n]{a^m}.$$

Hence $a^{\frac{m}{n}}$ is the n^{th} root of a^m, and in future we shall always take $a^{\frac{m}{n}} = \sqrt[n]{a^m}$

Cor. $a^{\frac{1}{n}} = \sqrt[n]{a^1} = \sqrt[n]{a}.$

Thus, $a^{\frac{1}{2}} = \sqrt{a}$, $a^{\frac{1}{3}} = \sqrt[3]{a}$, $a^{\frac{1}{4}} = \sqrt[4]{a}$, and so on.

Since $a^{\frac{1}{n}} \times a^{\frac{1}{n}} \times a^{\frac{1}{n}} \times \ldots\ldots\ldots m$ factors $= a^{\frac{1}{n}\times m} = a^{\frac{m}{n}}$

and $\quad a^{\frac{1}{n}} = \sqrt[n]{a},$

$$\left(\sqrt[n]{a}\right)^m = a^{\frac{m}{n}}, \text{ or } a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m$$

Thus, $a^{\frac{m}{n}}$ may be interpreted either as the n^{th} root of the mth power of a, or as the mth power of the n^{th} root of a.

3. Meaning of a^0.

By the law of indices,

$$a^0 \times a^m = a^{0+m} = a^m.$$

Dividing both sides by a^m, we have

$$\frac{a^0 \times a^m}{a^m} = \frac{a^m}{a^m}$$

$$\therefore \qquad a^0 = 1.$$

Hence for all finite values of a, $a^0 = 1$.

4. Meaning of a^{-m}.

By the law of indices,

$$a^m \times a^{-m} = a^{m-m} = a^0 = 1.$$

Dividing both sides by a^m, we have

$$a^{-m} = \frac{1}{a^m}$$

Also $$a^m = \frac{1}{a^{-m}}.$$

Cor. $\dfrac{a^{-m}}{b^{-n}} = a^{-m} \times \dfrac{1}{b^{-n}} = \dfrac{1}{a^m} \times b^n = \dfrac{b^n}{a^m}.$

Hence any quantity with a *negative* index denotes the **reciprocal** of the same with the same *positive* index and *vice-versa*.

Example 1. Find the value of $27^{\frac{4}{3}}$.

$$27^{\frac{4}{3}} = \left(\sqrt[3]{27}\right)^4 = 3^4 = 81.$$

Example 2. Find the value of $9^{-\frac{3}{2}}$.

$$9^{-\frac{3}{2}} = \frac{1}{9^{\frac{3}{2}}} = \frac{1}{(\sqrt{9})^3}$$

$$= \frac{1}{3^3} = \frac{1}{27}.$$

Example 3. Multiply together $\sqrt[4]{a^3}$, $a^{\frac{3}{2}}, \frac{1}{a^{-4}}$ and $a^{-\frac{1}{4}}$.

$$\text{The required product} = a^{\frac{3}{4}} \times a^{\frac{3}{2}} \times a^4 \times a^{-\frac{1}{4}}$$
$$= a^{\frac{3}{4}+\frac{3}{2}+4-\frac{1}{4}}$$
$$= a^6.$$

Example 4. Express $\sqrt[4]{a^{-3}} \div (\sqrt[6]{a})^{-5}$ without radical signs and negative indices.

$$\sqrt[4]{a^{-3}} \div \left(\sqrt[6]{a}\right)^{-5} = a^{-\frac{3}{4}} \div a^{-\frac{5}{6}}$$
$$= \frac{1}{a^{\frac{3}{4}}} \div \frac{1}{a^{\frac{5}{6}}}$$
$$= \frac{a^{\frac{5}{6}}}{a^{\frac{3}{4}}} = a^{\frac{5}{6}-\frac{3}{4}}$$
$$= a^{\frac{1}{12}}.$$

EXERCISE 70.

Express the following without fractional and negative indices:

1. $a^{\frac{3}{5}}$. 2. $x^{-\frac{3}{5}}$. 3. $x^{-\frac{2}{3}} \times 6a^{-\frac{1}{2}}$.

4. $9p^{-2} \times p^{-\frac{3}{4}}$. 5. $m^{-\frac{2}{3}} \div a^{-\frac{3}{2}}$. 6. $\frac{4}{x^{-\frac{2}{3}}}$.

7. $\sqrt[5]{m^3} \div \sqrt[5]{x^{-2}}$. 8. $\sqrt[2m]{x^{-7}} \times \sqrt{a^5}$.

9. $\sqrt[4m]{x^{12}} \div \sqrt[2m]{x^{-5}}$.

Express the following without radical signs and negative indices:

10. $(\sqrt[5]{x})^7$. 11. $(\sqrt[6]{m})^{-9}$. 12. $\frac{1}{\sqrt[3]{m^{-2}}}$. 13. $\frac{1}{(\sqrt[3]{a})^{-4}}$.

14. $\sqrt[4]{x^6} \div (\sqrt[6]{x})^{-1}$. 15. $\sqrt[4]{m^{-5}} \div (\sqrt[8]{m})^{-6}$.

Find the value of :

16. $4^{-\frac{1}{2}}$. 17. $8^{\frac{2}{3}}$. 18. $16^{\frac{3}{4}}$. 19. $64^{\frac{2}{3}}$.

20. $32^{-\frac{3}{5}}$. 21. $27^{-\frac{2}{3}}$. 22. $125^{\frac{2}{3}}$. 23. $81^{-\frac{3}{4}}$.

24. $128^{-\frac{3}{7}}$. 25. $343^{-\frac{2}{3}}$. 26. $\sqrt[5]{243^2}$ 27. $\frac{1}{9^{-2}}$.

28. $\left(\frac{1}{216}\right)^{-\frac{2}{3}}$. 29. $\frac{1}{\sqrt{16^{\frac{3}{2}}}}$. 30. $\sqrt[3]{(16^{-\frac{3}{4}})(125^{-2})}$

Simplify:

31. $16^{\frac{1}{2}}+49^{\frac{1}{2}}$. 32. $\frac{1}{8^{-\frac{1}{3}}}+\frac{1}{27^{-\frac{1}{3}}}$ 33. $\frac{(2\cdot5)^3}{10}$

34. $\frac{x+x^{-1}}{x-1}$. 35. $\frac{a^{2m+n}\times a^{3m-6n}}{a^{4m-3n}}$.

5. **To prove that $(a^m)^n=a^{mn}$ for all values of m a**

Case I. When n is a positive integer.

$$(a^m)^n=a^m\times a^m\times a^m\text{......to } n \text{ factors}$$
$$=a^{m+m+m+\dots\text{ to } n \text{ terms}}=a^{mn}.$$

Case II. When n is a positive fraction.

Let $n=\frac{p}{q}$, where p and q are positive integers,

Now $$(a^m)^n=(a^m)^{\frac{p}{q}}=\sqrt[q]{(a^m)^p}$$
$$=\sqrt[q]{a^{mp}}\quad=a^{\frac{mp}{q}}$$
$$=a^{mn}.$$

Case III. When n is any negative quantity.

Let $n=-p$, then p is a positive quantity.

Now $$(a^m)^n=(a^m)^{-p}=\frac{1}{(a^m)^p}$$
$$=\frac{1}{a^{mp}}=a^{-mp}$$
$$=a^{mn}.$$

Example 1. $(9^{\frac{2}{3}})^{\frac{3}{4}}=9^{\frac{2}{3}\times\frac{3}{4}}=9^{\frac{1}{2}}=\sqrt{9}=3.$

Example 2. $\left(27^{-\frac{2}{3}}\right)^{\frac{1}{2}}=27^{-\frac{2}{3}\times\frac{1}{2}}=27^{-\frac{1}{3}}=\dfrac{1}{27^{\frac{1}{3}}}=\dfrac{1}{\sqrt[3]{27}}=\dfrac{1}{3}.$

6. To prove that $a^nb^n=(ab)^n$ for all values of n.

Case I. When n is a positive integer.

$$a^nb^n=(a\times a\times a \ldots\ldots \text{to } n \text{ factors})\times(b\times b\times b \ldots \text{to } n \text{ factors})$$
$$=(ab\times ab\times ab \ldots \text{to } n \text{ factors})$$
$$=(ab)^n.$$

Case II. When n is a positive fraction.

Let $n=\dfrac{p}{q}$, where p and q are both positive integers, then $p=nq$.

Now $(ab)^n=(ab)^{\frac{p}{q}}=\sqrt[q]{(ab)^p}$
$$=\sqrt[q]{a^pb^p}=\sqrt[q]{a^{nq}b^{nq}}$$
$$=\sqrt[q]{(a^n)^q(b^n)^q}=\sqrt[q]{(a^nb^n)^q}$$
$$=(a^nb^n)^{\frac{q}{q}}=a^nb^n.$$

Case III. When n is any negative quantity.

Let $n=-p$, then p is a positive quantity.

Now $(ab)^n=(ab)^{-p}=\dfrac{1}{(ab)^p}$
$$=\frac{1}{a^pb^p}=a^{-p}b^{-p}=a^nb^n.$$

Hence generally $(abc\ldots)^n=a^n\times b^n\times c^n\ldots$ for all values of n.

Cor. $\dfrac{a^n}{b^n}=a^nb^{-n}=a^n(b^{-1})^n=(ab^{-1})^n=\left(\dfrac{a}{b}\right)^n.$

Example 3. $\left(8^{-2}\times 9^{\frac{3}{4}}\right)^{-\frac{2}{3}}=\left(8^{-2}\right)^{-\frac{2}{3}}\times\left(9^{\frac{3}{4}}\right)^{-\frac{2}{3}}$
$$=8^{\frac{4}{3}}\times 9^{-\frac{1}{2}}=(\sqrt[3]{8})^4\times\frac{1}{\sqrt{9}}$$
$$=2^4\times\frac{1}{3}=\frac{16}{3}.$$

Example 4. $\sqrt[3]{a^{-4}b^2} \times \sqrt{a^3b} \times \sqrt[6]{ab^{-3}}$

$$= a^{-\frac{4}{3}} b^{\frac{2}{3}} \times a^{\frac{3}{2}} b^{\frac{1}{2}} \times a^{\frac{1}{6}} b^{-\frac{1}{2}}$$
$$= a^{-\frac{4}{3}+\frac{3}{2}+\frac{1}{6}} \times b^{\frac{2}{3}+\frac{1}{2}-\frac{1}{2}}$$
$$= a^{\frac{1}{3}} \times b^{\frac{2}{3}}$$
$$= \sqrt[3]{ab^2}.$$

EXERCISE 71.

Simplify:

1. $(x^{-\frac{2}{3}})^6$. 2. $(x^{-\frac{3}{4}}y^{\frac{5}{8}})^{\frac{3}{4}}$. 3. $(x^{-\frac{1}{3}}b^{-2})^{-3}$.

4. $(x^6y^{\frac{5}{4}})^{-\frac{4}{3}}$. 5. $(\sqrt[4]{m^3n^4})^8$. 6. $(\sqrt[8]{a^{12}b^{-8}})^{-4}$

7. $\sqrt[4]{m^6 \sqrt[4]{m^{-3}}}$ 8. $\sqrt{x^{-6}y^8} \times \sqrt[4]{x^4y^{-12}}$.

9. $\sqrt{a^{-1}\sqrt{a^3\sqrt{a^{-4}}}}$. 10. $\sqrt[4]{a^{-4}\sqrt{b^5}} \times \sqrt{a\sqrt[4]{b^3}}$.

11. $(27a^3 \div 64b^{-3})^{\frac{2}{3}}$ 12. $(125a^3 \div 27b^{-3})^{-\frac{2}{3}}$.

13. $\{(a^{\frac{1}{3}}b^{-\frac{2}{3}})^{-6} \times (a^{-\frac{1}{3}}b^{\frac{2}{3}})^{-6}\}^{\frac{1}{4}}$

14. $\sqrt{xy^{-2}z^3} \div \sqrt[3]{(x^3y^2z^{-3})}^{-1}$

15. $\left\{(x^{a+b-c} \times x^{a-b+c})^b\right\}^c$.

16. $\left(\dfrac{x^m}{x^n}\right)^{m+n} \times \left(\dfrac{x^n}{x^l}\right)^{n+l} \times \left(\dfrac{x^l}{x^m}\right)^{l+m}$

17. Which is greater $(2^3)^2$ or 2^{3^2}, and by how much?

7. Algebraical operations involving fractional and negative indices.

As in the processes of multiplication, division, factorisation and square root, it is necessary to arrange the terms in ascending or descending powers of some common letter, the student should have sufficient practice in such arrangements while dealing with expressions involving fractional and negative indices.

Example 1. Arrange the following expressions in ascending powers of x :

(i) $x+x^0+x^{-1}+x^{-2}+x^2$.

(ii) $3x^{\frac{1}{2}}+x^{-1}+4x^{-\frac{1}{2}}+x^{\frac{1}{4}}+x^0$.

(iii) $x^{\frac{1}{8}}-2x^{-\frac{3}{4}}+3x^{-\frac{1}{4}}+1-7x^2+5x$.

(i) Since $0>-1$ and $-1>-2$, the proper arrangement of the terms in the first expression would be

$$x^{-2}+x^{-1}+x^0+x+x^2.$$

(ii) Since $0>-\frac{1}{2}$ and $-\frac{1}{2}>-1$, the proper arrangement of the terms in the second expression would be

$$x^{-1}+4x^{-\frac{1}{2}}+x^0+x^{\frac{1}{4}}+3x^{\frac{1}{2}}.$$

(iii) Since $1=x^0$ and $0>-\frac{1}{4}$ and $-\frac{1}{4}>-\frac{3}{4}$, the proper arrangement of the terms in the third expression would be

$$-2x^{-\frac{3}{4}}+3x^{-\frac{1}{4}}+1+x^{\frac{1}{8}}+5x-7x^2.$$

Ex. Arrange the above three expressions in descending powers of x.

Example 2. Multiply $ab^{-1}+1+a^2b^{-2}$ by $a^2b^{-2}+1-ab^{-1}$.

Arranging both the expressions in ascending powers of a and proceeding as usual, we have

$$\begin{array}{llll}
1 & +ab^{-1} & +a^2b^{-2} & \\
1 & -ab^{-1} & +a^2b^{-2} & \\
\hline
1 & +ab^{-1} & +a^2b^{-2} & \\
 & -ab^{-1} & -a^2b^{-2} & -a^3b^{-3} \\
 & & +a^2b^{-2} & +a^3b^{-3} \quad +a^4b^{-4} \\
\hline
1 & & +a^2b^{-2} & \qquad\qquad +a^4b^{-4}.
\end{array}$$

Example 3. Divide $11x^{\frac{1}{5}}-6+x^{\frac{3}{5}}-6x^{\frac{2}{5}}$ by $x^{\frac{1}{5}}-3$.

Arranging the dividend and divisor according to descending powers of x and proceeding as usual, we have

$$
\begin{array}{r}
x^{\frac{1}{5}}-3)\,x^{\frac{3}{5}}-6x^{\frac{2}{5}}+11x^{\frac{1}{5}}-6\,(x^{\frac{2}{5}}-3x^{\frac{1}{5}}+2 \\
\underline{x^{\frac{3}{5}}-3x^{\frac{2}{5}}}\qquad\qquad\qquad\qquad\qquad\qquad \\
-3x^{\frac{2}{5}}+11x^{\frac{1}{5}}\qquad\qquad\qquad\qquad \\
\underline{-3x^{\frac{2}{5}}+9x^{\frac{1}{5}}}\qquad\qquad\qquad\qquad \\
2x^{\frac{1}{5}}-6\qquad\qquad\qquad\qquad \\
\underline{2x^{\frac{1}{5}}-6}\qquad\qquad\qquad\qquad
\end{array}
$$

EXERCISE 72.

1. Arrange the following expressions according to ascending powers of x:

(i) $4x^2+1+3x^{-1}+5x^{-2}-7x$.

(ii) $3+6x^{-3}+5x^3+4x^{-2}+7x^{-1}$.

(iii) $x+4+6x^{-\frac{1}{2n}}+5x^{\frac{1}{n}}+3x^{\frac{1}{2n}}-x^{-\frac{1}{n}}$, where n is a positive integer.

(iv) $7x^{\frac{1}{n}}-2+x+6x^{-\frac{1}{3n}}+5x^{\frac{1}{2n}}$, where n is a positive integer.

2. Arrange the following expressions according to descending powers of y:

(i) $2+x^{-2}y^2+x^2y^{-2}$.

(ii) $x^{\frac{2}{3}}y^{-\frac{2}{3}}-1+x^{-\frac{2}{3}}y^{\frac{2}{3}}$.

(iii) $3y^{-3}-xy^{-2}+1-5x^2y^{-1}+7x^3$.

Multiply :

3. $a^{\frac{2}{3}}-1+a^{-\frac{2}{3}}$ by $a^{\frac{1}{3}}+a^{-\frac{1}{3}}$.

4. $a-a^{\frac{1}{2}}b^{\frac{1}{2}}+b$ by $a^{\frac{1}{2}}+b^{\frac{1}{2}}$.

5. $a+3b^{\frac{1}{2}}+4c^{\frac{1}{3}}$ by $a-3b^{\frac{1}{2}}+4c^{\frac{1}{3}}$.

6. $a^{\frac{2}{3}}+2a^{\frac{1}{2}}+3a^{\frac{1}{3}}+2a^{\frac{1}{6}}+1$ by $1+a^{\frac{1}{3}}-2a^{\frac{1}{6}}$.

7. $x^{-1}+y^{-1}+x^{-\frac{1}{2}}y^{-\frac{1}{2}}$ by $x^{-1}-x^{-\frac{1}{2}}y^{-\frac{1}{2}}+y^{-1}$.

8. $a^{\frac{2}{3}}+b^{\frac{2}{3}}+c^{\frac{2}{3}}-a^{\frac{1}{3}}b^{\frac{1}{3}}-a^{\frac{1}{3}}c^{\frac{1}{3}}-b^{\frac{1}{3}}c^{\frac{1}{3}}$ by $a^{\frac{1}{3}}+b^{\frac{1}{3}}+c^{\frac{1}{3}}$.

9. $a^{\frac{1}{2}}-3a^{\frac{1}{3}}b^{\frac{1}{6}}+3a^{\frac{1}{6}}b^{\frac{1}{3}}-b^{\frac{1}{2}}$ by $a^{\frac{1}{3}}-2a^{\frac{1}{6}}b^{\frac{1}{6}}+b^{\frac{1}{3}}$.

10. $x^{\frac{5}{8}}+x^{\frac{1}{4}}y^{-\frac{3}{8}}+y^{-\frac{5}{8}}+x^{\frac{3}{8}}y^{-\frac{1}{4}}+x^{\frac{1}{8}}y^{-\frac{1}{2}}+x^{\frac{1}{2}}y^{-\frac{1}{8}}$
by $x^{\frac{3}{8}}+x^{\frac{1}{8}}y^{-\frac{1}{4}}-y^{-\frac{3}{8}}-x^{\frac{1}{4}}y^{-\frac{1}{8}}$.

Divide:

11. $a^{\frac{2}{3}}b^{-\frac{2}{3}}+1-a^{\frac{2}{3}}-b^{-\frac{2}{3}}$ by $1+b^{-\frac{1}{3}}+a^{\frac{1}{3}}+a^{\frac{1}{3}}b^{-\frac{1}{3}}$.

12. $x^{\frac{4}{3}}+a^{\frac{2}{3}}x^{\frac{2}{3}}+a^{\frac{4}{3}}$ by $x^{\frac{2}{3}}+a^{\frac{1}{3}}x^{\frac{1}{3}}+a^{\frac{2}{3}}$.

13. $x^2y^{-2}+x^{-2}y^2+2$ by $x^{\frac{2}{3}}y^{-\frac{2}{3}}+x^{-\frac{2}{3}}y^{\frac{2}{3}}-1$.

14. $1-x^{-\frac{1}{2}}+2x^{\frac{3}{4}}+x^{\frac{3}{2}}$ by $1+x^{-\frac{1}{4}}+x^{\frac{3}{4}}$.

15. $x+a$ by $x^{\frac{1}{5}}+a^{\frac{1}{5}}$. 16. $x^{-1}+y^{-1}$ by $x^{-\frac{1}{3}}+y^{-\frac{1}{3}}$.

17. $8a^{\frac{3}{2}}+b^{-\frac{3}{2}}-c+6a^{\frac{1}{2}}b^{-\frac{1}{2}}c^{\frac{1}{3}}$ by $2a^{\frac{1}{2}}+b^{-\frac{1}{2}}-c^{\frac{1}{3}}$.

Example 4. Find the square of $a^{\frac{1}{5}}-b^{\frac{1}{7}}+c^{\frac{1}{3}}$.

$$(a^{\frac{1}{5}}-b^{\frac{1}{7}}+c^{\frac{1}{3}})^2=(a^{\frac{1}{5}})^2+(-b^{\frac{1}{7}})^2+c^{\frac{1}{3}})^2+2(a^{\frac{1}{5}})(-b^{\frac{1}{7}})+$$
$$2(a^{\frac{1}{5}})(c^{\frac{1}{3}})+2(-b^{\frac{1}{7}})(c^{\frac{1}{3}})=a^{\frac{2}{5}}+b^{\frac{2}{7}}+c^{\frac{2}{3}}-2a^{\frac{1}{5}}b^{\frac{1}{7}}+2a^{\frac{1}{5}}c^{\frac{1}{3}}-2b^{\frac{1}{7}}c^{\frac{1}{3}}$$

Find the square of:

18. $a^{\frac{1}{3}}+b^{\frac{1}{3}}$. 19. $a+a^{-1}$. 20. $a+a^{\frac{1}{2}}-a^{\frac{1}{3}}$.

21. $a^{\frac{1}{3}}-2a^{\frac{1}{2}}+a^{\frac{5}{6}}$. 22. $1+a^{-\frac{1}{3}}+a^{\frac{1}{3}}$. 23. $a^{\frac{3}{4}}-a^{\frac{1}{2}}b^{-\frac{1}{4}}+b^{\frac{1}{2}}$.

Example 5. Resolve into factors $a^{\frac{2}{3}}-b^{\frac{3}{2}}$:

$$a^{\frac{3}{2}}-b^{\frac{3}{2}}=(a^{\frac{1}{2}})^3-(b^{\frac{1}{2}})^3=(a^{\frac{1}{2}}-b^{\frac{1}{2}})\{(a^{\frac{1}{2}})^2+(a^{\frac{1}{2}})(b^{\frac{1}{2}})+(b^{\frac{1}{2}})^2\}$$
$$=(a^{\frac{1}{2}}-b^{\frac{1}{2}})(a+a^{\frac{1}{2}}b^{\frac{1}{2}}+b).$$

24. Do questions 3, 4, 7, 8, 16 and 17 by formulæ.

25. Multiply $a^{2n}+a^nb^{-n}+b^{-2n}$ by a^n-b^{-n}, using a formula.

26. Divide $a^{\frac{3n}{2}}-b^{-\frac{3n}{2}}$ by $a^{\frac{n}{2}}-b^{-\frac{n}{2}}$ by factors.

27. Find the L.C.M. of $a^{\frac{3}{2}}+b^{\frac{3}{2}}$, $a^{\frac{3}{2}}-b^{\frac{3}{2}}$, $a^{\frac{1}{2}}+b^{\frac{1}{2}}$, $a-b$ by factors.

28. Find the continued product of $1-x^{\frac{1}{2}}, 1+x^{\frac{1}{2}}, 1+x$.

Example 6. Simplify (i) $(x^{2^{n-1}})^2$, (ii) $(x^{3^{n-1}})^9$.

(i) $(x^{2^{n-1}})^2 = [x^{(2^{n-1})}]^2 = x^{2\cdot 2^{n-1}} = x^{2^{n-1+1}} = x^{2^n}$.

(ii) $(x^{3^{n-1}})^9 = [x^{(3^{n-1})}]^9 = x^{9\cdot 3^{n-1}} = x^{3^2\cdot 3^{n-1}}$

$= x^{3^{n-1+2}} = x^{3^{n+1}}$.

Simplify:

29. $(x^{2^{n+1}})^4$.

30. $(x^{3^{n+1}})^9$.

31. $27^n \div 3^{3n+1}$.

32. $[(a^m)^{m-\frac{1}{m}}]^{\frac{1}{m+1}}$.

33. $(x^{2^{n-1}} + a^{2^{n-1}})(x^{2^{n-1}} - a^{2^{n-1}})$.

34. $(a^{2^n} - b^{2^n}) \div (a^{2^{n-1}} + b^{2^{n-1}})$.

35. Compare $\left[(2^2)^2\right]^2$ and $2^{2^{2^2}}$.

36. Find the value of $3^m.3^n$ when $m = 2^4$ and $n = 2^3$.

Example 7. Simplify $\dfrac{2^{m+1}.3^{2m-n}.5^{m+n}.6^n}{6^m.10^{n+2}.15^m}$.

$$\text{The expression} = \frac{2^{m+1}\,3^{2m-n}.5^{m+n}.(3^n.2^n)}{(3^m.2^m).(5^{n+2}.2^{n+2}).5^m.3^m}$$

$$= \frac{2^{m+n+1}.3^{2m+n-n}.5^{m+n}}{2^{m+n+2}.3^{m+m}.5^{m+n+2}}$$

$$= \left(\frac{2^{m+n+1}}{2^{m+n+2}}\right).\left(\frac{3^{2m}}{3^{2m}}\right).\left(\frac{5^{m+n}}{5^{m+n+2}}\right)$$

$$= \frac{1}{2}.1.\frac{1}{5^2} = \frac{1}{2}.\frac{1}{25} = \frac{1}{50}.$$

Simplify:

37. $\dfrac{2^n.6^{m+1}.10^{m-n}.15^{m+n-2}}{4^m.3^{2m+n}.25^{m-1}}$.

38. $\dfrac{3^{n+4} - 6.3^{n+1}}{3^{n+2}}$.

39. $\dfrac{2^n.(2^{n-1})^n}{2^{n+1}.2^{n-1}} \times \dfrac{2\cdot 2^n}{(2^n)^n}$.

40. $\dfrac{3\cdot 4 \times 10^4 \times 5\cdot 7 \times 10^{-6} \times 4\cdot 4 \times 10^{-3}}{1\cdot 87 \times 10^{-4} \times 9\cdot 5 \times 10^3 \times 3 \times 10^{-4}}$.

41. $\dfrac{6^{-n}\times 36^{2n-2}}{6^{3n-2}\times 12^{-1}}$.

42. $\dfrac{3^{n+1}}{(3^n)^{n+1}} \div \dfrac{9^{n+1}}{(3^{n+1})^{n-1}}$.

Prove that

*43. $\dfrac{2^n+2^{n-1}}{2^{n+1}-2^n}=\dfrac{3}{2}$.

*44. $\dfrac{5^3.2^{\frac{1}{4}}.10^{-\frac{1}{4}}}{15^{\frac{3}{4}}\cdot 6^{-\frac{3}{4}}.4^{\frac{3}{8}}}=25$.

Example 8. $\left(\dfrac{a-b}{a^{\frac{1}{2}}-b^{\frac{1}{2}}}-\dfrac{a^{\frac{3}{2}}-b^{\frac{3}{2}}}{a-b}\right)^{-1}$.

The expression $=\left[\dfrac{(a^{\frac{1}{2}}+b^{\frac{1}{2}})(a^{\frac{1}{2}}-b^{\frac{1}{2}})}{a^{\frac{1}{2}}-b^{\frac{1}{2}}}-\dfrac{(a^{\frac{1}{2}}-b^{\frac{1}{2}})(a+a^{\frac{1}{2}}b^{\frac{1}{2}}+b)}{(a^{\frac{1}{2}}-b^{\frac{1}{2}})(a^{\frac{1}{2}}+b^{\frac{1}{2}})}\right]^{-1}$

$$=\left[(a^{\frac{1}{2}}+b^{\frac{1}{2}})-\frac{a+a^{\frac{1}{2}}b^{\frac{1}{2}}+b}{a^{\frac{1}{2}}+b^{\frac{1}{2}}}\right]^{-1}$$

$$=\left[\frac{(a^{\frac{1}{2}}+b^{\frac{1}{2}})^2-(a+a^{\frac{1}{2}}b^{\frac{1}{2}}+b)}{a^{\frac{1}{2}}+b^{\frac{1}{2}}}\right]^{-1}$$

$$=\left[\frac{a+2a^{\frac{1}{2}}b^{\frac{1}{2}}+b-a-a^{\frac{1}{2}}b^{\frac{1}{2}}-b}{a^{\frac{1}{2}}+b^{\frac{1}{2}}}\right]^{-1}$$

$$=\left[\frac{a^{\frac{1}{2}}b^{\frac{1}{2}}}{a^{\frac{1}{2}}+b^{\frac{1}{2}}}\right]^{-1}=\frac{a^{\frac{1}{2}}+b^{\frac{1}{2}}}{a^{\frac{1}{2}}b^{\frac{1}{2}}}.$$

Simplify :

45. $\dfrac{\left(p+\dfrac{1}{q}\right)^m\left(p-\dfrac{1}{q}\right)^n}{\left(q+\dfrac{1}{p}\right)^m\left(q-\dfrac{1}{p}\right)^n}$.

46. $\dfrac{x-x^{-1}}{x-1}$.

47. $\dfrac{x^3-y^{-3}}{x-y^{-1}}$.

48. $\dfrac{x^2}{(1-x)^n}+\dfrac{2}{(1-x)^{n-1}}-\dfrac{1}{(1-x)^{n-2}}$.

49. $\left(\dfrac{x^a}{x^b}\right)^{a+b}\cdot\left(\dfrac{x^b}{x^c}\right)^{b+c}\cdot\left(\dfrac{x^c}{x^a}\right)^{c+a}$.

50. $(2x^{\frac{1}{2}}+x^{-\frac{1}{2}})^2-(2x^{\frac{1}{2}}-x^{-\frac{1}{2}})^2$.

*51. $\dfrac{x^2+y^2-x^{-2}-y^{-2}}{x^2y^2-x^{-2}y^{-2}}+\dfrac{(x-x^{-1})(y-y^{-1})}{xy+x^{-1}y^{-1}}$.

*52. $a^{-2}(bc^{-1}-b^{-1}c)+b^{-2}(ca^{-1}-c^{-1}a)+c^{-2}(ab^{-1}-a^{-1}b)$.

Prove that

*53. $\dfrac{y^{-1}}{x^{-1}+y^{-1}}+\dfrac{y^{-1}}{x^{-1}-y^{-1}}=\dfrac{2xy}{y^2-x^2}$.

*54. $\left(1-\dfrac{1}{n}\right)^{-n}=\left(1+\dfrac{1}{n-1}\right)^n$.

*55. Simplify $\dfrac{1}{1+a^{y-x}+a^{z-x}}+\dfrac{1}{1+a^{x-y}+a^{z-y}}$

$$+\frac{1}{1+a^{y-z}+a^{x-z}}.$$

$\left[Hint. \quad \dfrac{1}{1+a^{y-x}+a^{z-x}}=\dfrac{a^x}{a^x+a^y+a^z}.\right]$

*56. If $x=2^{\frac{1}{3}}-2^{-\frac{1}{3}}$, prove that $2x^3+6x=3$.

[*Hint.* Cube both sides $x=2^{\frac{1}{3}}-2^{-\frac{1}{3}}$, and put x for $(2^{\frac{1}{3}}-2^{-\frac{1}{3}})$.]

*57. If $a^l=b^m=c^n$ and $b^2=ac$, then $\dfrac{1}{l}+\dfrac{1}{n}=\dfrac{2}{m}$.

58. If $a^x=b^y$ and $b^x=a^y$, then $a=b$.

59. If $a^x=b$, $b^y=c$, $c^z=a$, then $xyz=1$.

*60. If $x^y=y^x$, prove that $\left(\dfrac{x}{y}\right)^{\frac{x}{y}}=x^{\frac{x}{y}-1}$

[*Hint.* Proceed analytically: $\left(\dfrac{x}{y}\right)^{\frac{x}{y}}=x^{\frac{x}{y}-1}$

if $\dfrac{(x^{\frac{x}{y}})}{(y^{\frac{x}{y}})}=\dfrac{(x^{\frac{x}{y}})}{x}$

if $y^{\frac{x}{y}}=x$

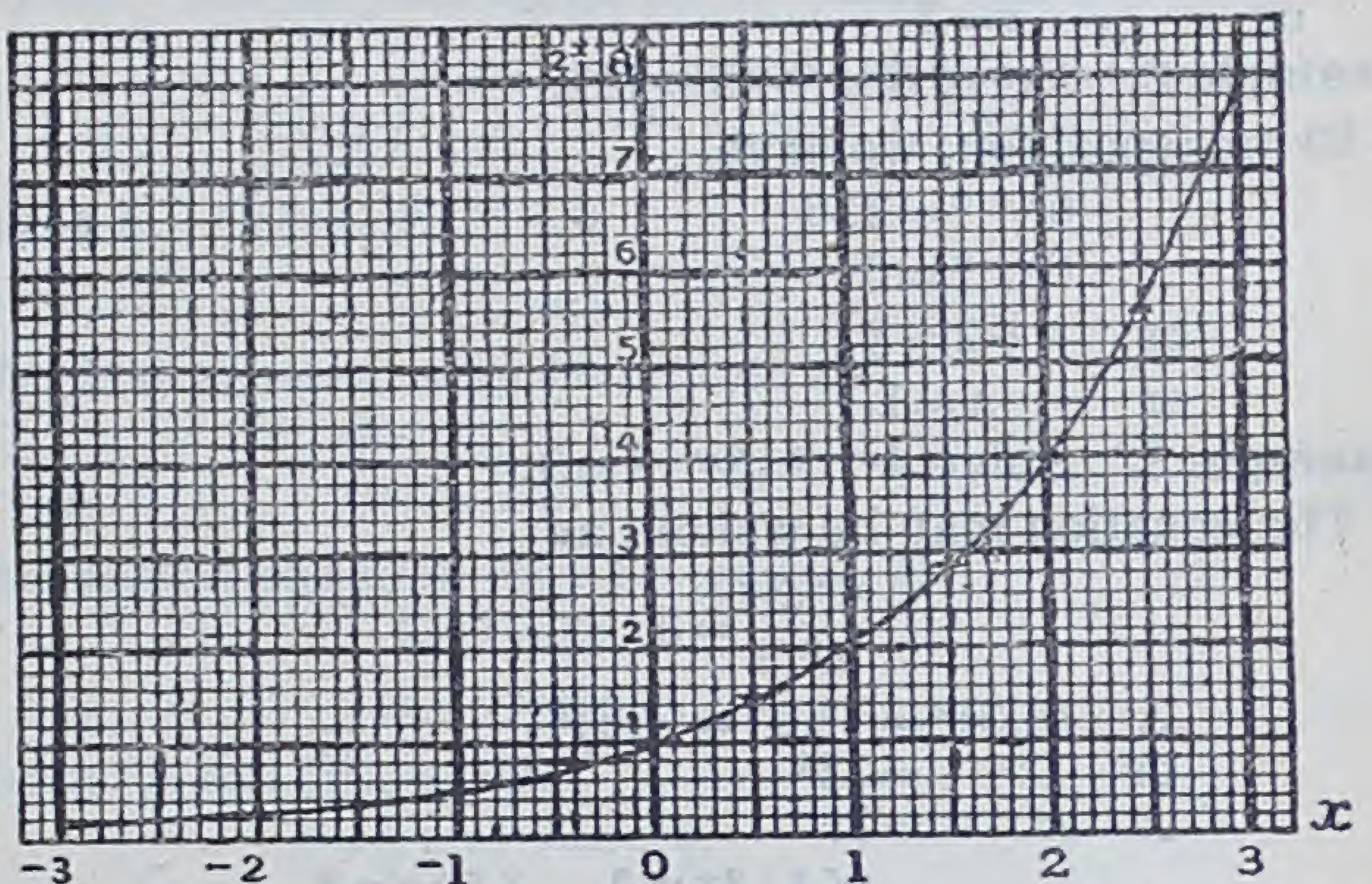
if $y^x=x^y$.]

8. To draw the graph of 2^x from $x=-3$ to $x=+3$.

Putting $x=-3$, -2, -1, 0, $\cdot5$, 1, $1\cdot5$, 2, $2\cdot5$ and 3 in 2^x and calculating its values, we prepare the following table :

x	-3	-2	-1	0	$\cdot5$	1	$1\cdot5$	2	$2\cdot5$	3
2^x	$\cdot125$	$\cdot25$	$\cdot5$	1	$1\cdot41$	2	$2\cdot83$	4	$5\cdot66$	8

Taking 1 cm. as a unit along x-axis and $\cdot5$ cm. as a unit along the other, and plotting these values, we obtain the following graph :



We note that

(i) 2^x increases as x increases. As there is no limit to the value of x, there is no limit to the value of 2^x.

(ii) For $x=0$, $2^x=1$,
for x positive, $2^x > 1$
for x negative, $2^x < 1$.

(iii) 2^x is positive for all values of x.

Ex. (i) From the graph, read off the approximate value of $2^{\cdot2}$ and $2^{-\cdot5}$

(ii) Use the graph to solve $2^x=5$.

9. Exponential Equations. The equations in which the unknown occurs as an index or exponent are called **exponential equations.** Thus $a^x = n$ is an exponential equation.

Example 1. Solve $9^x = 3^{x+2}$.

As $$9^x = (3^2)^x = 3^{2x},$$

$\therefore$ the equation can be written as below :
$$3^{2x} = 3^{x+2}.$$

As the bases of this equation are equal, therefore its indices are also equal.

Hence $$2x = x+2$$
or $$x = 2.$$

Example 2. Solve $6^{x+2} + 5 = 221$.

By transposition, we have
$$6^{x+2} = 216$$
$$\therefore \quad 6^{x+2} = 6^3$$
$$\text{or} \quad x+2 = 3$$
$$\text{or} \quad x = 1.$$

Example 3. Solve $4^{x-2} . 3^{x-1} = 3$.

The equation may be written as
$$\frac{4^x . 3^x}{4^2 . 3} = 3$$
$$\therefore \quad (4.3)^x = 4^2 . 3^2 = (4.3)^2$$
$$\text{or} \quad 12^x = 12^2$$
$$\therefore \quad x = 2.$$

Example 4. Solve $\left(\frac{4}{5}\right)^{2x-3} = \left(\frac{5}{4}\right)^{x-3}$.

As $\frac{5}{4} = \frac{1}{\left(\frac{5}{4}\right)^{-1}} = \left(\frac{4}{5}\right)^{-1}$ $\quad\therefore\quad \left(\frac{5}{4}\right)^{x-3} = \left(\frac{4}{5}\right)^{3-x}$

$$\therefore \quad \left(\frac{4}{5}\right)^{2x-3} = \left(\frac{4}{5}\right)^{3-x}$$
$$\therefore \quad 2x-3 = 3-x$$
$$\text{or} \quad 3x = 6$$
$$\text{or} \quad x = 2.$$

EXERCISE 73.

Solve the equations :

1. $2^{x+5}=4^{x+1}$. 2. $9^{x+1}=3^{x+7}$. 3. $5^{x+2}+6=631$.

4. $3^{2x-1}+5=86$. 5. $(\sqrt{6})^{x+4}=(\sqrt[3]{6})^{2x+3}$.

6. $(\sqrt[6]{16})^{2x+9}=(\sqrt[11]{64})^{2x+7}$. 7. $(\sqrt[4]{25})^{2x-1}=(\sqrt[6]{125})^{x+8}$.

8. $7^{x-1}.\ 3^{x-2}=7$. 9. $3^{x-2}.\ 5^{x-1}=5$. 10. $9^{x}=\dfrac{9}{3^{x}}$.

11. $5^{3x-4}=1$. [*Hint.* $1=5^{0}$.] 12. $a^{x-3}=b^{x-3}$.

[*Hint.* Divide both sides by b^{x-3} and put $1=\left(\dfrac{a}{b}\right)^{0}$.]

13. $\left(\dfrac{5}{7}\right)^{x-1}=\left(\dfrac{7}{5}\right)^{x-2}$. 14. $\left(\dfrac{3}{7}\right)^{2x-1}=\left(\dfrac{7}{3}\right)^{x-7}$.

Example 5. Solve $3^{2x-y}=9$

$$5^{x+y}=1.$$

The first equation may be written as

$$3^{2x-y}=3^{2}$$

or $$2x-y=2 \qquad \ldots \qquad \ldots \qquad \text{(i)}$$

The second equation may be written as

$$5^{x+y}=5^{0}$$

$\therefore$ $$x+y=0 \qquad \ldots \qquad \ldots \qquad \text{(ii)}$$

From (i) and (ii), we get

$$x=\tfrac{2}{3} \text{ and } y=-\tfrac{2}{3}.$$

Solve the equations :

*15. $\left.\begin{aligned} 3^{x}.\ 9^{y} &= 81 \\ 2x-y &= 8. \end{aligned}\right\}$

*16. $\left.\begin{aligned} 2^{x+1} &= 4^{y-3} \\ 3^{x-2} &= 9^{2y-3x}. \end{aligned}\right\}$

*17. $\left.\begin{aligned} a^{y-x} &= a^{z+1} \\ a^{2y+3} &= a^{3x+z} \\ x+y+z &= 3. \end{aligned}\right\}$

*18. $\left.\begin{aligned} a^{x+z}.\ a^{1-y} &= 1 \\ b^{y-z}.\ b^{2z-x} &= b \\ c^{y+1}.\ c^{x-z} &= c^{3}. \end{aligned}\right\}$

*19. $\left.\begin{aligned} 3^{x+y+z} &= 27^{x-y+z} \\ 4^{3y+2} &= 16^{x+z} \\ 2^{2x+2z+y} &= 4^{3x+y}. \end{aligned}\right\}$

*20. $\left.\begin{aligned} a^{x}.a^{y+1} &= a^{7} \\ a^{2y}.\ a^{3x+5} &= a^{20}. \end{aligned}\right\}$

*21. $\left.\begin{aligned} 2^{x-y}.\ 3^{x-2y} &= 2^y \\ 3^{x-2}.\ 5^{x-1} &= 5^{3y+1}.\ 3^{3y}. \end{aligned}\right\}$

[*Hint.* Put the 1st equation in the form $(2.3)^x = (2.3)^{2y}$ and the second equation in the form $(3.5)^{x-2} = (3.5)^{3y}$.

*22. $\left.\begin{aligned} 3^{2x-y}.\ 4^{2x} &= 4^y. \\ 4^{3x+1}.\ 5^{3x-2y+1} &= 4^{2y}. \end{aligned}\right\}$

10. Theorem. If $\dfrac{a}{b} = \dfrac{c}{d} = \dfrac{e}{f} = \ldots\ldots$

then each of the ratios $= \left(\dfrac{pa^n + qc^n + re^n + \ldots}{pb^n + qd^n + rf^n + \ldots}\right)^{\frac{1}{n}}$.

where $p, q, r \ldots\ldots\ldots n$ are any quantities whatever.

Let $\dfrac{a}{b} = \dfrac{c}{d} = \dfrac{e}{f} = \ldots = k.$

Then $a = bk,\quad c = dk,\ e = fk\ldots\ldots$

$$pa^n = pb^n k^n$$

$$qc^n = qd^n k^n$$

$$re^n = rf^n k^n$$

etc., etc.

Hence $pa^n + qc^n + re^n + \ldots\ldots = (pb^n + qd^n + rf^n + \ldots)k^n$

$$\frac{pa^n + qc^n + re^n + \ldots}{pb^n + qd^n + rf^n + \ldots} = k^n.$$

Taking n^{th} root of both sides, we have

$$\left(\frac{pa^n + qc^n + re^n + \ldots}{pb^n + qd^n + rf^n + \ldots}\right)^{\frac{1}{n}} = (k^n)^{\frac{1}{n}}$$

$$= k = \frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \ldots$$

Example. If $\dfrac{a}{b} = \dfrac{c}{d} = \dfrac{e}{f}$, prove that each of these ratios is equal to $\sqrt{\left(\dfrac{3a^{-2} - 5c^{-2} + 7e^{-2}}{3b^{-2} - 5d^{-2} + 7f^{-2}}\right)^{-1}}$

Let $\dfrac{a}{b} = \dfrac{c}{d} = \dfrac{e}{f} = k.$

$\therefore\quad a = bk,\ c = dk,\ e = fk.$

$\therefore \quad 3a^{-2}=3b^{-2}k^{-2}$

$\quad -5c^{-2}=-5d^{-2}k^{-2}$

$\quad 7e^{-2}=7f^{-2}k^{-2}.$

$\therefore \quad 3a^{-2}-5c^{-2}+7e^{-2}=k^{-2}(3b^{-2}-5d^{-2}+7f^{-2})$

or $\left(\dfrac{3a^{-2}-5c^{-2}+7e^{-2}}{3b^{-2}-5d^{-2}+7f^{-2}}\right)=k^{-2}$

or $\left(\dfrac{3a^{-2}-5c^{-2}+7e^{-2}}{3b^{-2}-5d^{-2}+7f^{-2}}\right)^{-1}=(k^{-2})^{-1}=k^2.$

Taking the sq. root of both sides, we have

$$\sqrt{\left(\frac{3a^{-2}-5c^{-2}+7e^{-2}}{3b^{-2}-5d^{-2}+7f^{-2}}\right)^{-1}}=\sqrt{k^2}=k.$$

$$\therefore \quad \frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\sqrt{\left(\frac{3a^{-2}-5c^{-2}+7e^{-2}}{3b^{-2}-5d^{-2}+7f^{-2}}\right)^{-1}}.$$

EXERCISE 74.

If $\dfrac{a}{b}=\dfrac{c}{d}=\dfrac{e}{f}$, prove that each of these ratios is equal to

1. $\sqrt{\dfrac{a^2-c^2+e^2}{b^2-d^2+f^2}}$.

2. $\left(\dfrac{a^3+2c^3-3e^3}{b^3+2d^3-3f^3}\right)^{\frac{1}{3}}$.

3. $\sqrt{\dfrac{5a^2-7c^2+6e^2}{5b^2-7d^2+6f^2}}$.

4. $\left(\dfrac{3a^3-4c^3+5e^3}{3b^3-4d^3+5f^3}\right)^{\frac{1}{3}}$.

5. $\sqrt[4]{\left(\dfrac{2a^{-4}+5c^{-4}+7e^{-4}}{2b^{-4}+5d^{-4}+7f^{-4}}\right)^{-1}}$.

6. $\left(\dfrac{la^7-mc^7+ne^7}{lb^7-md^7+nf^7}\right)^{\frac{1}{7}}$.

If $\dfrac{a}{b}=\dfrac{c}{d}$, prove that

7. $\sqrt{\dfrac{a^2+c^2}{b^2+d^2}}=\dfrac{ma+nc}{mb+nd}$.

8. $\dfrac{(\sqrt{a}+\sqrt{b})^2}{(\sqrt{c}+\sqrt{d})^2}=\dfrac{a-b}{c-d}$.

9. $\dfrac{\sqrt{3a^2+4c^2}}{\sqrt[3]{5a^3-6c^3}}=\dfrac{\sqrt{3b^2+4d^2}}{\sqrt[3]{5b^3-6d^3}}$.

If $\dfrac{a}{b}=\dfrac{c}{d}=\dfrac{e}{f}$, prove that

10. $\sqrt{\dfrac{pa^2+qc^2+re^2}{pb^2+qd^2+rf^2}}=\sqrt[3]{\dfrac{ace}{bdf}}$.

11. $\sqrt[3]{\dfrac{a^2c-c^2e-e^2a}{b^2d-d^2f-f^2b}}=\sqrt{\dfrac{a^2-c^2-e^2}{b^2-d^2-f^2}}$.

CHAPTER XVII

INVOLUTION AND EVOLUTION

1. Involution is the process of finding the powers of quantities. The powers of monomials have already been treated. Here, we propose to discuss the method of finding the powers of binomials, trinomials and multinomials.

We already know that

$$(x+a)(x+b)=x^2+(a+b)x+ab \text{ and}$$

$$(x+a)(x+b)(x+c)=x^3+(a+b+c)x^2+(ab+bc+ca)x+abc.$$

Here we observe that

(i) the degree of the product is the same as the number of factors;

(ii) the first term is the product of the first terms of the factors;

(iii) the co-efficient of the second terms is the sum of the second terms of the factors;

(iv) the co-efficient of the third terms is the sum of the products of the second terms of the factors, taken two at a time;

(v) the last term is the product of the second terms of the factors.

Putting Σ for the sum of like terms, we have

$$(x+a)(x+b)(x+c)=x^3+x^2\Sigma a+x\Sigma ab+abc.$$

Similarly,

$$(x+a)(x+b)(x+c)(x+d)=x^4+x^3\Sigma a+x^2\Sigma ab+x\Sigma abc+abcd.$$

Example 1. Find the continued product of

$$(x+1)(x+2)(x+3)(x-4).$$

Putting a for 1, b for 2, c for 3 and d for -4, we get the product

$$=x^4+x^3\Sigma a+x^2\Sigma ab+x\Sigma abc+abcd.$$

$$\Sigma a=1+2+3-4=2,$$

$\Sigma ab = 2+3-4+6-8-12 = -13,$

$\Sigma abc = 6-8-24-12 = -38,$

$abcd = -24,$

$\therefore$ the product $= x^4 + 2x^3 - 13x^2 - 38x - 24.$

Example 2. Find the continued product of

$$(x+a+b)(x-a+b)(x+a-b).$$

The expression $= \{x+(a+b)\}\{x+(-a+b)\}\{x+(a-b)\}.$

The co-efficient of x^2 in the product

$$=(a+b)+(-a+b)+(a-b)=a+b,$$

the co-efficient of x in the product

$$=(a+b)(-a+b)+(a+b)(a-b)+(-a+b)(a-b)$$
$$=-a^2+b^2+a^2-b^2-a^2+2ab-b^2$$
$$=-(a-b)^2,$$

the last term $=(a+b)(-a+b)(a-b)$

$$=-(a^3-a^2b-ab^2+b^3).$$

$\therefore$ the product

$$=x^3+x^2(a+b)-x(a-b)^2-(a^3-a^2b-ab^2+b^3).$$

EXERCISE 75.

Find the continued product of :

1. $(x-1)(x-2)(x+4)$.
2. $(x-2)(x+3)(x+4)$.
3. $(x+2)(x+3)(x+4)$.
4. $(x-2)(x-3)(x-4)$.
5. $(x-a)(x-b)(x-c)$.
6. $(x+1)(x+2)(x-4)(x-5)$.
7. $(x-2)(x-3)(x+4)(x+5)$.
8. $(x-a)(x-b)(x-c)(x-d)$.
9. $(x+p+q)(x+p-q)(x-p+q)$.
10. $(x+y+1)(x+y-1)(x-y+1)$.

Example 3. Expand $(x+a)^4$.

$(x+a)^4=(x+a)(x+a)(x+a)(x+a)$

In this case the second term of each factor is a

The co-efficient of x^3 in the product $=a+a+a+a=4a$,

the co-efficient of x^2 in the product

$$=a.a+a.a+a.a+a.a+a.a+a.a=6a^2,$$

the co-efficient of x in the product

$$=a.a.a+a.a.a+a.a.a+a.a.a=4a^3,$$

the last term $=a.a.a.a.=a^4$.

$\therefore\ (x+a)^4 = x^4 + 4x^3a + 6x^2a^2 + 4xa^3 + a^4.$

Expand as in example 3:

11. $(x+a)^5$. 12. $(x+a)^6$.

13. $(x-a)^4$. 14. $(x-a)^5$.

2. Expansion of Binomials.

We already know that

$$\left.\begin{array}{l}(x+a)^3 = x^3 + 3x^2a + 3xa^2 + a^3 \\ (x-a)^3 = x^3 - 3x^2a + 3xa^2 - a^3\end{array}\right\}$$

$$\left.\begin{array}{l}(x+a)^4 = x^4 + 4x^3a + 6x^2a^2 + 4xa^3 + a^4 \\ (x-a)^4 = x^4 - 4x^3a + 6x^2a^2 - 4xa^3 + a^4\end{array}\right\}$$

$$\left.\begin{array}{l}(x+a)^5 = x^5 + 5x^4a + 10x^3a^2 + 10x^2a^3 + 5xa^4 + a^5 \\ (x-a)^5 = x^5 - 5x^4a + 10x^3a^2 - 10x^2a^3 + 5xa^4 - a^5.\end{array}\right\}$$

From the above cases, we observe that:

(i) The number of terms in the expansion is greater than the index of the binomial by one.

(ii) The first and the last terms in the expansion are respectively x and a, each raised to the same power as the binomial.

(iii) In each successive term, the index of x decreases and that of a increases by one.

(iv) The sum of the indices of x and a in any term is equal to the index of the binomial.

(v) The co-efficient of the first term is one, and that of each succeeding term is obtained by multiplying the co-efficient of the preceding term by the index of x in that term and dividing the product by the *number* of terms already obtained. The co-efficient of the last term is one.

(vi) The co-efficients of the terms, equi-distant from the middle, are equal.

(vii) If the sign between the two terms of the binomial is positive, all the terms in the expansion are positive, and if negative, then the odd terms are positive and the even terms are negative.

Example 4. Expand $(x+a)^8$.

The total number of terms in the expansion $=9$,

the first term $=x^8$,

the second term $=\frac{1.8}{1}x^7a=8x^7a$,

the third term $=\frac{8.7}{2}x^6a^2=28x^6a^2$,

the fourth term $=\frac{28.6}{3}x^5a^3=56x^5a^3$,

the fifth term $=\frac{56.5}{4}x^4a^4=70x^4a^4$.

The numerical co-efficients of the sixth, seventh, eighth and ninth terms are respectively 56, 28, 8 and 1.

Hence, we have

$$(x+a)^8=x^8+8x^7a+28x^6a^2+56x^5a^3+70x^4a^4+56x^3a^9$$
$$+28x^2a^6+8xa^7+a^8.$$

Example 5. Expand $(2a-3b)^5$.

The total number of terms in the expansion $=6$,

the first term $=(2a)^5$,

the second term $=\frac{1.5}{1}(2a)^4(3b)=5(2a)^4(3b)$,

the third term $=\frac{5.4}{2}(2a)^3(3b)^2=10(2a)^3(3b)^2$.

The numerical co-efficients of the fourth, fifth and sixth terms are respectively 10, 5 and 1.

Hence, we have

$$(2a-3b)^5=(2a)^5-5(2a)^4(3b)+10(2a)^3(3b)^2-10(2a)^2(3b)$$
$$+5(2a)(3b)^4-(3b)^5$$
$$=32a^5-240a^4b+720a^3b^2-1080a^2b^3+810ab^4-243b^5.$$

Expand :

15. $(x+1)^5$. 16. $(a-b)^6$. 17. $(p+q)^7$.
18. $(1-x)^6$. 19. $(2+m)^5$. 20. $(3-a)^4$.
21. $(2x-1)^5$. 22. $(1-3x)^6$. 23. $(1-x)^{10}$.
24. $(2+3x)^6$. 25. $(3a-2b)^5$. 26. $\left(a+\frac{1}{a}\right)^5$.
27. $(a+\frac{1}{2}b)^4$. 28. $\left(x-\frac{1}{x}\right)^7$.

Simplify :

29. $(x+1)^4-(x-1)^4$. 30. $(x-1)^5+(x+1)^5$.
31. $(x+a)^6-(x-a)^6$. 32. $(3p+4q)^4-(3p-4q)^4$.

Example 6. Find the co-efficient of x^2 in the expansion of $(x+2)^3(x-1)^4$.

The given expression

$$=(x^3+6x^2+12x+8)(x^4-4x^3+6x^2-4x+1).$$

The co-efficient of x^2 can be obtained, by multiplying $(+6x^2)$ by $(+1)$, $(+12x)$ by $(-4x)$ and $(+8)$ by $(+6x^2)$.

Hence, the required co-efficient

$$=(+6)(+1)+(+12)(-4)+(+8)(+6)$$
$$=6-48+48=6.$$

Find the co-efficient of :

33. x^2 in the expansion of $(2x-5)^2(3x+1)^4$.
34. x^3 in the expansion of $(x+1)^5(x-2)^3$.
35. x^4 in the expansion of $(1-x)^6\left(2x+\frac{3}{x}\right)^2$.
*36. x^5 in the expansion of $(1+3x+3x^2+x^3)^3$.
*37. x^6 in the expansion of $(x^3-x^2-x+1)^3$.

NOTE. If x is small as compared with 1, $(1+x)^n$ is approximately equal to $1+nx+\frac{n(n-1)}{2}x^2$.

Example 7. Find an approximate value of $(1002)^5$.

We have $(1002)^5=(1000+2)^5=(1000)^5\left(1+\frac{2}{1000}\right)^5$.

$$\text{Now}\left(1+\frac{2}{1000}\right)^5=1+5\left(\frac{2}{1000}\right)+\frac{5.4}{2}\left(\frac{2}{1000}\right)^2+\ldots\ldots$$
$$=1+\frac{1}{100}+\frac{4}{100{,}000}+\ldots\ldots$$
$$=1{\cdot}01004\ldots\ldots\ldots\text{(approximately.)}$$

[In the expansion of $\left(1+\frac{2}{1000}\right)^5$, we neglect the terms beyond the third, because their value is very small as compared with 1.]

Hence $(1002)^5=(1000)^5\,(1{\cdot}01004)$
$=10^{15}(1{\cdot}01004)$
$=1{,}010{,}040{,}000{,}000{,}000$ (approximately.)

Find an approximate value of :

***38.** $(1{\cdot}005)^5$. ***39.** $(1{\cdot}02)^{10}$. ***40.** $(1{\cdot}0003)^{25}$.

***41.** Re. 1 at 4% per annum compound interest after 10 years amounts to Rs. $(1+{\cdot}04)^{10}$. Calculate the value of this amount to 3 decimal places, by the above method.

EVOLUTION

3. Since $(+4)\times(+4)=+16$
and $(-4)\times(-4)=+16$
$\therefore$ $\sqrt{+16}=+4$ as well as -4.
Similarly, $\sqrt{+x^2}=+x$ as well as $-x$.
Or briefly, $\sqrt{+16}=\pm4$ and $\sqrt{+x^2}=\pm x$.

Thus, *the square root of a positive quantity is both positive and negative.*

Since no two quantities, *both positive* or *both negative*, can give us a negative product, therefore a *negative quantity cannot have a real square root.*

Examples

(i) $\sqrt{+16a^4x^6}=\pm4a^2x^3$, for $(\pm4a^2x^3)^2=+16a^4x^6$.
(ii) $\sqrt[3]{-8a^3x^6}=-2ax^2$, for $(-2ax^2)^3=-8a^3x^6$.
(iii) $\sqrt[3]{+27x^6}=+3x^2$, for $(+3x^2)^3=+27x^6$.
(iv) $\sqrt[4]{+16a^8}=\pm2a^2$, for $(\pm2a^2)^4=+16a^8$.

(v) $\sqrt[5]{-32a^{10}} = -2a^2$, for $(-2a^2)^5 = -32a^{10}$.

(vi) $\sqrt[5]{+32a^{10}} = +2a^2$, for $(+2a^2)^5 = +32a^{10}$.

Hence the rules:

(i) Any *even* root of a positive quantity may be either positive or negative :

(ii) Any *odd* root of a quantity has the same sign as that of the quantity itself.

(iii) A negative quantity cannot have a real square root.

EXERCISE 76.

Find *mentally* the square root of:

1. $a^4b^{16}c^2$. 2. $64a^6b^8c^{10}$. 3. $\dfrac{81a^6b^2c^4}{x^{10}y^{12}z^{16}}$.

4. $1{\cdot}21.\dfrac{a^{14}\,b^{18}}{c^6\,d^{10}}$. 5. $\dfrac{1{\cdot}44a^6(x-y)^4}{{\cdot}36\,b^8(x+y)^6}$.

Find *mentally* the cube root of:

6. $125a^6b^3c^9$. 7. $-64x^3b^6a^{15}$.

8. $343\cdot\dfrac{a^3b^6}{x^9y^{12}}$. 9. $-\dfrac{27}{64}\cdot\dfrac{a^6b^9}{x^{12}y^{18}}$.

Evaluate *mentally* :

10. $\sqrt[4]{16x^4y^8z^{12}}$. 11. $\sqrt[7]{-128a^{14}b^{21}}$.

12. $\sqrt[5]{\dfrac{32}{x^5y^{10}}}$. 13. $\sqrt[5]{-\dfrac{x^{10}y^{25}}{243}}$.

SQUARE ROOT

4. Since $a^2+2ab+b^2=(a+b)^2$

and $a^2-2ab+b^2=(a-b)^2$

$\therefore\ \sqrt{(a^2+2ab+b^2)}=\sqrt{(a+b)^2}=a+b$

and $\sqrt{(a^2-2ab+b^2)}=\sqrt{(a-b)^2}=a-b$.

Thus, we can write down the square root of a trinomial, when it can be put in the form of

$$a^2+2ab+b^2 \text{ or } a^2-2ab+b^2.$$

Example 1. Find the square root of $25x^2-30xy+9y^2$.

Reducing the expression to the standard form, we have

$$\sqrt{(25x^2-30xy+9y^2)}=\sqrt{(5x)^2-2(5x)(3y)+(3y)^2}$$
$$=\sqrt{(5x-3y)^2}=\pm(5x-3y).$$

Example 2. Find the square root of $\frac{x^4}{4}-3x^2y+9y^2$.

Reducing the expression to the standard form, we have

$$\sqrt{\left(\frac{x^4}{4}-3x^2y+9y^2\right)}=\sqrt{\left(\frac{x^2}{2}\right)^2-2.\left(\frac{x^2}{2}\right)(3y)+(3y)^2}$$
$$=\sqrt{\left(\frac{x^2}{2}-3y\right)^2}=\pm\left(\frac{x^2}{2}-3y\right).$$

EXERCISE 77.

Find the square root of the following:

1. $4a^2-12ab+9b^2$.
2. $16x^4+40x^2+25$.
3. $9x^4+30x^2y^2+25y^4$.
4. $81a^2-18ab^2+b^4$.
5. a^6-2a^3+1.
6. $1-12x+36x^2$.
7. $x^2-x+\frac{1}{4}$.
8. $x^4y^2+x^2y+\frac{1}{4}$.
9. $m^4-\frac{2}{3}m^2+\frac{1}{9}$.
10. $\frac{a^2}{4b}-2+\frac{4b^2}{a^2}$.
11. $\frac{x^4}{y^4}-1+\frac{y^4}{4x^4}$.
12. $x^4-\frac{1}{2}x^2y^2+\frac{y^4}{16}$.
13. $\frac{1}{9}a^2-\frac{1}{6}ab+\frac{1}{16}b^2$.
14. $\frac{1}{4x^2}+\frac{1}{3y}+\frac{x^2}{9y^2}$.
15. $a^4-a^3b+\frac{1}{4}a^2b^2$.

Example 3. Find the square root of

$$4(a+b)^2-4(a^2-b^2)+(a-b)^2.$$

Reducing the expression to the standard form, we have

$\{2(a+b)\}^2-2\{2(a+b)(a-b)\}+(a-b)^2$.

$\therefore$ the square root $=2(a+b)-(a-b)$

$=a+3b$.

Find the square root of :

16. $4(x^2+1)^2+4(x^2+1)+1.$

17. $(3a^2+1)^2+14(3a^2+1)+49.$

18. $\left(\frac{x}{y}-2\right)^2+4\left(\frac{x}{y}-2\right)+4.$

19. $4(a+b)^2-12(a^2-b^2)+9(a-b)^2.$

20. $(a^4+1)^2-(a^4-1)^2$ 21. $\left(\frac{a+b}{a-b}\right)^2-2+\left(\frac{a-b}{a+b}\right)^2.$

22. $9(a+b)^2+6(a+b)(x+y)+x^2+2xy+y^2.$

Example 4. Find the square root of

$$x^2+\frac{1}{x^2}+12\left(x+\frac{1}{x}\right)+38.$$

The expression

$$=\left(x^2+2+\frac{1}{x^2}\right)+2.6\left(x+\frac{1}{x}\right)+36\ [\because 38=2+36.]$$

$$=\left(x+\frac{1}{x}\right)^2+2.6\left(x+\frac{1}{x}\right)+6^2=\left\{\left(x+\frac{1}{x}\right)+6\right\}^2$$

$\therefore$ the square root $=x+6+\dfrac{1}{x}.$

Example 5. Find the square root of

$$\left(x^2+\frac{1}{x^2}\right)^2-6\left(x+\frac{1}{x}\right)^2+21.$$

$$\text{The expression}=\left(x^2+\frac{1}{x^2}\right)^2-6\left(x^2+\frac{1}{x^2}+2\right)+21$$

$$=\left(x^2+\frac{1}{x^2}\right)^2-6\left(x^2+\frac{1}{x^2}\right)-12+21$$

$$=\left(x^2+\frac{1}{x^2}\right)^2-6\left(x^2+\frac{1}{x^2}\right)+9$$

$$=\left(x^2+\frac{1}{x^2}\right)^2-2.3\left(x^2+\frac{1}{x^2}\right)+3^2$$

$$=\left\{\left(x^2+\frac{1}{x^2}\right)-3\right\}^2$$

$\therefore$ the sq. root $=x^2-3+\dfrac{1}{x^2}.$

Find the square root of:

23. $x^2+\frac{1}{x^2}+8\left(x+\frac{1}{x}\right)+18.$

24. $x^2+\frac{1}{x^2}+4\left(x+\frac{1}{x}\right)+6.$

25. $x^4+\frac{1}{x^4}+2\left(x^2+\frac{1}{x^2}\right)+3.$

26. $x^2+\frac{1}{x^2}+2\left(x-\frac{1}{x}\right)-1.$

27. $\left(x+\frac{1}{x}\right)^2-4\left(x-\frac{1}{x}\right).$

28. $\left(x^2+\frac{1}{x^2}\right)^2-4\left(x+\frac{1}{x}\right)^2+12.$

29. $\left(x-\frac{1}{x}\right)^2+4\left(x+\frac{1}{x}\right)+8.$

30. $\left(x^2+\frac{1}{x^2}\right)^2-4\left(x-\frac{1}{x}\right)^2-4.$

Example 6. Find the square root of

$$\frac{y^2}{4}-xy+x^2-4x+4+2y.$$

Arranging the terms according to descending powers of x, we have

$$x^2-4x-xy+4+2y+\frac{y^2}{4}$$

$$=x^2-2x\left(2+\frac{y}{2}\right)+\left(2+\frac{y}{2}\right)^2=\left\{x-\left(2+\frac{y}{2}\right)\right\}^2$$

$\therefore$ the required square root $=x-2-\frac{y}{2}$.

Example 7. Find the square root of

$$x^4-6x^3+19x^2-30x+25.$$

Here, break up $19x^2$ into two parts, such that the first two terms and one of them may be a perfect square.

Thus, the expression

$$=x^4-6x^3+9x^2+10x^2-30x+25$$

$$=(x^2-3x)^2+2.5(x^2-3x)+5^2=\{(x^2-3x)+5\}^2$$

$\therefore$ the required square root $=x^2-3x+5$.

Find the square root of :

31. $x^4-4x^3+6x^2-4x+1$.

32. $4x^4+12x^3-11x^2-30x+25$.

33. $x^4+2x^3y+3x^2y^2+2xy^3+y^4$.

34. $9x^4-30x^3y+49x^2y^2-40xy^3+16y^4$.

35. $\dfrac{x^2}{y^2}+\dfrac{y^2}{x^2}+3+\dfrac{2x}{y}+\dfrac{2y}{x}$.

36. $x^2+\dfrac{y^2}{16}+z^2+\dfrac{xy}{2}-2xz-\dfrac{yz}{2}$.

37. $\dfrac{x^2}{y^2}+\dfrac{2x}{z}+\dfrac{y^2}{z^2}+\dfrac{2y}{x}+\dfrac{z^2}{x^2}+\dfrac{2z}{y}$.

38. $(ab+bc+ca)^2-4abc(a+c)$.

[*Hint.* $4abc(a+c)=4ac(ab+bc)$. Put x for $(ab+bc)$.]

39. $(2x^2+5xy+2y^2)(6x^2+5xy+y^2)(3x^2+7xy+2y^2)$.

40. $\left(\dfrac{y+z}{6}\right)^2+\dfrac{x(x-y-z)}{9}$.

[*Hint.* From the second fraction take $\dfrac{x^2}{9}$.]

41. $x^2+2x+1+\dfrac{2}{x-1}+\dfrac{1}{x^4-2x^2+1}$.

42. $(a-b)^4-2(a^2+b^2)(a-b)^2+2(a^4+b^4)$.

Find the fourth root of :

43. $x^4-4x^3+6x^2-4x+1$.

44. Shew that $(x+1)(x+2)(x+3)(x+4)+1$ is a perfect square.

45. Shew that $(x+3)(x+4)(x+5)(x+6)+1$ is a perfect square.

Example 8. Find the square root of $4^m+9^n-2^{m+1}.3^n$.

The expression $=(2^2)^m+(3^2)^n-2.2^m.3^n$

$$=(2^m)^2+(3^n)^2-2.2^m.3^n$$
$$=(2^m-3^n)^2$$

$\therefore$ the sq. root $=2^m-3^n$.

Find the square root of :

46. $x^{2m}-4x^{m+n}+4x^{2n}$.

47. $16^m+9^n+2^{2m+1}.3^n$.

48. $25x^{2m}-20x^{m+n}+4x^{2n}$.

5. General rule for finding the square root.

So far we have been extracting the square root of Algebraic expressions by factors, but there are expressions whose square roots cannot be conveniently determined by factors. Thus we stand in need of a method which is applicable in all cases.

Such a method is based upon the following formulæ :

$$(a+b)^2=a^2+(2a+b)b \quad \ldots \quad \ldots \quad \ldots \quad \text{(i)}$$

$$(a+b+c)^2=a^2+(2a+b)b+(2a+2b+c)c \quad \ldots \quad \text{(ii)}$$

$$(a+b+c+d)^2=a^2+(2a+b)b+(2a+2b+c)c$$
$$+(2a+2b+2c+d)d \quad \ldots \quad \text{(iii)}$$

Suppose we have to find the square root of $a^2+2ab+b^2$.

Here, the first term of the sq. root is obviously $=\sqrt{a^2}=a$.

If a^2, or the square of the first term of the sq. root, is subtracted from the given expression, the remainder $=2ab+b^2$.

The second term of the square root (*i.e.*, b) can be derived from the second term of the expression by dividing it by twice the first term of the square root.

Now this term must be such as to give us $2ab+b^2$ when multiplied by $(2a+b)$, *i.e.*, twice the first term of the square root plus the second term.

As it is so, therefore b is the second term of the square root.

$\therefore$ $a+b$ is the square root of $a^2+2ab+b^2$.

The process explained above is abbreviated below :

Steps of process.

(i) The first term of the sq. root $= \sqrt{a^2} = a$.

(ii) The square of the first term of the sq. root $= a^2$.

(iii) When a^2 is subtracted from the given expression, the remainder $= +2ab+b^2$.

(iv) The second term of the sq. root $= +2ab \div 2a = +b$.

(v) $(+2a+b)b = +2ab+b^2$.

i.e., the 2nd term of the sq. root multiplied by the sum of twice the first term and the second term is equal to the remainder.

Abbreviated process.

$$\begin{array}{l|l}
a & a^2+2ab+b^2\,(a+b \\
 & a^2 \\ \hline
2a+b & +2ab+b^2 \\
 & +2ab+b^2 \\ \hline
 & \quad\times \qquad \times
\end{array}$$

Example 1. Find the square root of $9x^2-24xy+16y^2$.

First term $= \sqrt{9x^2} = 3x$.

2nd term $= -24xy \div 6x$
$= -4y$.

$$\begin{array}{l|l}
3x & 9x^2-24xy+16y^2\,(3x-4y \\
 & 9x^2 \\ \hline
6x-4y & -24xy+16y^2 \\
 & -24xy+16y^2 \\ \hline
 & \quad\times \qquad \times
\end{array}$$

If we have to find the square root of a multinomial, first we arrange it in descending powers of one of its letters and apply formula (i) or (ii), as the case may be. We find the first two terms of the square root, as explained in example 1. To find the third term, we divide the highest term of the second remainder by twice the first term of the square root already obtained; the quotient thus got is the third term of the square root, provided it would satisfy the condition given below:

(twice 1st term + twice 2nd term + 3rd term) $\times$ 3rd term
= the second remainder.

Example 2. Find the square root of $x^4-6x^3+4-12x+13x^2$.

The expression when arranged in descending powers of x is $x^4-6x^3+13x^2-12x+4$.

$$
\begin{array}{r|l}
 & x^4-6x^3+13x^2-12x+4\,(\,x^2-3x+2 \\
x^2 & x^4 \\
\hline
2x^2-3x & -6x^3+13x^2-12x+4 \\
 & -6x^3+9x^2 \\
\hline
2x^2-6x+2 & \quad +4x^2-12x+4 \\
 & \quad +4x^2-12x+4 \\
\hline
 & \quad \times \quad \times \quad \times
\end{array}
$$

(i) Here the 1st term $=\sqrt{x^4}=x^2$.

(ii) When $(x^2)^2$ or x^4 is subtracted from the expression, the first remainder $=-6x^3+13x^2-12x+4$.

(iii) $-6x^3 \div$ *twice* $x^2=-3x$, which is the 2nd term of the square root.

(iv) Multiply (*twice* x^2-3x) by $-3x$ and subtract the product from the first remainder; the result $+4x^2-12x+4$ is the second remainder.

(v) $+4x^2 \div$ *twice* $x^2=2$, which is the 3rd term of the square root.

(vi) Multiply [*twice* x^2+ *twice* $(-3x)+2$] by 2; the product is equal to the second remainder.

$\therefore$ x^2-3x+2 is the required sq. root.

Example 3. Find the square root of

$$x^4+\frac{4}{x^2}-2+4x-x^3+\frac{x^2}{4}.$$

In arranging the terms in descending powers of x, -2 must be placed before $\frac{4}{x^2}$.

Arranging the terms in descending powers of x and proceeding as usual, we have

$$x^4 - x^3 + \frac{x^2}{4} + 4x - 2 + \frac{4}{x^2}\left(x^2 - \frac{x}{2} + \frac{2}{x}\right.$$

$$\begin{array}{l|l}
x^2 & x^4 \\
\hline
2x^2 - \frac{x}{2} & -x^3 + \frac{x^2}{4} + 4x - 2 + \frac{4}{x^2} \\
 & -x^3 + \frac{x^2}{4} \\
\hline
2x^2 - x + \frac{2}{x} & +4x - 2 + \frac{4}{x^2} \\
 & +4x - 2 + \frac{4}{x^2} \\
\hline
 & \times \quad \times \quad \times
\end{array}$$

$\therefore$ the square root $= x^2 - \frac{x}{2} + \frac{2}{x}$.

Example 4. Find the square root of

$$x^{\frac{3}{2}} + xy^{-\frac{1}{2}} - 2x^{\frac{5}{4}}y^{-\frac{1}{4}} - 2x^{\frac{1}{2}}y^{\frac{1}{4}} + 2x^{\frac{3}{4}}y^{\frac{1}{2}} + y.$$

Arranging the terms according to descending powers of x and proceeding as usual, we have

$$x^{\frac{3}{4}} - x^{\frac{1}{2}}y^{-\frac{1}{4}} + y^{\frac{1}{2}}$$

$$\begin{array}{l|l}
x^{\frac{3}{4}} & x^{\frac{3}{2}} - 2x^{\frac{5}{4}}y^{-\frac{1}{4}} + xy^{-\frac{1}{2}} + 2x^{\frac{3}{4}}y^{\frac{1}{2}} - 2x^{\frac{1}{2}}y^{\frac{1}{4}} + y \\
 & x^{\frac{3}{2}} \\
\hline
2x^{\frac{3}{4}} - x^{\frac{1}{2}}y^{-\frac{1}{4}} & -2x^{\frac{5}{4}}y^{-\frac{1}{4}} + xy^{-\frac{1}{2}} \\
 & -2x^{\frac{5}{4}}y^{-\frac{1}{4}} + xy^{-\frac{1}{2}} \\
\hline
2x^{\frac{3}{4}} - 2x^{\frac{1}{2}}y^{-\frac{1}{4}} + y^{\frac{1}{2}} & +2x^{\frac{3}{4}}y^{\frac{1}{2}} - 2x^{\frac{1}{2}}y^{\frac{1}{4}} + y \\
 & +2x^{\frac{3}{4}}y^{\frac{1}{2}} - 2x^{\frac{1}{2}}y^{\frac{1}{4}} + y \\
\hline
 & \times \quad \times \quad \times
\end{array}$$

$\therefore$ the square root $= x^{\frac{3}{4}} - x^{\frac{1}{2}}y^{-\frac{1}{4}} + y^{\frac{1}{2}}$.

EXERCISE 78.

Find the square root of:

1. $16x^2 - 56xy + 49y^2$.
2. $x^4 + 6x^3 + 15x^2 + 18x$
3. $16x^4 - 24x^3 + 25x^2 - 12x + 4$.
4. $x^4 - 2ax^3 + 5a^2x^2 - 4a^3x + 4a^4$
5. $x^6 - 12x^5 + 60x^4 - 160x^3 + 240x^2 - 192x + 64$.

6. $a^2x^4+4ax^3+2(3a^2+2)x^2+12ax+9a^2$.

7. $\dfrac{x^4}{64}+\dfrac{x^3}{8}-x+1$. 8. $9x^4-18x^3+9x+\dfrac{9}{4}$.

9. $\frac{4}{9}x^4+x^3+\frac{59}{48}x^2+\frac{3}{4}x+\frac{1}{4}$.

10. $9x^2-24x+19-\dfrac{4}{x}+\dfrac{1}{4x^2}$. 11. $x^4-2x^3+\dfrac{3x^2}{2}-\dfrac{x}{2}+\dfrac{1}{16}$.

12. $\frac{25}{4}+16a^2b^2+9a^4b^4-5ab-6a^3b^3$.

13. $\dfrac{a^2}{4b^2}-\dfrac{a}{b}+\dfrac{4b^2}{a^2}-1+\dfrac{4b}{a}$. 14. $\dfrac{x^2}{y^2}+\dfrac{y^2}{x^2}-\dfrac{x}{y}+\dfrac{y}{x}-\dfrac{7}{4}$.

15. $\dfrac{9x^2}{4y^2}+\dfrac{5}{2}+\dfrac{y^2}{4x^2}-\dfrac{3x}{y}-\dfrac{y}{x}$.

16. $4x^4+9\left(x^2+\dfrac{1}{x^2}\right)+12x^2\left(x+\dfrac{1}{x}\right)+18$.

17. $a^6+\dfrac{1}{4a^6}+2-2\left(a^3+\dfrac{1}{2a^3}\right)$.

18. $x^{2m+2n}+x^{2m-2n}+\frac{1}{4}+2x^{2m}-x^{m+n}-x^{m-n}$.

19. $\frac{9}{4}x^{2m+2}+4x^{2m-2}+1-6x^{2m}+3x^{m+1}-4x^{m-1}$.

20. $\dfrac{x^4}{y^4}+\dfrac{y^4}{x^4}-4\left(\dfrac{x^2}{y^2}+\dfrac{y^2}{x^2}\right)+6$. 21. $(x+y)^4+x^4+y^4$

22. $16x^{-\frac{3}{2}}+25+4x^{\frac{3}{2}}-12x^{\frac{3}{4}}-24x^{-\frac{3}{4}}$.

23. $\frac{1}{4}+\frac{1}{16}x^3+\frac{1}{3}x^{\frac{1}{2}}+\frac{1}{9}x-\frac{1}{6}x^2-\frac{1}{4}x^{\frac{3}{2}}$.

24. $x^2y^{-2}+2xy^{-1}+x^{-2}y^2+3+2x^{-1}y$.

25. $a^{\frac{4}{3}}+a^{-\frac{2}{3}}+1+2a^{\frac{1}{3}}+2a^{\frac{2}{3}}+2a^{-\frac{1}{3}}$.

Find the fourth root of :

26. $1-4x+6x^2-4x^3+x^4$.

27. $16x^4-96x^3y+216x^2y^2-216xy^3+81y^4$.

Example 5. Find the square root of $1.10^4+4.10^3+6.10^2+4.10+1$, taking 10 as the variable.

$$
\begin{array}{rl}
 & 1.10^2+2.10\ \ +1 \\
\cline{2-2}
 & 1.10^4+4.10^3+6.10^2+4.10+1\,(\\
1.10^2\times 1.10^2= & 1.10^4 \\
\cline{2-2}
 & \qquad +4.10^3+6.10^2 \\
(2.10^2+2.10)\times 2.10= & \qquad \underline{+4.10^3+4.10^2} \\
 & \qquad\qquad +2.10^2+4.10+1 \\
(2.10^2+4.10+1)\times 1= & \qquad\qquad \underline{+2.10^2+4.10+1} \\
 & \qquad\qquad \times \qquad \times \qquad \times
\end{array}
$$

The above process can be reduced to the arithmetical form by omitting the superfluous powers of 10, as shown below

$$
\begin{array}{r|l}
 & \quad 1\ \ 2\ \ 1 \\
 & \overline{1\,4\,6\,4\,1} \\
1\times 1= & 1 \\ \hline
 & \ \ 4\,6 \\
(20+2)\times 2= & \ \ 4\,4 \\ \hline
 & \quad 2\,4\,1 \\
(240+1)\times 1= & \quad 2\,4\,1 \\ \hline
 & \quad * * *
\end{array}
$$

28. Find the square root of $4x^4+8x^3+8x^2+4x+1$, and *hence* extract the square root of 48841.

29. Find the square root of $x^4+2x^3+7x^2+6x+9$, and *hence* extract the square root of 12769.

Example 6. Find the square root of $1-4x$ up to 4 terms.

$$
\begin{array}{l|l}
 & 1-4x\ (\ 1-2x-2x^2-4x^3 \\
1 & 1 \\ \hline
 & -4x \\
2-2x & -4x+4x^2 \\ \hline
 & -4x^2 \\
2-4x-2x^2 & -4x^2+8x^34+x^4 \\ \hline
 & -8x^3-4x^4 \\
2-4x-4x^2-4x^3 & -8x^3+16x^4+16x^5+16x^6. \\ \hline
\end{array}
$$

$\therefore$ the square root up to 4 terms $=1-2x-2x^2-4x^3$.

Find up to 4 terms the square root of :

30. a^2+x^2. 31. $1+x-x^2$.

Example 7. Find (i) what may be added to, (ii) what may be subtracted from, or (iii) what value may be given to m, so that the expression $16x^4-16x^3-20x^2+12x+m$ may be a perfect square ?

$$
\begin{array}{l|l}
 & 16x^4-16x^3-20x^2+12x+m\ (\ 4x^2-2x-3 \\
4x^2 & 16x^4 \\ \hline
 & -16x^3-20x^2 \\
8x^2-2x & -16x^3+\ \ 4x^2 \\ \hline
 & -24x^2+12x+m \\
8x^2-4x-3 & -24x^2+12x+9 \\ \hline
 & \qquad\qquad m-9
\end{array}
$$

Obviously, the given expression will be a perfect square

(i) if $-m+9$ is added to it, or

(ii) if $m-9$ is subtracted from it, or

(iii) if $m-9=0$, or $m=9$.

32. What may be added to, or subtracted from $9x^4+12x^3+7x^2+2x+a$, so that it may be a perfect square ?

33. For what value of x will $x^4-12x^3+38x^2-9x-5$ be a perfect square ?

34. Find the condition so that x^2+px+q may be a perfect square.

*35. Find the values of a and b for which $x^4+4x^3+10x^2+ax+b$ is a perfect square.

CUBE ROOT

6. The cube root of an expression is found out by the application of the formulæ:

$$(a+b)^3=a^3+3a^2b+3ab^2+b^3 \quad \text{... (i)}$$
$$(a-b)^3=a^3-3a^2b+3ab^2-b^3 \quad \text{... (ii)}$$

Example. Find the cube root of $x^3-6x^2+12x-8$.

Reducing the expression to the standard form, we have $x^3-6x^2+12x-8=(x)^3-3(2)(x)^2+3(2)^2(x)-(2)^3=(x-2)^3$.

$\therefore$ the cube root $=x-2$.

EXERCISE 79.

Find by formulæ the cube root of the following:

1. $x^3+12x^2+48x+64$.
2. $x^3-9x^2+27x-27$.
3. $8x^3-12x+\dfrac{6}{x}-\dfrac{1}{x^3}$.
4. $27x^3-27x^2y+9xy^2-y^3$.

CHAPTER XVIII

SURDS

1. When a root of a number *cannot be exactly obtained*, it is called an **irrational quantity, or a surd.** Thus, $\sqrt{2}$, $\sqrt[3]{4}$, $\sqrt{\frac{3}{5}}$ are all surds.

If a root of an Algebraical expression cannot be denoted without the use of a fractional index, it is called an **irrational quantity** or a **surd.**

Thus $\quad \sqrt{x^2+xy+y^2}$, $\sqrt[3]{x^3+y^3}$, $\sqrt{\dfrac{x}{y}}$ are all surds.

In our mathematical work, we frequently meet with numbers and expressions which are in the *form of surds* but are *not really surds.*

Thus, $\sqrt{25}$ and $\sqrt{x^2+2xy+y^2}$ are in the surd form but they are not really surds; for $\sqrt{25}=5$ and $\sqrt{x^2+2xy+y^2}=x+y$.

A product partly rational and partly surd may be expressed as a *complete surd*, as illustrated below:

Example 1. $3\sqrt{5}=(3^2)^{\frac{1}{2}}.\,5^{\frac{1}{2}}$

$$=9^{\frac{1}{2}}.5^{\frac{1}{2}}=(9.5)^{\frac{1}{2}}$$
$$=\sqrt{9.5}=\sqrt{45}.$$

Example 2. $4\sqrt[3]{6}=(4^3)^{\frac{1}{3}}.6^{\frac{1}{3}}$

$$=64^{\frac{1}{3}}.6^{\frac{1}{3}}$$
$$=(64.6)^{\frac{1}{3}}$$
$$=\sqrt[3]{384}.$$

A surd may sometimes be expressed as the product of a rational quantity and a surd, as illustrated below:

Example 3. $\sqrt{50}=\sqrt{25\times 2}$

$$=(5^2\times 2)^{\frac{1}{2}}$$
$$=(5^2)^{\frac{1}{2}}\times 2^{\frac{1}{2}}$$
$$=5\sqrt{2}.$$

Example 4. $\sqrt[3]{56}=\sqrt[3]{8\times 7}=(2^3\times 7)^{\frac{1}{3}}$

$$=(2^3)^{\frac{1}{3}}\times 7^{\frac{1}{3}}=2\times 7^{\frac{1}{3}}$$
$$=2\sqrt[3]{7}.$$

EXERCISE 80.

Express as a complete surd:

1. $4\sqrt{6}$.	2. $5\sqrt[3]{4}$.	3. $2\sqrt[4]{6}$.
4. $3\sqrt[4]{5}$.	5. $2\sqrt[5]{3}$.	6. $3\sqrt[5]{4}$.
7. $x\sqrt[n]{y}$.	8. $x^2\sqrt[p]{y}$.	9. $x^3\sqrt[5]{y^2}$.

Simplify:

10. $\sqrt{48}$.	11. $\sqrt{63}$.	12. $\sqrt{44}$.
13. $\sqrt[3]{250}$.	14. $\sqrt[4]{567}$.	15. $\sqrt[5]{80}$.
16. $\sqrt[3]{x^6y}$.	17. $\sqrt[n]{x^{3n}.a}$.	18. $\sqrt[a]{x^{4a}y^2}$.
19. $\sqrt[3]{-5120}$.	20. $\sqrt[3]{250x^7y^4}$.	

Surds are said to be **similar** when they have, or can be reduced so as to have the same irrational factors.

Thus, $3\sqrt{5}$, $7\sqrt{5}$ are similar surds, and $3\sqrt{125}$ and $7\sqrt{5}$ are also similar surds, for $3\sqrt{125}=15\sqrt{5}$.

The reduction of surds, to similar surds is necessary for the processes of addition and subtraction, as illustrated below

Example 1. $\sqrt[3]{128}+\sqrt[3]{54}=\sqrt[3]{64\times 2}+\sqrt[3]{27\times 2}$

$$=\sqrt[3]{4^3\times 2}+\sqrt[3]{3^3\times 2}$$
$$=4\sqrt[3]{2}+3\sqrt[3]{2}$$
$$=\sqrt[3]{2}(4+3)=7\sqrt[3]{2}.$$

Example 2. $\sqrt[4]{768}-\sqrt[4]{243}=\sqrt[4]{256\times 3}-\sqrt[4]{81\times 3}$

$$=\sqrt[4]{4^4\times 3}-\sqrt[4]{3^4\times 3}$$
$$=4\sqrt[4]{3}-3\sqrt[4]{3}$$
$$=\sqrt[4]{3}(4-3)$$
$$=\sqrt[4]{3}.$$

Surds are said to be of the **same order** when they have all got the same root-symbol. Thus, $\sqrt{7}$, $\sqrt{x-y}$, $\sqrt{5^3}$, $\sqrt{a^5}$ are all surds of the same (the *second*) order.

$\sqrt[3]{2}$, $\sqrt[3]{5}$, $\sqrt[3]{a^2}$, $\sqrt[3]{x+y}$ are all surds of the *third* order.

$\sqrt[n]{a}$, $\sqrt[n]{x^2}$, $\sqrt[n]{x^2-y^2}$ are all surds of the nth order.

Surds of the second order are called **quadratic surds** and of the third order are called **cubic surds.**

Example 3. Express $\sqrt{a^3}$, $\sqrt[3]{a^4}$ and $\sqrt[4]{a^5}$ as surds of the same order.

The given surds are of the 2nd, 3rd and 4th orders. The required order will be indicated by the L.C.M. of 2, 3, 4, or it will be of the 12th order. Thus,

$$\sqrt{a^3}=a^{\frac{3}{2}}=a^{\frac{18}{12}}=\sqrt[12]{a^{18}},$$

$$\sqrt[3]{a^4}=a^{\frac{4}{3}}=a^{\frac{16}{12}}=\sqrt[12]{a^{16}},$$

$$\text{and } \sqrt[4]{a^5}=a^{\frac{5}{4}}=a^{\frac{15}{12}}=\sqrt[12]{a^{15}}.$$

Hence the required surds are

$$\sqrt[12]{a^{18}}.\ \sqrt[12]{a^{16}} \text{ and } \sqrt[12]{a^{15}}.$$

Example 4. Which is greater $\sqrt[4]{5}$ or $\sqrt[3]{4}$?

The common order being 12^{th}

$$\sqrt[4]{5}=5^{\frac{1}{4}}=5^{\frac{3}{12}}=\sqrt[12]{5^3}=\sqrt[12]{125}$$

$$\text{and} \quad \sqrt[3]{4}=4^{\frac{1}{3}}=4^{\frac{4}{12}}=\sqrt[12]{4^4}=\sqrt[12]{256};$$

$$\therefore \quad \sqrt[3]{4} \text{ is greater than } \sqrt[4]{5}.$$

Example 5. Arrange $\sqrt{2}$, $\sqrt[3]{4}$ and $\sqrt[4]{6}$ in descending order of magnitude.

The common order being 12^{th}

$$\sqrt{2}=2^{\frac{1}{2}}=2^{\frac{6}{12}}=\sqrt[12]{2^6}=\sqrt[12]{64},$$

$$\sqrt[3]{4}=4^{\frac{1}{3}}=4^{\frac{4}{12}}=\sqrt[12]{4^4}=\sqrt[12]{256},$$

$$\sqrt[4]{6}=6^{\frac{1}{4}}=6^{\frac{3}{12}}=\sqrt[12]{6^3}=\sqrt[12]{216}.$$

Hence the required arrangement is $\sqrt[3]{4}$, $\sqrt[4]{6}$ and $\sqrt{2}$.

EXERCISE 81.

Reduce to the simplest form:

1. $4\sqrt{5}+3\sqrt{20}-\sqrt{45}$.
2. $\sqrt{72}-2\sqrt{32}+\sqrt{8}$.
3. $\sqrt{12}-\sqrt{27}+\sqrt{75}$.
4. $2\sqrt[3]{4}+5\sqrt[3]{32}-\sqrt[3]{108}$.
5. $2\sqrt[3]{40}+3\sqrt[3]{625}-4\sqrt[3]{320}$.
6. $4\sqrt[3]{-27}+\sqrt[3]{-8}+5\sqrt[3]{-64}$.
7. $\sqrt{20x^3}+\sqrt{80x^3}-45\sqrt{x^2y}$.
8. $\sqrt{a^2x^3}-\sqrt{a^4x}-\sqrt{9a^4x}$.
9. $3\sqrt[3]{8a^4}-\sqrt[3]{125a^4}+\sqrt[3]{27a^4}$.

Reduce to surds of the same order:

10. $\sqrt{6}, \sqrt[4]{3}, \sqrt[3]{2}$.
11. $\sqrt[3]{2}, \sqrt[3]{4}, \sqrt[6]{8}$.
12. $\sqrt{x^3}, \sqrt[3]{x^2}, \sqrt[4]{x}$.

Which is greater:

13. $\sqrt[3]{5}$ or $\sqrt[4]{6}$?
14. $\sqrt{2}$ or $\sqrt[3]{3}$?
15. $\sqrt[3]{4}$ or $\sqrt[4]{5}$?
16. $\sqrt[3]{6}$ or $\sqrt[4]{10}$?

Arrange according to descending order of magnitude:

17. $\sqrt{2}, \sqrt[3]{4}, \sqrt[4]{5}$.
18. $\sqrt{3}, \sqrt[3]{5}, \sqrt[4]{6}$.
19. $2\sqrt{3}, 3\sqrt{2}, \sqrt[5]{7}$.
20. $\sqrt[3]{6}, \sqrt[4]{8}, \sqrt[6]{12}$.

2. Multiplication and division of surds.

Case I. When the surds are of the *same* order.

Example 1. $\sqrt[5]{8}\times\sqrt[5]{15}=8^{\frac{1}{5}}\times 15^{\frac{1}{5}}=(8\times 15)^{\frac{1}{5}}$

$$=(120)^{\frac{1}{5}}=\sqrt[5]{120}.$$

Example 2. $\sqrt[3]{10}\div\sqrt[3]{4}=10^{\frac{1}{3}}\div 4^{\frac{1}{3}}$

$$=\left(\frac{10}{4}\right)^{\frac{1}{3}}=\sqrt[3]{\frac{5}{2}}.$$

Case II. When the surds are of *different* orders.

Example 3. $\sqrt[6]{8}\times\sqrt[4]{3}=8^{\frac{1}{6}}\times 3^{\frac{1}{4}}=8^{\frac{2}{12}}\times 3^{\frac{3}{12}}$

$$=(8^2)^{\frac{1}{12}}\times(3^3)^{\frac{1}{12}}=(8^2\times 3^3)^{\frac{1}{12}}$$

$$=(64\times 27)^{\frac{1}{12}}=\sqrt[12]{1728}.$$

Example 4. $\sqrt[9]{8} \div \sqrt[6]{6} = 8^{\frac{1}{9}} \div 6^{\frac{1}{6}}$

$$= 8^{\frac{2}{18}} \div 6^{\frac{3}{18}}$$
$$= (8^2)^{\frac{1}{18}} \div (6^3)^{\frac{1}{18}}$$
$$= (64)^{\frac{1}{18}} \div (216)^{\frac{1}{18}}$$
$$= \left(\frac{64}{216}\right)^{\frac{1}{18}} = \sqrt[18]{\frac{8}{27}}.$$

Case III. When the given surds have the *same* quantity under the radical sign.

Example 5. $\sqrt[4]{3} \times \sqrt[6]{3} = 3^{\frac{1}{4}} \times 3^{\frac{1}{6}}$

$$= 3^{\frac{1}{4}+\frac{1}{6}} = 3^{\frac{5}{12}}$$
$$= \sqrt[12]{3^5} = \sqrt[12]{243}.$$

Example 6. $\sqrt[3]{5} \div \sqrt[4]{5} = 5^{\frac{1}{3}} \div 5^{\frac{1}{4}}$

$$= 5^{\frac{1}{3}-\frac{1}{4}} = 5^{\frac{1}{12}}$$
$$= \sqrt[12]{5}.$$

Case IV. When simplification of surds is a necessary step before multiplication or division.

Example 7. $4.\sqrt{72} \times 2.\sqrt{45}$

$$= 4.\sqrt{6^2.2} \times 2.\sqrt{3^2.5}$$
$$= 4.6.\sqrt{2} \times 2.3\sqrt{5}$$
$$= 24\sqrt{2} \times 6\sqrt{5}$$
$$= 144\sqrt{2} \times \sqrt{5}$$
$$= 144\sqrt{10}.$$

Example 8. $6\sqrt{75} \div 2\sqrt{63}$

$$= 6\sqrt{5^2 \times 3} \div 2.\sqrt{3^2 \times 7}$$
$$= 6.5\sqrt{3} \div 2.3\sqrt{7}$$
$$= 30\sqrt{3} \div 6\sqrt{7}$$
$$= \frac{30\sqrt{3}}{6\sqrt{7}} = 5\sqrt{\tfrac{3}{7}}.$$

EXERCISE 82.

Simplify :—

1. $\sqrt[3]{7}\times\sqrt[3]{9}$.
2. $\sqrt[4]{5}\times\sqrt[4]{12}$.
3. $\sqrt[5]{8}\times\sqrt[5]{18}$.
4. $\sqrt[5]{12}\div\sqrt[5]{8}$.
5. $\sqrt[6]{16}\div\sqrt[6]{12}$.
6. $\sqrt[4]{20}\div\sqrt[4]{15}$.
7. $\sqrt[3]{6}\times\sqrt[4]{9}$.
8. $\sqrt{15}\times\sqrt[3]{12}$.
9. $\sqrt[3]{21}\times\sqrt[5]{6}$.
10. $\sqrt[3]{20}\div\sqrt[4]{15}$.
11. $\sqrt[5]{18}\div\sqrt[6]{24}$.
12. $\sqrt[4]{48}\div\sqrt[3]{72}$.
13. $\sqrt{36}\times\sqrt{54}$.
14. $\sqrt{96}\times\sqrt{88}$.
15. $\sqrt[3]{128}\times\sqrt[3]{54}$.
16. $5\sqrt[3]{49}\times\sqrt[3]{14}$.
17. $\sqrt{18}\div\sqrt{200}$.
18. $5\sqrt[3]{6}\div3\sqrt{10}$.
19. $\sqrt{3}.\sqrt[3]{2}.\sqrt[6]{4}$.
20. $\sqrt{8}.\sqrt[4]{3}.\sqrt[3]{4}$.
21. $\sqrt{2}.\sqrt[3]{3}.\sqrt[6]{5}$.
22. $5\sqrt{12}\div10\sqrt{27}$.
23. $\sqrt[3]{56}\div\sqrt[3]{48}$.
24. $x\sqrt[3]{x^2y^5}$.
25. $2\sqrt{24}\times3\sqrt[4]{18}\times4\sqrt[6]{24}$.

§ 3. **Compound Surds.** An expression consisting of two or more simple surds connected by the signs + or − is called **a compound surd.** Thus. $6\sqrt[3]{7}$ and $2\sqrt[5]{8}$ being simple surds, $6\sqrt[3]{7}+2\sqrt[5]{8}$ and $6\sqrt[3]{7}-2\sqrt[5]{8}$ are compound surds.

The process of multiplication of compound surds is similar to that of compound algebraic expressions.

Example 1. Multiply $4\sqrt{3}+3\sqrt{2}$ by $2\sqrt{5}-5\sqrt{3}$.

$$(4\sqrt{3}+3\sqrt{2})(2\sqrt{5}-5\sqrt{3}).$$
$$=4\sqrt{3}\times2\sqrt{5}-4\sqrt{3}\times5\sqrt{3}+3\sqrt{2}\times2\sqrt{5}-3\sqrt{2}\times5\sqrt{3}.$$
$$=8\sqrt{15}-60+6\sqrt{10}-15\sqrt{6}.$$

Example 2. Find the square of $(5\sqrt{3}-6\sqrt{2})$.

$$(5\sqrt{3}-6\sqrt{2})^2=(5\sqrt{3})^2-2.5\sqrt{3}.6\sqrt{2}+(6\sqrt{2})^2$$
$$=75-60\sqrt{6}+72$$
$$=147-60\sqrt{6}.$$

EXERCISE 83.

Multiply :—

1. $\sqrt{2}+\sqrt{3}$ by $\sqrt{5}+\sqrt{7}$.
2. $\sqrt{5}+\sqrt{6}$ by $\sqrt{3}-\sqrt{2}$.
3. $4\sqrt{2}-5\sqrt{3}$ by $2\sqrt{2}+3\sqrt{3}$.
4. $3\sqrt{5}+6\sqrt{3}$ by $4\sqrt{7}-2\sqrt{5}$.
5. $6-2\sqrt{6}$ by $5-2\sqrt{6}$.
6. $5\sqrt{2}-2\sqrt{3}$ by $5\sqrt{2}-2\sqrt{3}$.

7. $6\sqrt{3}+3\sqrt{5}$ by $6\sqrt{3}-3\sqrt{5}$.

8. $\sqrt{5}+\sqrt{3}+\sqrt{6}$ by $\sqrt{5}+\sqrt{3}-\sqrt{6}$.

9. $\sqrt[3]{4}+\sqrt[3]{9}+\sqrt[3]{36}$ by $\sqrt[3]{2}+\sqrt[3]{3}$.

Find the continued product of :

10. $(2-\sqrt{3}), (\sqrt{3}+2), (3-\sqrt{2})$.

11. $(4+\sqrt{6}), (\sqrt{3}+1), (\sqrt{6}-\sqrt{3})$.

12. $(1+\sqrt{2}+\sqrt{3}), (1+\sqrt{2}-\sqrt{3}), (1-\sqrt{2}+\sqrt{3}),$
$(-1+\sqrt{2}+\sqrt{3})$.

Find the square of :

13. $2\sqrt{6}+3\sqrt{5}$.

14. $3\sqrt{7}-4\sqrt{6}$.

15. $3\sqrt{3}-4$.

16. $x-\sqrt{x^2-1}$.

17. $a^2+\sqrt{a^2-b^2}$.

18. $\sqrt{a^2+2b^2}-\sqrt{a^2-2b^2}$.

19. $2\sqrt{8x-1}+3\sqrt{1-4x}$.

20. $3\sqrt{a^2+b^2}+4\sqrt{a^2-b^2}$.

21. Find the cube of $1+2\sqrt{x}$.

4. **Rationalising Factor.** When the product of two irrational expressions or surds is rational, each is said to be the **rationalising factor** of the other.

Thus $\sqrt[3]{5}$ is the rationalising factor of $\sqrt[3]{25}$ and $\sqrt[3]{25}$ is the rationalising factor of $\sqrt[3]{5}$ for $\sqrt[3]{25}\times\sqrt[3]{5}=\sqrt[3]{125}=5$, and $(\sqrt{5}+\sqrt{3})$ is the rationalising factor of $(\sqrt{5}-\sqrt{3})$ and $(\sqrt{5}-\sqrt{3})$ is the rationalising factor of $(\sqrt{5}+\sqrt{3})$, for $(\sqrt{5}+\sqrt{3})(\sqrt{5}-\sqrt{3})=5-3=2$.

The rationalising factor of a monomial surd can be found easily, as illustrated below :

The rationalising factor of $\sqrt[3]{a^2}$ or $a^{\frac{2}{3}}$ is obviously $a^{\frac{1}{3}}$ or $\sqrt[3]{a}$, for $\sqrt[3]{a^2}\times\sqrt[3]{a}=a^{\frac{2}{3}}\times a^{\frac{1}{3}}=a^{\frac{2}{3}+\frac{1}{3}}=a$.

Similarly, the rationalising factor of $\sqrt[3]{a^2}.\ \sqrt{b}.\ \sqrt[3]{c}$ or $a^{\frac{2}{3}}b^{\frac{1}{2}}c^{\frac{1}{3}}$ is $a^{\frac{1}{3}}b^{\frac{1}{2}}c^{\frac{2}{3}}$ or $\sqrt[3]{a}.\ \sqrt{b}.\ \sqrt[3]{c^2}$, for $\sqrt[3]{a^2}.\ \sqrt{b}.\ \sqrt[3]{c}\times\sqrt[3]{a}.\ \sqrt{b}.\ \sqrt[3]{c^2}$

$$=a^{\frac{2}{3}}b^{\frac{1}{2}}c^{\frac{1}{3}}\times a^{\frac{1}{3}}b^{\frac{1}{2}}c^{\frac{2}{3}}$$
$$=a^{\frac{2}{3}+\frac{1}{3}}\ b^{\frac{1}{2}+\frac{1}{2}}\ c^{\frac{1}{3}+\frac{2}{3}}$$
$$=a.b.c.$$

From these and similar examples, we can establish the following rule for finding the rationalising factor of monomial surds:

Rule. (i) **Change the radical sign of the surd into the fractional index.**

(ii) Find the *least fraction* which when added to this index will make it an integer—such a fraction is the index of the rationalising factor.

Example 1. Find the rationalising factor of $7\sqrt[5]{2^3}.\sqrt[4]{3^5}.\sqrt[3]{4^2}$.

$7\sqrt[5]{2^3}\cdot\sqrt[4]{3^5}\cdot\sqrt[3]{4^2}=7.2^{\frac{3}{5}}.\,3^{\frac{5}{4}}.\,4^{\frac{2}{3}}$.

Since $\frac{3}{5}$ is the index of 2 and $\frac{3}{5}+\frac{2}{5}=1$,

$\therefore\ 2^{\frac{2}{5}}$ or $\sqrt[5]{2^2}$ is the rationalising factor of $\sqrt[5]{2^3}$.

Since $\frac{5}{4}$ is the index of 3 and $\frac{5}{4}+\frac{3}{4}=2$,

$\therefore\ 3^{\frac{3}{4}}$ or $\sqrt[4]{3^3}$ is the rationalising factor of $\sqrt[4]{3^5}$.

Since $\frac{2}{3}$ is the index of 4 and $\frac{2}{3}+\frac{1}{3}=1$,

$\therefore\ 4^{\frac{1}{3}}$ or $\sqrt[3]{4}$ is the rationalising factor of $\sqrt[3]{4^2}$.

Or the rationalising factor of $7.\sqrt[5]{2^3}.\sqrt[4]{3^5}.\sqrt[3]{4^2}$ is $\sqrt[5]{2^2}.\sqrt[4]{3^3}.\sqrt[3]{4}$.

The rationalising factor of a binomial quadratic surd can be found by merely changing the sign between the two terms, from $-$ into $+$ and from $+$ into $-$.

Example 2. Find the rationalising factor of $a\sqrt{x}+b\sqrt{y}$.

Since $(a\sqrt{x}+b\sqrt{y})(a\sqrt{x}-b\sqrt{y})=(a\sqrt{x})^2-(b\sqrt{y})^2$
$$=a^2x-b^2y,$$

$\therefore\ (a\sqrt{x}-b\sqrt{y})$ is the rationalising factor of
$$(a\sqrt{x}+b\sqrt{y}).$$

The rationalising factor of a binomial qua dratic surd is called its **conjugate**.

EXERCISE 84.

Write down *mentally* the rationalising factor of:

1. $2\sqrt{2}$, $\sqrt{45}$, $\sqrt{54}$, $\sqrt{a^3b}$, $\sqrt[3]{a}$, $\sqrt{x-1}$, $\sqrt[4]{a^2b^3}$
2. $\sqrt{3}.\sqrt{2}$. 3. $\sqrt[3]{4}.\sqrt{3}$. 4. $\sqrt{2}.\sqrt{3}.\sqrt[3]{6}$.

5. $\sqrt{3}+\sqrt{2}$. 6. $\sqrt{5}-\sqrt{3}$ 7. $\sqrt{a}-\sqrt{b}$.
8. $a+\sqrt{b}$. 9. $\sqrt{5}.\sqrt[3]{6}.\sqrt[4]{7^3}$. 10. $\sqrt[3]{5}.\sqrt[3]{6^2}.\sqrt[4]{7^3}$.
11. $a\sqrt{x}-b\sqrt{y}$. 12. $4\sqrt{3}+2\sqrt{5}$. 13. $3\sqrt{2}-2\sqrt{3}$.
14. $\sqrt{a+x}-\sqrt{a-x}$. 15. $x+\sqrt{x^2-1}$.
16. $\sqrt{x^2+1}-\sqrt{x^2-1}$.

Example 3. Simplify $5\sqrt{3}\div4\sqrt{2}$.

$$5\sqrt{3}\div4\sqrt{2}=\frac{5\sqrt{3}}{4\sqrt{2}}.$$

Rationalising the denominator, we get

$$\frac{5\sqrt{3}}{4\sqrt{2}}=\frac{5\sqrt{3}\times\sqrt{2}}{4\sqrt{2}\times\sqrt{2}}=\frac{5\sqrt{6}}{4.2}=\frac{5\sqrt{6}}{8}.$$

Example 4. Given $\sqrt{3}=1{\cdot}732$, find the value of $\frac{8}{2\sqrt{3}}$ up to 3 places of decimals.

Rationalising the denominator of this fraction, we have

$$\frac{8}{2\sqrt{3}}=\frac{8\times\sqrt{3}}{2\sqrt{3}\times\sqrt{3}}=\frac{8\sqrt{3}}{2.3}=\frac{8\sqrt{3}}{6}$$

$$=\frac{8\times1{\cdot}732}{6}=\frac{13{\cdot}856}{6}=2{\cdot}309.$$

Simplify:

17. $5\sqrt{3}\div7\sqrt{5}$. 18. $7\sqrt[3]{4}\div5\sqrt[3]{2}$.
19. $4\sqrt[3]{5}\div4\sqrt[3]{2}$. 20. $2\sqrt[4]{5}\div6\sqrt[4]{3}$.

Given $\sqrt{2}=1{\cdot}414$, $\sqrt{3}=1{\cdot}732$ and $\sqrt{5}=2{\cdot}236$ find the value of the following fractions up to 3 places of decimals:

21. $\frac{4}{5\sqrt{3}}$. 22. $\frac{3}{2\sqrt{2}}$. 23. $\frac{7}{4\sqrt{3}}$.
24. $\frac{1}{3\sqrt{5}}$. 25. $\frac{3}{2\sqrt{5}}$. 26. $\frac{6}{5\sqrt{3}}$.

Example 5. Given $\sqrt{3}=1{\cdot}732$, find the value of $\frac{2+\sqrt{3}}{2-\sqrt{3}}$.

Rationalising the denominator of $\frac{2+\sqrt{3}}{2-\sqrt{3}}$, we have

$$\text{the fraction} = \frac{(2+\sqrt{3})(2+\sqrt{3})}{(2-\sqrt{3})(2+\sqrt{3})}$$

$$= \frac{4+3+4\sqrt{3}}{4-3} = 7+4\sqrt{3}$$

$$= 7+4\times 1{\cdot}732 = 7+6{\cdot}928$$

$$= 13{\cdot}928.$$

Example 6. Rationalise the denominator of $\dfrac{a+\sqrt{a^2-1}}{a-\sqrt{a^2-1}}$.

$$\text{The fraction} = \frac{(a+\sqrt{a^2-1})(a+\sqrt{a^2-1})}{(a-\sqrt{a^2-1})(a+\sqrt{a^2-1})}$$

$$= \frac{a^2+a^2-1+2a\sqrt{a^2-1}}{a^2-(a^2-1)}$$

$$= \frac{2a^2-1+2a\sqrt{a^2-1}}{a^2-a^2+1} = 2a^2+2a\sqrt{a^2-1}-1.$$

Given $\sqrt{2}=1{\cdot}414$, $\sqrt{3}=1{\cdot}732$, $\sqrt{5}=2{\cdot}236$ and $\sqrt{6}=2{\cdot}449$, find the value of :

27. $\dfrac{\sqrt{2}+1}{\sqrt{2}-1}$. 28. $\dfrac{1}{\sqrt{3}-2}$. 29. $\dfrac{\sqrt{5}+2}{\sqrt{5}-2}$.

30. $\dfrac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$. $\dfrac{7}{3\sqrt{3}-2\sqrt{2}}$. 32. $\dfrac{15}{4\sqrt{3}-3\sqrt{2}}$.

33. Solve $x\sqrt{2}+x=7$ up to 3 places of decimals.

34. Evaluate $\left(x+\dfrac{1}{x}\right)^2$ if $x=\sqrt{3}-\sqrt{2}$.

35. Evaluate $x^4+\dfrac{1}{x^4}$ when $x=3+\sqrt{8}$.

Reduce to an equivalent fraction with a rational denominator :

36. $\dfrac{2+3\sqrt{2}}{3-2\sqrt{2}}$. 37. $\dfrac{4\sqrt{2}+\sqrt{5}}{3\sqrt{3}-2\sqrt{2}}$. 38. $\dfrac{a+\sqrt{a^2-b^2}}{a-\sqrt{a^2-b^2}}$.

39. $\dfrac{\sqrt{x+1}+\sqrt{x-1}}{\sqrt{x+1}-\sqrt{x-1}}$. 40. $\dfrac{\sqrt{x^2+1}+\sqrt{x^2-1}}{\sqrt{x^2+1}-\sqrt{x^2-1}}$.

The rationalising factor of a binomial cubic surd can be found by the application of the identity $(a\pm b)(a^2\mp ab+b^2)=a^3\pm b^3$.

Example 7. Find the rationalising factor of $\sqrt[3]{a}+\sqrt[3]{b}$.

$\sqrt[3]{a}+\sqrt[3]{b}=a^{\frac{1}{3}}+b^{\frac{1}{3}}$.

Since $\left(a^{\frac{1}{3}}+b^{\frac{1}{3}}\right)\left\{\left(a^{\frac{1}{3}}\right)^2-a^{\frac{1}{3}}b^{\frac{1}{3}}+\left(b^{\frac{1}{3}}\right)^2\right\}$

$$=\left(a^{\frac{1}{3}}\right)^3+\left(b^{\frac{1}{3}}\right)^3=a+b,$$

$\therefore \left(a^{\frac{1}{3}}\right)^2-a^{\frac{1}{3}}b^{\frac{1}{3}}+\left(b^{\frac{1}{3}}\right)^2$ or $\sqrt[3]{a^2}-\sqrt[3]{ab}+\sqrt[3]{b^2}$ is the required rationalising factor.

Write down *mentally* the rationalising factor of :

41. $\sqrt[3]{5}-\sqrt[3]{2}$. 42. $\sqrt[3]{4}+\sqrt[3]{3}$. 43. $\sqrt[3]{6}-\sqrt[3]{5}$.
44. $\sqrt[3]{6}+\sqrt[3]{3}$. 45. $\sqrt[3]{x}-\sqrt[3]{y}$. 46. $\sqrt[3]{x}+\sqrt[3]{y}$.

The rationalising factors of homogeneous binomial surds of higher orders can be found by the application of the following identities :

(i) $(a-b)(a^{n-1}+a^{n-2}b+a^{n-3}b^2+\dots+ab^{n-2}+b^{n-1})=a^n-b^n$, for all values of n.

(ii) $(a+b)(a^{n-1}-a^{n-2}b+a^{n-3}b^2-\dots+ab^{n-2}-b^{n-1})=a^n-b^n$ when n is *even*.

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(iii) $(a+b)(a^{n-1}-a^{n-2}b+a^{n-3}b^2-\dots-ab^{n-2}+b^{n-1})=a^n+b^n$ when n is *odd*.

Example 8. Find the rationalising factor $\sqrt[5]{a}-\sqrt[5]{b}$.

$$\sqrt[5]{a}-\sqrt[5]{b}=a^{\frac{1}{5}}-b^{\frac{1}{5}}.$$

In $(x-y)(x^4+x^3y+x^2y^2+xy^3+y^4)\equiv x^5-y^5$,

if we put $a^{\frac{1}{5}}$ for x and $b^{\frac{1}{5}}$ for y, we get

$$(a^{\frac{1}{5}}-b^{\frac{1}{5}})(a^{\frac{4}{5}}+a^{\frac{3}{5}}b^{\frac{1}{5}}+a^{\frac{2}{5}}b^{\frac{2}{5}}+a^{\frac{1}{5}}b^{\frac{3}{5}}+b^{\frac{4}{5}})$$
$$=(a^{\frac{1}{5}})^5-(b^{\frac{1}{5}})^5=a-b,$$

$\therefore \quad (a^{\frac{4}{5}}+a^{\frac{3}{5}}b^{\frac{1}{5}}+a^{\frac{2}{5}}b^{\frac{2}{5}}+a^{\frac{1}{5}}b^{\frac{3}{5}}+b^{\frac{4}{5}})$, or

$\sqrt[5]{a^4}+\sqrt[5]{a^3b}+\sqrt[5]{a^2b^2}+\sqrt[5]{ab^3}+\sqrt[5]{b^4}$ is the required rationalising factor.

Write down *mentally* the rationalising factor of:

47. $\sqrt[4]{3}-\sqrt[4]{2}$.
48. $\sqrt[5]{3}-\sqrt[5]{2}$.
49. $\sqrt[6]{5}-\sqrt[6]{3}$.
50. $\sqrt[4]{3}+\sqrt[4]{2}$.
51. $\sqrt[5]{3}+\sqrt[5]{2}$.
52. $\sqrt[6]{5}+\sqrt[6]{3}$.

Example 9. Find the rationalising factor of $1+\sqrt{2}+\sqrt{3}$.

$$1+\sqrt{2}+\sqrt{3}=\{(1+\sqrt{2})+\sqrt{3}\}.$$

Multiplying the expression by $(1+\sqrt{2})-\sqrt{3}$, we get

$$\{(1+\sqrt{2})+\sqrt{3}\}\{(1+\sqrt{2})-\sqrt{3}\}=(1+\sqrt{2})^2-(\sqrt{3})^2$$
$$=1+2+2\sqrt{2}-3$$
$$=2\sqrt{2}.$$

Again, multiplying $2\sqrt{2}$ by $\sqrt{2}$, we get

$$2\sqrt{2}\times\sqrt{2}=4$$

$\therefore$ the rationalising factor is $(1+\sqrt{2}-\sqrt{3})\sqrt{2}$.

Find the rationalising factor of:

53. $1-\sqrt{2}+\sqrt{3}$.
54. $\sqrt{3}+\sqrt{2}-1$.
55. $\sqrt{3}-\sqrt{2}+\sqrt{5}$.
56. $\sqrt{2}+\sqrt{3}-\sqrt{5}$.
57. $3\sqrt{5}-2\sqrt{3}+1$.
58. $3\sqrt{2}+2\sqrt{3}-3\sqrt{6}$.

Example 10. Simplify $\dfrac{1}{2-\sqrt{3}}-\dfrac{1}{\sqrt{3}+\sqrt{2}}+\dfrac{4}{3-\sqrt{5}}$.

$$\text{The expression}=\frac{2+\sqrt{3}}{(2-\sqrt{3})(2+\sqrt{3})}-\frac{\sqrt{3}-\sqrt{2}}{(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2})}$$
$$+\frac{4(3+\sqrt{5})}{(3-\sqrt{5})(3+\sqrt{5})}=\frac{2+\sqrt{3}}{4-3}-\frac{\sqrt{3}-\sqrt{2}}{3-2}+\frac{4(3+\sqrt{5})}{9-5}$$
$$=2+\sqrt{3}-\sqrt{3}+\sqrt{2}+3+\sqrt{5}$$
$$=5+\sqrt{2}+\sqrt{5}.$$

Example 11. Simplify $\dfrac{9}{\sqrt{6}+\sqrt{12}+\sqrt{24}-\sqrt{48}-\sqrt{3}}$.

$$\text{The denominator}=\sqrt{6}+\sqrt{12}+\sqrt{24}-\sqrt{48}-\sqrt{3}$$
$$=\sqrt{6}+2\sqrt{3}+2\sqrt{6}-4\sqrt{3}-\sqrt{3}$$
$$=3\sqrt{6}-3\sqrt{3}=3\sqrt{3}(\sqrt{2}-1).$$

$$\therefore \quad \text{the fraction} = \frac{9}{3\sqrt{3}(\sqrt{2}-1)} = \frac{3\sqrt{3}\times\sqrt{3}}{3\sqrt{3}(\sqrt{2}-1)}$$

$$= \frac{\sqrt{3}}{\sqrt{2}-1} = \frac{\sqrt{3}(\sqrt{2}+1)}{(\sqrt{2}-1)(\sqrt{2}+1)}$$

$$= \frac{\sqrt{3}(\sqrt{2}+1)}{2-1} = \sqrt{3}(\sqrt{2}+1).$$

Simplify :

59. $\dfrac{1}{a+\sqrt{a^2-1}} + \dfrac{1}{a-\sqrt{a^2-1}}$.

60. $\dfrac{1}{a-\sqrt{a^2-b^2}} - \dfrac{1}{a+\sqrt{a^2-b^2}}$.

61. $\dfrac{5}{3\sqrt{8}+\sqrt{12}-\sqrt{27}-\sqrt{32}}$.

62. $\dfrac{\sqrt{3}}{\sqrt{5}+1} - \dfrac{\sqrt{5}}{2(\sqrt{3}-1)} + \dfrac{1}{\sqrt{5}-\sqrt{3}}$.

63. $\dfrac{2\sqrt{6}}{\sqrt{2}+\sqrt{3}} + \dfrac{3\sqrt{2}}{\sqrt{3}+\sqrt{6}} - \dfrac{4\sqrt{3}}{\sqrt{6}+\sqrt{2}}$.

64. $\dfrac{a+\sqrt{a^2-1}}{a-\sqrt{a^2-1}} - \dfrac{a-\sqrt{a^2-1}}{a+\sqrt{a^2-1}}$.

65. $\dfrac{\sqrt{x^2+1}+\sqrt{x^2-1}}{\sqrt{x^2+1}-\sqrt{x^2-1}} + \dfrac{\sqrt{x^2+1}-\sqrt{x^2-1}}{\sqrt{x^2+1}+\sqrt{x^2-1}}$.

66. Find the value of $(2+\sqrt{3})^{-3}-(2-\sqrt{3})^{-3}$.

5. Properties of quadratic surds.

Theorem I. *The square root of a quantity cannot be partly rational and partly irrational.*

For, if possible, let

$\sqrt{x}=a+\sqrt{b}$; where $\sqrt{b}$ is a *true surd*.

Squaring both sides, we get

$x=a^2+b+2a\sqrt{b}$.

Hence $\quad \sqrt{b}=\dfrac{x-a^2-b}{2a}$,

i.e., a surd is equal to a rational quantity, which is impossible.

Theorem II. If $x+\sqrt{y}=a+\sqrt{b}$, *where* x *and* a *are both rational and* $\sqrt{y}$ *and* $\sqrt{b}$ *are true surds, then* $x=a$ *and* $y=b$.

For, if x is not equal to a than let x be greater than a and be equal to $a+h$,

$$\therefore \quad (a+h)+\sqrt{y}=a+\sqrt{b}$$
$$\therefore \quad h+\sqrt{y}=\sqrt{b}$$
$$\text{or} \quad \sqrt{b}=h+\sqrt{y}.$$

Thus, the square root of a rational quantity is partly rational and partly irrational, which is impossible by theorem I.

Hence $x=a$ and consequently $\sqrt{y}=\sqrt{b}$.

Ex. Tell the flaw in the following process:

$$\text{Since} \quad 3+\sqrt{25}=6+\sqrt{4}$$
$$\therefore \quad 3=6 \text{ and } 25=4.$$

NOTE. It is important to remember that theorem II holds good only when $\sqrt{y}$ and $\sqrt{b}$ are *true surds* and not merely in surd form, as in the above example.

The method for finding the square root of expressions of the form $a+\sqrt{b}$, where $\sqrt{b}$ is a true surd, is based upon theorem II and is illustrated below by means of a few examples.

Example 1. Find the square root of $31+4\sqrt{21}$.

Let $\sqrt{(31+4\sqrt{21})}=\sqrt{a}+\sqrt{b}$.

Squaring both sides, we get

$$31+4\sqrt{21}=a+b+2\sqrt{ab}$$

$$\therefore \quad \text{by theorem II, } a+b=31 \quad \ldots \quad \ldots \quad \text{(i)}$$
$$\text{and} \quad 2\sqrt{ab}=4\sqrt{21} \quad \ldots \quad \ldots \quad \text{(ii)}$$
$$\therefore \quad a-b=\sqrt{(a+b)^2-4ab}$$
$$=\sqrt{(31)^2-(4\sqrt{21})^2}$$
$$=\sqrt{625}=25 \quad \ldots \quad \ldots \quad \text{(iii)}$$

By adding (i) and (iii), we get

$$2a=56 \text{ or } a=28$$

and by substituting the value of a in (i), we get

$$b=3.$$

$$\therefore \qquad \sqrt{31+4\sqrt{21}}=\sqrt{28}+\sqrt{3}$$
$$=2\sqrt{7}+\sqrt{3}.$$

Example 2. Find the square root of $21-8\sqrt{5}$.

Let $\sqrt{21-8\sqrt{5}}=\sqrt{a}-\sqrt{b}$.

Squaring both sides, we get

$$21-8\sqrt{5}=a+b-2\sqrt{ab}$$

$\therefore \qquad a+b=21 \qquad \dots \qquad \dots \qquad \dots \qquad$ (i)

and $\qquad 2\sqrt{ab}=8\sqrt{5} \qquad \dots \qquad \dots \qquad \dots \qquad$ (ii)

$$\therefore \qquad a-b=\sqrt{(a+b)^2-4ab}$$
$$=\sqrt{(21)^2-(8\sqrt{5})^2}$$
$$=\sqrt{441-320}$$

$=\sqrt{121}=11. \qquad \dots \qquad \dots \qquad \dots \qquad$ (iii)

From equations (i) and (iii), we get

$$a=16 \text{ and } b=5$$

$$\therefore \qquad \sqrt{21-8\sqrt{5}}=\sqrt{16}-\sqrt{5}$$
$$=4-\sqrt{5}.$$

NOTE. In most cases the square root of expressions of the form $a+\sqrt{b}$ can be found by inspection, as illustrated below.

Example 3. Find the square root of $7+2\sqrt{10}$.

Here, we have to find two numbers which when added give 7 and when multiplied give 10.

They are obviously 2 and 5.

Thus, $\qquad \sqrt{7+2\sqrt{10}}=\sqrt{5}+\sqrt{2}.$

EXERCISE 85.

Find the square root of:

1. $52+14\sqrt{3}$. 2. $17-12\sqrt{2}$. 3. $41+12\sqrt{5}$.
4. $19+8\sqrt{3}$. 5. $14-3\sqrt{20}$. 6. $94+6\sqrt{245}$.
7. $43-30\sqrt{2}$. 8. $27-10\sqrt{2}$. 9. $87-12\sqrt{42}$.

Find by *inspection* the square root of :—

10. $7+4\sqrt{3}$. **11.** $8+2\sqrt{15}$. **12.** $5+2\sqrt{6}$.

13. $12+2\sqrt{35}$. **14.** $21-8\sqrt{5}$. **15.** $8-2\sqrt{7}$.

16. If $\sqrt{(21-6\sqrt{10})}=\sqrt{x}-\sqrt{y}$ find the value of x and y.

Example 4. Find the square root of $\sqrt{32}-\sqrt{24}$.

$$\sqrt{32}-\sqrt{24}=\sqrt{2}(\sqrt{16}-\sqrt{12})$$
$$=\sqrt{2}(4-2\sqrt{3})$$

$$\therefore\quad \sqrt{(32-\sqrt{24})}=\sqrt[4]{2}\sqrt{(4-2\sqrt{3})}.$$

Now proceeding as in example 2, we find that

$$\sqrt{4-2\sqrt{3}}=\sqrt{3}-1$$

$$\therefore\quad \sqrt{(\sqrt{32}-\sqrt{24})}=\sqrt[4]{2}(\sqrt{3}-1).$$

Find the square root of :

17. $\sqrt{18}+\sqrt{10}$. **18.** $\sqrt{27}-\sqrt{24}$.

19. $\sqrt{75}+\sqrt{72}$. **20.** $\sqrt{128}-\sqrt{120}$.

21. $5\sqrt{3}+2\sqrt{18}$. **22.** $6\sqrt{5}-\sqrt{175}$.

Find the fourth root of :

23. $68+48\sqrt{2}$. **24.** $7-4\sqrt{3}$.

25. $28-16\sqrt{3}$. **26.** $124-32\sqrt{15}$.

Example 5. Simplify $\sqrt{2}(1+\sqrt{3})\sqrt{(2-\sqrt{3})}$.

Extracting the square root of $2-\sqrt{3}$, we find

$$\sqrt{2-\sqrt{3}}=\frac{1}{\sqrt{2}}(\sqrt{3}-1)$$

$$\therefore\ \text{the expression}=\sqrt{2}(1+\sqrt{3})\frac{1}{\sqrt{2}}(\sqrt{3}-1)=(\sqrt{3}+1)(\sqrt{3}-1)$$
$$=3-1=2.$$

Simplify :

27. $\sqrt{6-\sqrt{(17+12\sqrt{2})}}$. **28.** $\dfrac{2-\sqrt{3}}{\sqrt{(7-4\sqrt{3})}}$.

29. $(1+\sqrt{3})(\sqrt{2}-\sqrt{3})\sqrt{(5+2\sqrt{6})}$.

30. $\dfrac{\sqrt{(9+2\sqrt{20})}+\sqrt{(9-2\sqrt{20})}}{\sqrt{(9+2\sqrt{20})}-\sqrt{(9-2\sqrt{20})}}$.

*31. $\dfrac{\sqrt{(5+2\sqrt{6})}+\sqrt{(5-2\sqrt{6})}}{\sqrt{(5+2\sqrt{6})}-\sqrt{(5-2\sqrt{6})}}+\dfrac{\sqrt{(5+2\sqrt{6})}-\sqrt{(5-2\sqrt{6})}}{\sqrt{(5+2\sqrt{6})}+\sqrt{(5-2\sqrt{6})}}$.

*32. $\dfrac{3}{\sqrt{(7+2\sqrt{10})}}+\dfrac{2}{\sqrt{(8-2\sqrt{15})}}-\dfrac{5}{\sqrt{(7+2\sqrt{6})}}$.

*33. Evaluate $\sqrt{x}-\dfrac{1}{\sqrt{x}}$ when $x=3+2\sqrt{2}$.

*34. Find the value of $\dfrac{1+x}{1+\sqrt{(1+x)}}+\dfrac{1-x}{1-\sqrt{(1-x)}}$

when $x=\dfrac{\sqrt{3}}{2}$.

[*Hint.* $1+x=1+\dfrac{\sqrt{3}}{2}=\dfrac{2+\sqrt{3}}{2}$ or $\dfrac{4+2\sqrt{3}}{4}$

and $\sqrt{1+x}=\sqrt{\left(\dfrac{4+2\sqrt{3}}{4}\right)}=\dfrac{\sqrt{3}+1}{2}$.

Similarly, $1-x=\dfrac{2-\sqrt{3}}{2}$ or $\dfrac{4-2\sqrt{3}}{4}$

and $\sqrt{1-x}=\dfrac{\sqrt{3}-1}{2}$.]

*Example 6. Find the square root of $12+4\sqrt{3}-4\sqrt{5}-2\sqrt{15}$.

Let $\sqrt{(12+4\sqrt{3}-4\sqrt{5}-2\sqrt{15})}=\sqrt{x}+\sqrt{y}-\sqrt{z}$.

Then squaring both sides, we have

$12+4\sqrt{3}-4\sqrt{5}-2\sqrt{15}=x+y+z+2\sqrt{xy}-2\sqrt{xz}-2\sqrt{yz}$

$\therefore\quad x+y+z=12 \quad \ldots \quad \ldots \quad \ldots \quad$ (i)

$2\sqrt{xy}=4\sqrt{3} \quad \ldots \quad \ldots \quad \ldots \quad$ (ii)

$2\sqrt{xz}=4\sqrt{5} \quad \ldots \quad \ldots \quad \ldots \quad$ (iii)

$2\sqrt{yz}=2\sqrt{15} \quad \ldots \quad \ldots \quad \ldots \quad$ (iv)

From (ii) and (iii), we get

$4x\sqrt{yz}=16\sqrt{15} \quad \ldots \quad \ldots \quad \ldots \quad$ (v)

But $2\sqrt{yz}=2\sqrt{15}$ $\therefore$ $2x=8$, or $x=4$.

From (ii), $y=3$, and from (iii), $z=5$.

Also $x+y+z=4+3+5=12$.

$\therefore\quad \sqrt{(12+4\sqrt{3}-4\sqrt{5}-2\sqrt{15})}=2+\sqrt{3}-\sqrt{5}$.

Find the square root of :

*35. $6+\sqrt{8}+\sqrt{12}+\sqrt{24}$.

*36. $10-\sqrt{24}+2\sqrt{10}-2\sqrt{15}$.

*37. $9+2\sqrt{3}-2\sqrt{5}-2\sqrt{15}$.

*38. $15-4\sqrt{5}-4\sqrt{6}+2\sqrt{30}$.

6. Equations involving surds.

Example 1. Solve $\sqrt{x+16}=\sqrt{x}+2$.

Squaring both sides, we have

$$x+16=x+4+4\sqrt{x}.$$

Hence $\qquad 4\sqrt{x}=12$

$\therefore \qquad \sqrt{x}=3$

$\therefore \qquad x=9.$

Example 2. Solve $\sqrt{x+2}+\sqrt{x-3}=\sqrt{4x-3}$.

Squaring both sides, we have

$$x+2+x-3+2\sqrt{(x^2-x-6)}=4x-3.$$

By transposition, $\qquad 2\sqrt{(x^2-x-6)}=2x-2$

$$=2(x-1)$$

$\therefore \qquad \sqrt{(x^2-x-6)}=x-1.$

Squaring both sides, we have

$$x^2-x-6=x^2-2x+1.$$

By transposition, $\qquad x=7.$

Example 3. Solve $x+\sqrt{x^2+\sqrt{(4x^2-1)}}=1$.

By transposition, $\qquad \sqrt{x^2+\sqrt{(4x^2-1)}}=1-x.$

Squaring both sides, we have

$$x^2+\sqrt{(4x^2-1)}=1-2x+x^2$$

$\therefore \qquad \sqrt{(4x^2-1)}=1-2x.$

Squaring both sides, we have

$$4x^2-1=1-4x+4x^2.$$

By transposition, $\quad 4x=2$

$\therefore \qquad x=\frac{1}{2}.$

EXERCISE 86.

Solve the equations :

1. $\sqrt{3x-2}=4.$
2. $\sqrt{x+1}=3\sqrt{x-1}.$
3. $\sqrt{x-1}=1-\sqrt{x}.$
4. $\sqrt{x+7}=1+\sqrt{x}.$
5. $\sqrt{x+11}=11-\sqrt{x}.$
6. $\sqrt{5(x+2)}=2+\sqrt{5x}.$
7. $\sqrt{3x+4}+4-\sqrt{3x}=0.$
8. $\sqrt{16x^2-25}=3-4x.$
9. $x+a+\sqrt{x^2-a^2}=2a.$
10. $\sqrt{x}+\sqrt{7+x}=\dfrac{21}{\sqrt{7+x}}.$
11. $\sqrt{x}+\sqrt{a+x}=\dfrac{2a}{\sqrt{a+x}}.$
12. $\sqrt[3]{7-4x}=3.$
13. $\sqrt[4]{x-3}=\sqrt[8]{x^2-2x+11}.$
14. $\sqrt[5]{x+4}=\sqrt[10]{x^2+5x+25}.$
15. $\sqrt{x}-\sqrt{4+x}=\dfrac{2}{\sqrt{x}}.$
16. $\sqrt{x}+\sqrt{x-\sqrt{1-x}}=1.$
17. $\sqrt{x+11}-\sqrt{x-4}=3.$
18. $\sqrt{2x+1}+\sqrt{2x-7}=4.$
19. $\sqrt{3x+1}-\sqrt{3x-11}=2.$
20. $\sqrt{5x-1}=1+\sqrt{5x-2}.$
21. $\sqrt{7x-10}+\sqrt{7x+1}=11.$
22. $\sqrt{x^2-3x+5}-\sqrt{x^2-x+1}=1.$
23. $\sqrt{x^2+11x+20}-\sqrt{x^2+5x-1}=3.$
24. $\sqrt{9x+1}=\sqrt{x+1}-\sqrt{4x+1}.$
25. $\sqrt{3x+1}+\sqrt{2(x-6)}=\sqrt{5x+9}.$
26. $\sqrt{x+5}+\sqrt{x+6}=\sqrt{4x+11}.$

Example 4. Solve $\sqrt{4x+9}-\sqrt{4x-7}=2.$

If we subtract $(4x-7)$ from $(4x+9)$, we get

$$(4x+9)-(4x-7)\equiv 16 \quad \ldots \quad \ldots \quad \text{[Identity.]}$$

Dividing this identity by the given equation, we get

$$\frac{(4x+9)-(4x-7)}{\sqrt{4x+9}-\sqrt{4x-7}}=\frac{16}{2}=8$$

or $\quad \sqrt{4x+9}+\sqrt{4x-7}=8 \quad \ldots \quad$ [Equivalent equation.]

SECTIONAL REVISION V

Test Papers

PAPER 1

1. (i) Prove in full (a) $(x^2)^5 = x^{10}$, (b) $(3x^2)^4 = 81x^8$.
   (ii) Write in shorter form (a) 2,700,000,
   (b) $2{\cdot}3 \times 2{\cdot}3 \times 2{\cdot}3 \times 1{,}000{,}000$.
   (iii) Simplify (a) $x^0 \times 1$. (b) $x^0 \times x$.
2. Find by expansion the approximate value of
   (i) $(1{\cdot}0001)^4$. (ii) $(1{\cdot}004)^5$.
3. Expand (i) $(3p+2q)^4$, (ii) $(3p-2q)^4$.
4. Find the values of a and b for which $x^4-4x^3-2x^2+ax+b$ may be a perfect square.
5. Find the square root of $57-28\sqrt{2}$.
6. Solve the equations $\left.\begin{aligned} 3^{x+y+z} &= 9^{x-y+z} \\ 4^{3y+2} &= 16^{2x+z} \\ 2^{2x+3y+z} &= 8^{x+2y}. \end{aligned}\right\}$

PAPER 2

1. Express the following facts without using the index notation:
   (i) the velocity of light is $30{\cdot}02 \times 10^{10}$ *m.m.* per second;
   (ii) the distance of a star named L Centuri is $\cdot 26 \times 10^{14}$ miles;
   (iii) $30 \times 10^4 \times 2{\cdot}4 \times 10^5 = 72 \times 10^9$.
2. (i) Compare the values of $\left[(3^2)^2\right]^2$ and $3^{2^{2^2}}$.
   (ii) Write down the values of $(1)^0$, $(1)^{-4}$, $(-1)^{-3}$, $(-1)^{-2}$.
3. If $a^x = m$, $a^y = n$ and $a^2 = (m^y n^x)^z$, shew that $xyz = 1$.
4. Simplify $(1+x)^6-(1-x)^6$.
5. Find the square root of $31-12\sqrt{3}$.
6. Solve the equations (i) $(\sqrt{8})^{x+2} = (\sqrt[5]{8})^{3x+1}$.
   (ii) $5^{x-1}.3^{x-2} = 5$.

PAPER 3

1. Express the following facts without using negative indices:

(i) the mass of an atom of hydrogen is $1{\cdot}6\times10^{-24}$ gr.,
(ii) the diameter of an atom of oxygen is 3×10^{-8} cm.;
(iii) the wave length of yellow light is 27×10^{-6} inches.

2. If $a=3$ and $b=2$ write down the values of:
(i) $(ab)^{-1}$, (ii) $(a+b)^{-1}$, (iii) $(a-b)^{-1}$, (iv) $a^{-1}+b^{-1}$.

3. Find the co-efficient of x^3 in the expansion of
$$(x+1)^5\ (x-2)^4.$$

4. Find the square root of $5\sqrt{5}+\sqrt{120}$.

5. If $\dfrac{a}{b}=\dfrac{c}{d}=\dfrac{e}{f}$, shew that each
$$=\sqrt[3]{\left(\frac{2a^{-3}-5c^{-3}+3e^{-3}}{2b^{-3}-5d^{-3}+3f^{-3}}\right)^{-1}}$$

6. Solve $\sqrt[3]{x+1}-\sqrt[3]{x-1}=\sqrt[3]{2}$.

PAPER 4

1. Write down the values of:
(i) 2^{-3}, (ii) 3^{-2}, (iii) $(\frac{1}{9})^{-2}$ (iv) $(\frac{2}{3})^{-4}$.

2. (a) If $16^x=1$, find x.

(b) If $10^{\cdot301}=2$ approximately, solve the equations:
(i) $10^x=200$; (ii) $5^x=10$.

3. Rupee one with compound interest @ 5% per annum after 12 years amounts to Rs. $(1+{\cdot}05)^{12}$.
Calculate the value of the amount to 3 places of decimals.

4. Simplify $\dfrac{2+\sqrt{3}}{\sqrt{2}+\sqrt{2+\sqrt{3}}}+\dfrac{2-\sqrt{3}}{\sqrt{2}-\sqrt{2-\sqrt{3}}}$.

5. Solve the equations:
(i) $\sqrt{x+9}=1+\sqrt{x-4}$.

(ii) $\sqrt{x+\sqrt{4x+\sqrt{1+16x}}}=1+\sqrt{x}$.

6. Eliminate x and y from the equations:
$$x+\frac{1}{x}=a,\qquad y+\frac{1}{y}=b,\qquad xy+\frac{1}{xy}=c.$$

PAPER 5

1. If $a=\sqrt{\dfrac{e-1}{e+1}}$, express $\dfrac{1-a}{1+a}$ in terms of e.

2. Simplify (i) $\sqrt{(2+\sqrt{3})}-\sqrt{(2-\sqrt{3})}$.

   (ii) $\dfrac{\sqrt{5}+2}{\sqrt{(9-4\sqrt{5})}}$.

3. Find the square root of $x^2+\dfrac{1}{x^2}+6\left(x-\dfrac{1}{x}\right)+7$.

4. Find the cube root of

$$x^6+3x^5-3x^4-11x^3+6x^2+12x-8.$$

5. Solve the equations :

   (i) $\sqrt{5x+4}+\sqrt{5x-1}=5$.

   (ii) $\dfrac{ax-1}{\sqrt{ax}+1}=4-\dfrac{\sqrt{ax}-1}{2}$.

6. Eliminate x, y and z from the equations:
   $(y+z)^2=ayz$, $(z+x)^2=bzx$, $(x+y)^2=cxy$.

PAPER 6

1. Write down the values of :

   (i) $(81)^{\frac{3}{4}}$, (ii) $(81)^{1\cdot 25}$, (iii) $\sqrt[5]{32^4}$, (iv) $(100)^{-1\cdot 5}$.

2. (i) Multiply $x^{\frac{1}{2}}+x^{\frac{1}{4}}y^{\frac{1}{4}}+y^{\frac{1}{2}}$ by $x^{\frac{1}{2}}-x^{\frac{1}{4}}y^{\frac{1}{4}}+y^{\frac{1}{2}}$.

   (ii) Divide $x-y$ by $x^{\frac{1}{4}}-y^{\frac{1}{4}}$.

3. Find the square root of :

   $9+3x+2y+\frac{1}{3}xy+\frac{1}{4}x^2+\frac{1}{9}y^2$.

4. Solve the equations (i) $3^{x-1}=3^{x-2}+6$.

   (ii) $\sqrt[3]{1+x}+\sqrt[3]{1-x}=\sqrt[3]{2}$.

5. (i) Rationalise the denominator of $\dfrac{\sqrt{5}-\sqrt{3}-2}{\sqrt{5}+\sqrt{3}+2}$.

   (ii) Simplify $\dfrac{1}{2(\sqrt{5}-\sqrt{3})}+\dfrac{\sqrt{3}}{(\sqrt{5}+1)}-\dfrac{\sqrt{5}}{2(\sqrt{3}+1)}$.

6. Eliminate y from $m=y^x$ and $n=x^y$

PAPER 7

1. (i) One factor of $a+b$ is $a^{\frac{1}{3}}+b^{\frac{1}{3}}$, write down the other.
(ii) Find the continued product of

$$x^{\frac{1}{4}}-y^{\frac{1}{4}},\ x^{\frac{1}{4}}+y^{\frac{1}{4}},\ x^{\frac{1}{2}}+y^{\frac{1}{2}}.$$

2. Expand (i) $(1+x)^6$, (ii) $(1-x)^6$.

3. Find the square root of $x^4+1+2x+x^3+\dfrac{1}{x^2}+\dfrac{x^2}{4}$.

4. Solve the equations (i) $\sqrt{x+2}+\sqrt{x-2}=\dfrac{4}{\sqrt{x+2}}$.

(ii) $\dfrac{\sqrt{x+3}+\sqrt{x-3}}{\sqrt{x+3}-\sqrt{x-3}}=\frac{1}{2}$.

5. (i) Evaluate $\sqrt[4]{(17-12\sqrt{2})}$ correct up to 3 decimal places.

(ii) Simplify $\dfrac{\left\{(x^m)^{\frac{1}{r}}\quad(x^n)^{-\frac{1}{p}}\right\}^{pr}}{\left\{\sqrt[n]{x^p}\ .\ \sqrt[m]{x^r}\right\}^{mn}}$.

6. Eliminate x from $x^3+\dfrac{1}{x^3}=a^3$ and $x+\dfrac{1}{x}=b$.

PAPER 8

1. Draw the graph of 2^x from $x=-2$ to $x=+2$ and read on it the approximate value of $2^{\cdot 5}$.

2. Solve the equations $\left.\begin{array}{l} 3^{4x-3y}\ .\ 5^{4x}=5^{3y} \\ 5^{2x-y+1}\ .\ 6^{2x+4}=6^{y}. \end{array}\right\}$

3. Expand (i) $(x+a)^5$, (ii) $(x-a)^5$.

4. Find the square root of $4x^4-12x^3+29x^2-30x+25$.

5. Simplify $\dfrac{2^{2n+1}-2^{n+2}+2}{2^{2n+1}-2^{n+1}}$.

6. Eliminate x and y from the equations:

$$a=\frac{1}{x}-x,\ b=\frac{1}{y}-y \text{ and } x^2+y^2=1.$$

*CHAPTER XX

QUADRATIC EQUATIONS, PROBLEMS AND GRAPHS

1. **Such** statements of equality as

$$3x^2=12$$
$$3x^2-3x=0$$
$$2x(x-2)=(2-x)(1-x)+5$$

are *equations* and not identities because they are not true for all values of x, *e.g.*, none of them is true for $x=7$.

When all the terms are transposed to the left-hand side, these equations read

$$3x^2-12=0$$
$$3x^2-3x=0$$
$$x^2-x-7=0.$$

Each of them involves the square of the variable x but no higher power and is called a **Quadratic Equation** in x.

The general form of a quadratic equation is

$$ax^2+bx+c=0$$

where a, b and c are independent of x.

When there is no term in x, as in $3x^2-12=0$, the equation is called a **pure quadratic**; when there is a term in x, as in $3x^2-3x=0$ or $x^2-x-7=0$, the equation is called an **adfected quadratic.**

Solution of Pure Quadratic

2. Illustrative examples:

Example 1. Solve the equation $7x^2=63$.

Dividing both sides by 7, $x^2=9$.

Taking the square root of each side, we have

$$x=\pm3.$$

For when $x=+3$ or -3, $7x^2=7\ (\pm3)^2=63$.

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* This Chapter is not included in the Punjab syllabus.

NOTE. Since the square root of $x^2 = \pm x$, it seems necessary to write the answer as $\pm x = \pm 3$; but on analysis, we find that it leads to

(i) $+x = +3$, (ii) $+x = -3$, (iii) $-x = +3$, (iv) $-x = -3$, out of which (i) and (iv) give the same solution *viz.*, $x = +3$

and (ii) ,, (iii) ,, ,, $x = -3$.

Example 2. Solve the equation $5(x-3)^2 = 180$.

Dividing both sides by 5, we have

$$(x-3)^2 = 36.$$

Taking the square root of both sides, we have

$$x - 3 = \pm 6$$

i.e. $x - 3 = +6$, $\therefore\ x = +9$

and $x - 3 = -6$, $\therefore\ x = -3$.

$\therefore$ the two roots are $+9$ and -3.

Example 3. Solve the equation $(2x-5)^2 = x^2 - 20x + 73$.

Simplifying the left-hand side, we have

$$4x^2 - 20x + 25 = x^2 - 20x + 73.$$

By transposition, $4x^2 - x^2 = -20x + 73 + 20x - 25$.

$\therefore$ $3x^2 = 48$.

Dividing both sides by 3, $x^2 = 16$.

Taking the square root of both sides,

$$x = \pm 4.$$

Example 4. Solve the equation $\dfrac{1}{1+x} + \dfrac{1}{1-x} = \dfrac{4}{1+x^2}$.

Simplifying the left-hand side, we have

$$\frac{1 - x + 1 + x}{1 - x^2} = \frac{4}{1 + x^2}$$

or $$\frac{2}{1 - x^2} = \frac{4}{1 + x^2}.$$

Dividing both sides by 2, we have

$$\frac{1}{1 - x^2} = \frac{2}{1 + x^2}.$$

By cross-multiplication, we get

$$1 + x^2 = 2 - 2x^2.$$

By transposition. $3x^2=1$

or $\qquad x^2=\frac{1}{3}$.

Taking the square root of both sides,

$$x=\pm\frac{1}{\sqrt{3}}.$$

EXERCISE 88.

Solve the following equations:

1. $3x^2+7=55$.
2. $10-2x^2=\frac{1}{2}x^2$.
3. $(x-5)^2=25$.
4. $(x+1)^2=2x+5$.
5. $(x-2)(x-5)=15-7x$.
6. $(2x-3)(x-1)=12-5x$.
7. $\dfrac{ax^2}{b}=\dfrac{a^3}{b}$.
8. $15x^2-7x^2=20+3x^2$.
9. $8x^2+11=5x^2+14$.
10. $29-10x^2=83-16x^2$.
11. $2x^2+17-5(x^2-2)=3(x^2+5)-12$.
12. $\dfrac{x^2+7}{x^2-5}=\dfrac{x^2-15}{x^2-3}$.
13. $\dfrac{4}{x-3}-\dfrac{4}{x+3}=\dfrac{1}{3}$.
14. $\dfrac{x^2-1}{5}+\dfrac{1-2x^2}{3}=x^2$
15. $\dfrac{1}{1+x}+\dfrac{1}{1-x}=\dfrac{6}{1+x^2}$.
16. $\dfrac{1}{a+x}+\dfrac{1}{a-x}=1$.
17. $12\left(\dfrac{x}{2}+1\right)^2=108$.

Solution by Factorisation

3. The solution of adfected quadratic equations depends on the principle that if $a\times b=0$, then either $a=0$, or $b=0$.

Example 1. Solve the equation $(x+2)(x-3)=0$.

Since the product of two factors $(x+2)$ and $(x-3)$ is equal to zero, either $x+2=0$, or $x-3=0$.

If $x+2=0$, $x=-2$; if $x-3=0$, $x=+3$.

$\therefore$ the required roots are -2 and $+3$.

For when $x=-2$, we have $(-2+2)(-2-3)=0\times-5=0$

and when $x=3$, ,, ,, $(3+2)(3-3)=5\times0=0$.

Example 2. Solve the equation $2x+5=3x^2$.

Reducing it to the standard form $ax^2+bx+c=0$, we have by transposition $\qquad 3x^2-2x-5=0$.

Factorising $(x+1)(3x-5)=0$

whence $x+1=0$ or $3x-5=0$

$\therefore$ the required roots are -1 and $+\frac{5}{3}$.

Verification. (1) When $x=-1$, $3x^2-2x-5$

$$=3.(-1)^2-2(-1)-5$$
$$=3+2-5=0.$$

(2) When $x=\frac{5}{3}$, $3x^2-2x-5$

$$=3.(\tfrac{5}{3})^2-2.\tfrac{5}{3}-5$$
$$=\frac{25}{3}-\frac{10}{3}-5=0.$$

Example 3. Solve the equation $x-\dfrac{9}{x}=15\left(1-\dfrac{3}{x}\right)$.

Multiplying both sides by x, we have

$$x^2-9=15x-45.$$

By transposition $x^2-15x+36=0.$

Factorising $(x-12)(x-3)=0,$

whence $x-12=0$, or $x-3=0$.

$\therefore$ the required roots are $+12$ and $+3$, which can be verified by substitution.

Example 4. Find the equation whose roots are $+7$ and -5.

The factor corresponding to the root $+7$ is $x-7$,

" " " " " -5 is $x+5$.

Hence $(x-7)(x+5)=0$, or $x^2-2x-35=0$ is the required equation.

EXERCISE 89.

Write down the roots of the following equations:

1. $(x-4)(x-6)=0$.
2. $(x+4)(x+6)=0$.
3. $(x-5)(x+7)=0$.
4. $(x+8)(x-9)=0$.
5. $(1-x)(3-x)=0$.
6. $(2-x)(5+x)=0$.
7. $(x-a)(x+b)=0$.
8. $(2x-3)(3x-4)=0$.
9. $(6x-1)(5x+2)=0$.
10. $(ax-b)(bx+c)=0$.
11. $\left(\dfrac{x}{3}+2\right)\left(3x-\dfrac{2}{3}\right)=0$.
12. $x(x+5)=0$.

13. $(x+4)^2=0.$ 14. $(3x-7)^2=0.$
15. $(3x+a)^2=0.$ 16. $(ax-b)^2=0.$
17. $\{x-(p+q)\}^2=0.$ 18. $\{x-(p+q)\}\{x+(p-q)\}=0.$

Solve the following equations:

19. $x^2-9x+14=0.$ 20. $x^2+10x+24=0.$
21. $x^2-x-12=0.$ 22. $1+7x+12x^2=0.$
23. $27+12x+x^2=0.$ 24. $5x^2+4x-1=0.$
25. $6x^2+6=13x.$ 26. $3x^2+3=10x.$
27. $x^2-5x=36.$ 28. $16x^2-159x=10.$
29. $8x^2+3=14x.$ 30. $5(3x^2-4)=44x.$

Frame the equations whose roots are:

31. $7, -3.$ 32. $\pm 4.$ 33. $+a, -b.$
34. $-12, +15.$ 35. $m+n, m-n.$

Note. In solving equations it is important to note the following two principles:

(i) When both sides of an equation are squared or multiplied by an expression containing the variable, new or extraneous roots are introduced into it. For example,

when $x-5=0$, $x=+5$ *only.*

If we take the square of $x=+5$, we get $x^2=+25$.

Taking the square root of both sides, we have

$$x=\pm 5.$$

The original root is $+5$, and -5 is the extraneous root, introduced by squaring both sides of the equation.

Again, if we multiply both sides of equation $x=+5$ by $x-3$, we get

$$x(x-3)=+5(x-3).$$

By transposition, $x(x-3)-5(x-3)=0,$

or $(x-3)(x-5)=0.$

Hence, the two roots are $+3$ and $+5$, out of which $+3$ is extraneous and introduced by multiplying both sides of $x=+5$ by $x-3$.

(ii) If both sides of an equation are divided by an expression containing the variable, we lose some roots, the number of such roots is the same as the degree of that expression. For example, let us suppose

$$(5x+2)(3x-4)=(5x+2)(4x-3).$$

Dividing both sides of the equation by $5x+2$, we have

$$3x-4=4x-3 \text{ or } x=-1.$$

If instead of dividing both sides by $5x+2$, we transpose all the terms to the left-hand side, we get

$$(5x+2)(3x-4)-(5x+2)(4x-3)=0$$

or $\quad (5x+2)\{(3x-4)-(4x-3)\}=0$

or $\quad (5x+2)(-x-1)=0$

or $\quad (5x+2)(x+1)=0$

$\therefore$ either $(5x+2)=0$ and $x=-\frac{2}{5}$,

or $\quad (x+1)=0$ and $x=-1$.

$\therefore$ the two roots are $-\frac{2}{5}$ and -1.

Thus by dividing both sides of the original equation by $5x+2$, we lose the root $x=-\frac{2}{5}$.

Method of Completing the Square

4. The method of completing the square is very important as it enables us to solve quadratic equations which cannot be easily done by the method of factorisation.

The method is illustrated in the following examples.

Example 1. Solve the equation $x^2-3x=18$.

Completing the square by adding to each side the square of half the co-efficient of x, we have

$$x^2-3x+(\tfrac{3}{2})^2=18+(\tfrac{3}{2})^2$$

$$\text{or } (x-\tfrac{3}{2})^2=\tfrac{81}{4}.$$

Taking the square root of both sides, we have

$$(x-\tfrac{3}{2})=\pm\tfrac{9}{2}$$

$$\therefore \quad x=\tfrac{3}{2}\pm\tfrac{9}{2}.$$

$\therefore$ the required roots are $+6, -3$.

Example 2. Solve the equation $6x^2+7x-20=0$.

By transposition, $6x^2+7x=20$.

Making the co-efficient of x^2 unity by dividing both sides by 6, we have

$$x^2+\tfrac{7}{6}x=\tfrac{10}{3}.$$

Adding the square of half the co-efficient of x to each side, *i.e.*, $(\tfrac{7}{12})^2$, we have

$$x^2+\tfrac{7}{6}x+(\tfrac{7}{12})^2=\tfrac{10}{3}+(\tfrac{7}{12})^2$$

$$\therefore \quad (x+\tfrac{7}{12})^2=\tfrac{529}{144}.$$

Taking the square root of both sides, we have

$$x+\tfrac{7}{12}=\pm\tfrac{23}{12}$$

$$\therefore \quad x=-\tfrac{7}{12}\pm\tfrac{23}{12}.$$

Hence the required roots are $-\tfrac{5}{2}$, $+\tfrac{4}{3}$.

EXERCISE 90.

Solve the following equations by completing the square :

1. $x^2-4x=45$.
2. $x^2-6x-65=0$.
3. $3x^2+13x=30$.
4. $6x^2-13x+6=0$.
5. $8x^2-14x-9=0$.
6. $15x^2-11x-12=0$.
7. $5x^2-2x-3=0$.
8. $4x^2-48x-25=0$.
9. $3x^2-51x+216=0$.
10. $ax^2-bx-c=0$.

Method of Solving by Formula

5. The method of completing the square, when applied to the general form $ax^2+bx+c=0$, gives us a well-known formula for finding the roots of the quadratic equation.

$$ax^2+bx+c=0.$$

Dividing the equation by a, we get

$$x^2+\frac{b}{a}x+\frac{c}{a}=0.$$

$$\therefore \quad x^2+\frac{b}{a}x=-\frac{c}{a}$$

$$\therefore \quad x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2=\left(\frac{b}{2a}\right)^2-\frac{c}{a}$$

$$\therefore \quad \left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$$

$$\therefore \quad x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a}$$

$$\therefore \quad x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

This formula should be stated in words and committed to memory.

Example. Solve the equation $5x^2-31x+30=0$ by the formula.

If the given equation is put in the general form $ax^2+bx+c=0$

$$a=+5,\ b=-31,\ c=30, \text{ and}$$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{31\pm\sqrt{(-31)^2-4.5.30}}{2.5}$$

$$=\frac{31\pm\sqrt{961-600}}{10}=\frac{31\pm\sqrt{361}}{10}$$

$$=\frac{31\pm19}{10}=5,\ \tfrac{6}{5}.$$

EXERCISE 91.

Solve the following equations by the formula:

1. $3x^2-5x-2=0.$
2. $3x^2-2x-1=0.$
3. $3x^2-7x+2=0.$
4. $12x^2-17x+6=0.$
5. $4x^2-65x+126=0.$
6. $10x+11=\dfrac{6}{x}.$
7. $\dfrac{5}{5-x}+\dfrac{8}{8-x}=3.$
8. $7x^2+17x+6=0.$
9. $5x^2-23x+12=0.$
10. $15x^2+2px-8p^2=0.$
11. $x^2+px=2(x+p).$
12. $pq(x^2-1)=(p^2-q^2)x.$

6. PROBLEMS INVOLVING QUADRATIC EQUATIONS

Since a quadratic equation has always two roots, the problems involving quadratic equations give us two different solutions. The student has to verify to see which of the two roots satisfies the particular problem.

Generally one of the two solutions will satisfy the given problem and the other solution will satisfy a similar problem, if properly interpreted.

Example 1. Find two consecutive even numbers such that the sum of their squares is 100.

Let x and $x+2$ be the two consecutive even numbers.

$$\therefore \quad x^2+(x+2)^2=100$$
$$\therefore \quad x^2+x^2+4x+4=100$$
$$\therefore \quad 2x^2+4x-96=0$$
$$\therefore \quad x^2+2x-48=0$$
$$\therefore \quad (x+8)(x-6)=0$$

Hence the roots are -8 and $+6$.

Verification. (1) When $x=-8$, $x+2=-6$ and
$(-8)^2+(-6)^2=100$.

(2) When $x=+6$, $x+2=+8$ and
$(+6)^2+(+8)^2=100$.

Example 2. The perimeter of a rectangle is 54 ft. and its area is 180 sq. ft. Find its length and breadth.

Since the perimeter $=54$ ft., $\therefore$ its semi-perimeter $=27$ ft.

Let the length be x ft. $\therefore$ the breadth $=(27-x)$ ft., and the area $=x\,(27-x)$ sq. ft.

$$\therefore \quad x(27-x)=180$$
$$\text{or} \quad x^2-27x+180=0$$
$$\therefore \quad (x-15)(x-12)=0$$

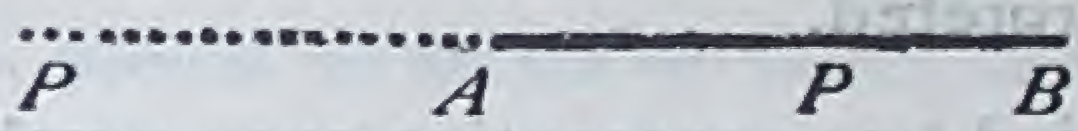
Hence the roots are 15 and 12.

Verification. (1) When the length is 15, breadth $=27-15=12$ and the area $=15\times12=180$.

(2) When the length is 12, breadth $=27-12=15$ and the area $=12\times15=180$.

Example 3. AB is a straight line whose length is a units. Find a point P in it such that $AB.PB=AP^2$.

Let $AP=x$ units, then $PB=(a-x)$ units.



$$\because \quad AB.PB=AP^2$$

$$\therefore \quad a(a-x)=x^2$$

or $$x^2+ax-a^2=0$$

$$\therefore \quad x=\frac{-a\pm\sqrt{a^2+4a^2}}{2}=\frac{-a\pm a\sqrt{5}}{2}.$$

Hence the two roots are $\frac{1}{2}(\sqrt{5}-1)a$ and $-\frac{1}{2}(\sqrt{5}+1)a$.

The positive value of x corresponds to the point P of internal division.

The negative value of x corresponds to the point P of external division.

EXERCISE 92.

1. Find a number which is equal to three times its square.

2. The sum of a number and its reciprocal is $\frac{25}{12}$. Find it.

3. The sum of the squares of two consecutive numbers is 365. Find them.

4. The product of two consecutive even numbers exceeds their sum by 142. Find them.

5. The sum of the squares of two consecutive odd numbers is 290. Find them.

6. Two numbers differ by 7 and their reciprocals differ by $\frac{7}{60}$. Find them.

7. The area of a rectangle is 255 sq. ft. If its length be diminished by 1 and breadth increased by 1, it becomes a square. Find its dimensions.

8. A straight line AB is a units in length, find a point P in it such that

(i) $AP^2 = 2\ AB.\ PB.$ (ii) $AP^2 = 2\ PB^2.$

(iii) $AP.\ PB = 3\ PB^2.$ (iv) $2\ AP.\ PB = (\frac{1}{2}\ AB)^2.$

9. The area of a rectangle is 240 sq. ft. and its diagonal is 26 ft. Find its sides.

10. The area of a right-angled triangle is 240 sq. ft. and its hypotenuse is 34 ft. Find its sides.

11. The area of a rectangle is equal to the area of a square whose side is 12 ft. longer than the breadth of the rectangle. If the length of the rectangle be increased by 25 ft. and the breadth diminished by 6 ft., its area remains unaltered. Find its dimensions.

12. A cyclist rode 132 miles in a number of hours which was less by 1 than the number of miles he rode per hour. Find the number of hours he rode.

13. A cyclist rode 75 miles at a uniform rate. If he had gone 3 miles per hour slower, he would have taken one hour and 15 minutes more. What was his rate?

14. A lawn is of rectangular shape. Its length is 24 yds. and breadth 16 yds. A path of uniform width whose area is equal to that of the lawn, goes round it. Find the width of the path.

15. The hypotenuse of a right-angled triangle is less than the sum of the other sides by 8 ft. and its area is 120 sq. ft. Find its sides.

16. If a cyclist had gone 3 miles per hour faster, he would have taken 1 hour and 20 minutes less to ride 80 miles. What time did he take?

17. The circumference of one wheel is 4 ft. more than that of another. If the larger wheel makes 220 revolutions less in a mile than the smaller, find the circumference of each wheel.

18. The front wheel of a carriage makes 60 revolutions more than the hind wheel in going 1,080 ft. If the circumference of the front wheel be increased by 2 ft., it will make 15 revolutions more than the hind wheel. Find the circumference of each wheel.

19. The length, breadth and height of a rectangular room are in the ratio of 8 : 6 : 5; if each of the dimensions be increased by a foot, the area of its four walls would be 1,408 sq. ft. Find the dimensions of the room.

20. A battalion of soldiers is formed into a solid square. If the number of soldiers be reduced by 16, it can be formed into a hollow square 6 deep, having in the front 16 soldiers more than before. What is the number of soldiers?

7. Graphs of Quadratic Functions and Equations.

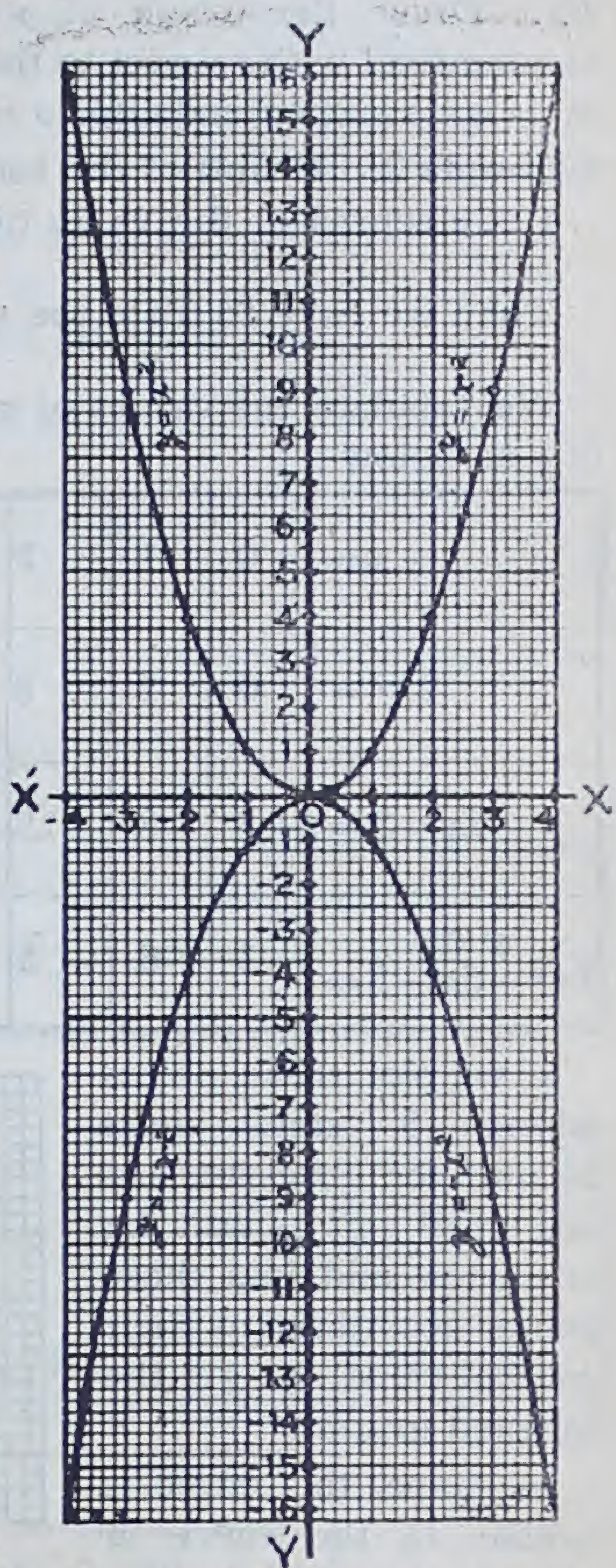
Example 1. Graphs of $y=x^2$ and $y=-x^2$.

Tabulating the values of x and the corresponding values of y in $y=x^2$, we have

$x=$	0	±1	±2	±3	±4	...
$y=$	0	1	4	9	16	...

If we plot these points to the scale $5mm.$ as unit for abscissæ and $4mm.$ as unit for ordinates and join these points, we get the annexed upper graph. (p 393).

It is important to note that for any two equal and opposite values of x, the corresponding values of y are equal and *positive.* Consequently the graph lies entirely above the x-axis and is symmetrical with respect to the y-axis.



As x increases indefinitely, y also increases indefinitely but more rapidly than x.

From this graph we can read off approximately the values of squares and square roots. For example, where $x = 1{\cdot}8$.

we read $y = 3{\cdot}25$ approximately,

$\therefore \quad (1{\cdot}8)^2 = 3{\cdot}25$ approximately.

Again, at the point where $y = 7{\cdot}75$, we read $x = \pm 2{\cdot}8$ approximately.

Hence $\sqrt{7{\cdot}75} = \pm 2{\cdot}8$ approximately.

Draw the graph of $y = -x^2$ with the same origin and axes as for $y = x^2$ and notice that this graph is exactly like the graph of $y = x^2$ but lies just below it and *appears to be its image in x-axis.*

Each of these graphs is a curve known as **parabola.**

Ex. Draw the graph of $y=2x^2$ and show that it is symmetrical with respect to the y-axis, lies throughout above the x-axis and extends up to infinity.

Example 2. Graph of the function $2x^2-5x-3$.

Let y represent the value of the functions, or

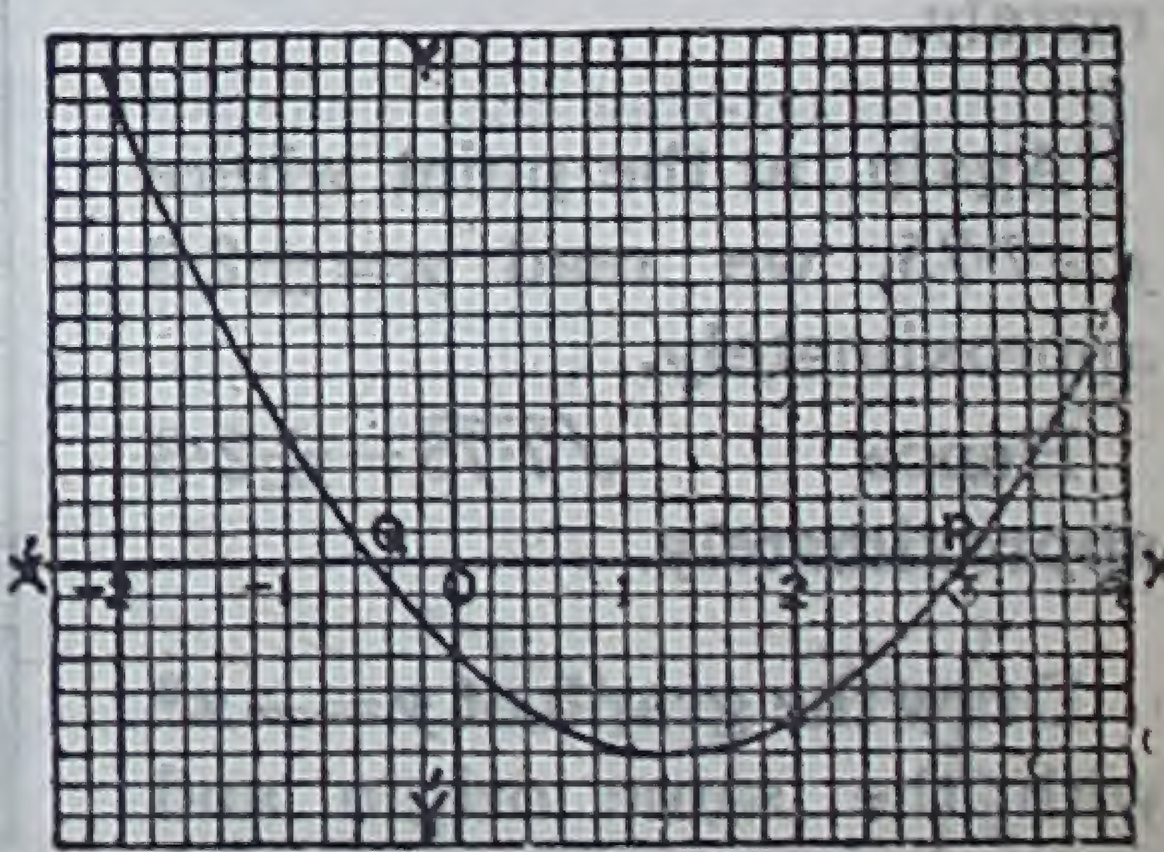
$$y=2x^2-5x-3.$$

Then we have to trace the graph of the equation

$$y=2x^2-5x-3.$$

We tabulate the values of x and the corresponding values of y as below :

$x=$	0	1	2	3	4	...	-1	-2	...
$2x^2=$	0	2	8	18	32	...	2	8	...
$-5x-3=$	-3	-8	-13	-18	-23	...	2	7	...
y or $2x^2-5x-3=$	-3	-6	-5	0	9	...	4	15	...



If we plot these points taking 5 small units as unit for abscissæ and one small unit as unit for ordinates and join these points according to their general trend, we get the annexed graph.

In shape, this curve is similar to the curve in example 1, but is differently placed with respect to the axes.

Try to answer the following questions :

1. What are the values of y when $x=1{\cdot}5$, $2{\cdot}5$, $-{\cdot}5$ $-1{\cdot}5$?
2. For what value of x is the value of y least ?
3. What is the least value of y ?
4. For what value of x is y equal to zero ?
5. Read the co-ordinates of points P and Q where the graph crosses the x-axis.
6. Can you use the graph to solve $2x^2-5x-3=0$?
7. For what values of x do we have $2x^2-5x-3=6$?

Example 3 Solve graphically the equation

$$2x^2-5x-3=0 \quad \ldots \quad (1)$$

First Method. Draw the graph of $y=2x^2-5x-3$ ... (2) as in example 2.

To solve the equation (1) we have to find the values of x which make $2x^2-5x-3$ equal to zero, *i.e.*, values of x which make $y=0$.

Now when $y=0$, the graph of (2) crosses the x-axis at the points P and Q.

The values of x at these points are 3 and $-\frac{1}{2}$.

Hence the roots of (1) are 3 and $-\frac{1}{2}$.

Or, **the roots of equation (1) are the abscissæ of points where the graph of (2) crosses the x-axis.**

Second Method. The equation $2x^2-5x-3=0$ is the same as the equation $2x^2=5x+3$.

Equate each side of the equation to y,

then $y=2x^2$ and $y=5x+3$.

Draw the graphs of $y=2x^2$ ... ... (1)

and $y=5x+3$ ... ... (2)

For (1) when

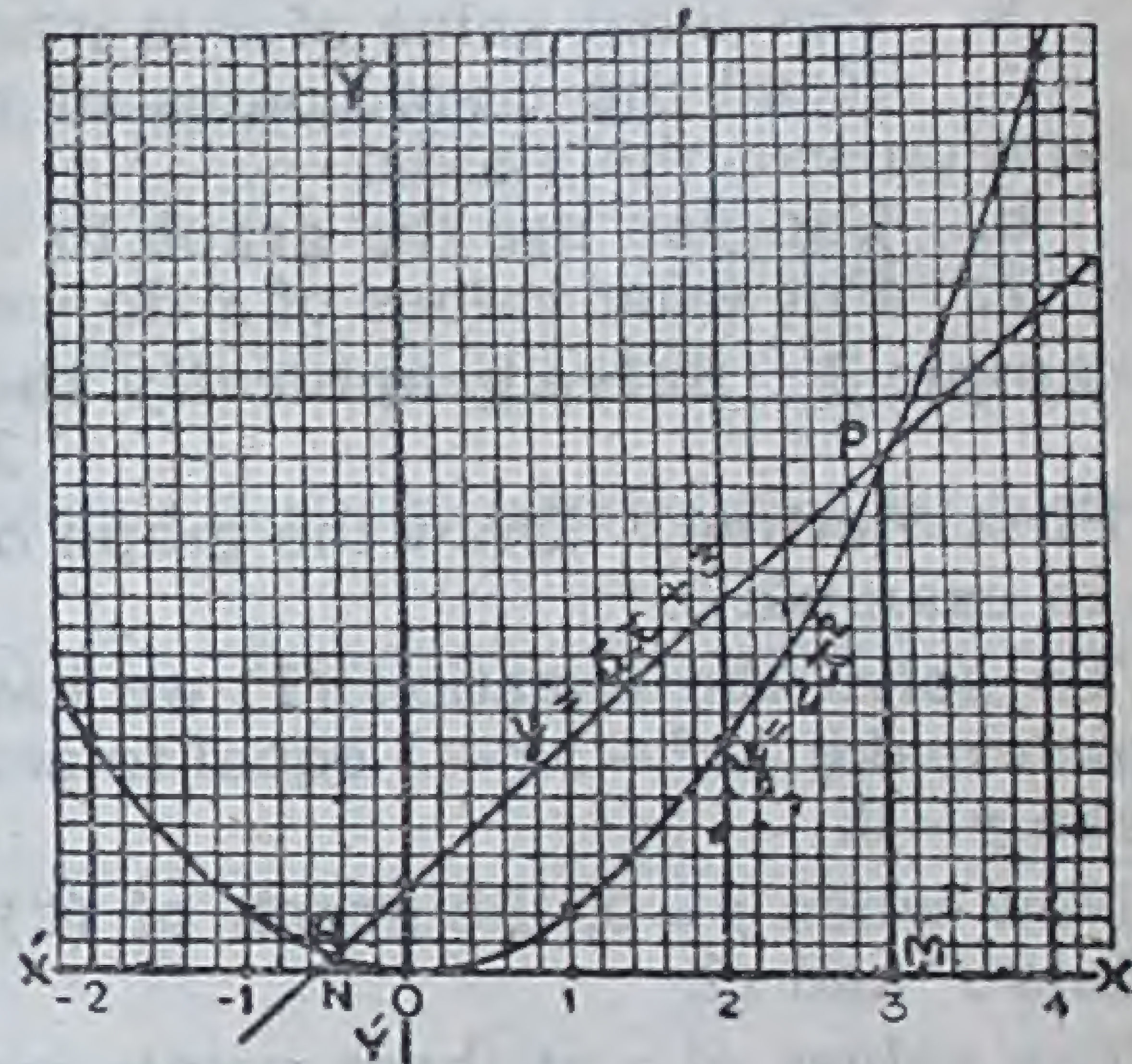
$x=$	0	1	2	3	4	...	-1	-2	...
$y=$	0	2	8	18	32	...	2	8	...

For (2) when

$x=$	0	1	4	...
$y=$	3	8	23	...

These graphs are drawn below to the same scale, taking 5 small units as unit of abscissæ and one small unit as unit of ordinates.

The graphs intersect at points P and Q. The value of x at the point $P = OM = 3$, *i.e.*, when $x = 3$, we have $2x^2 \equiv 5x + 3$.



Similarly, the value of x at $Q = ON = -\frac{1}{2}$, *i.e.*, when $x = -\frac{1}{2}$ we have $2x^2 \equiv 5x + 3$.

Hence the solutions of the equation are 3 and $-\frac{1}{2}$.

In other words, **the abscissæ of the points of intersection of the graphs of $y = 2x^2$ and $y = 5x + 3$ are the roots of the equation $2x^2 = 5x + 3$.**

Third Method. Another method which is simpler depends upon the use of the graph of $y = x^2$, as illustrated below:

To solve graphically $2x^2 - 5x - 3 = 0$,

let $y = x^2$ ... ... (1)

then $2y - 5x - 3 = 0$ ... ... (2)

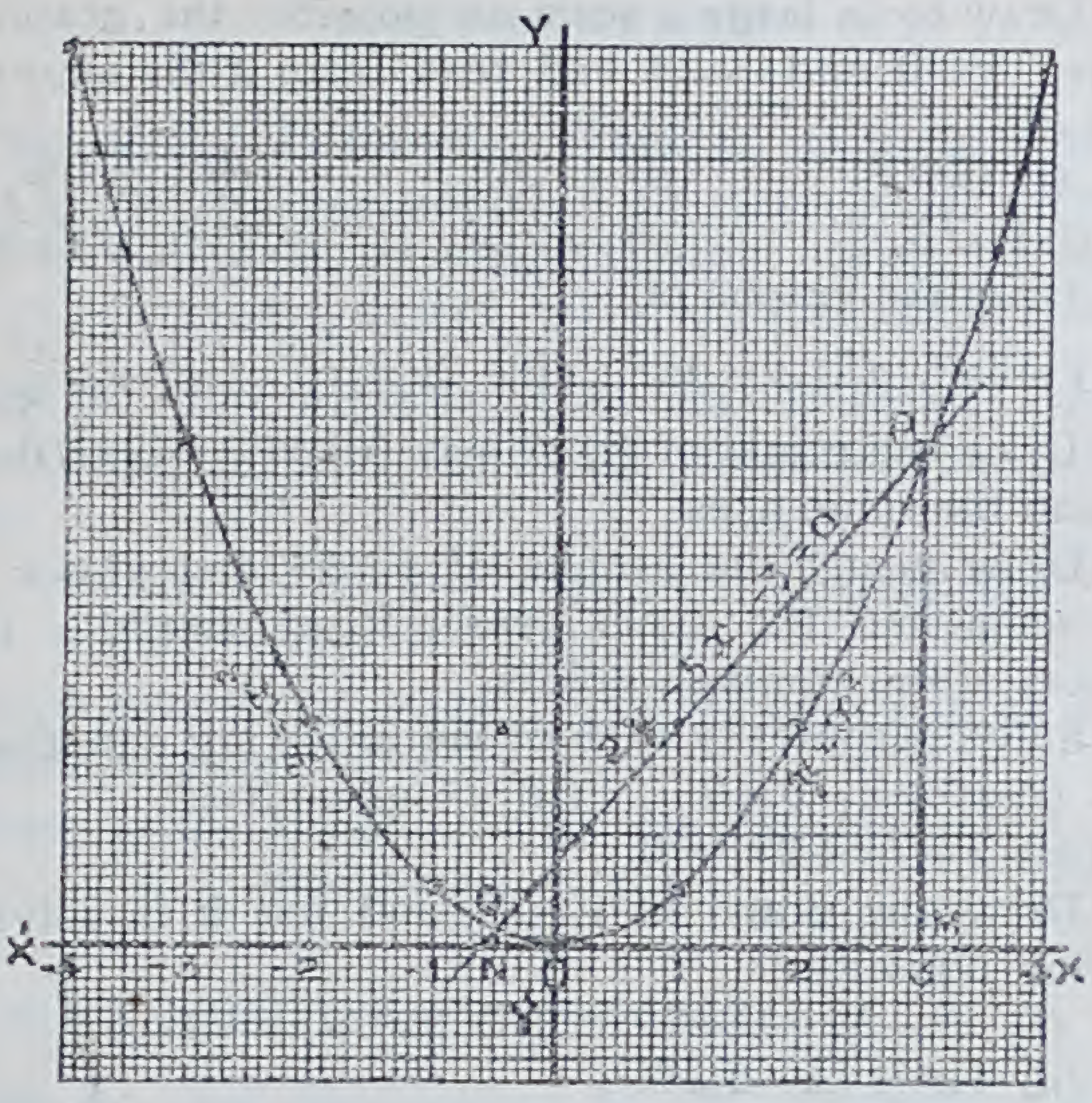
For (1) when

$x=$	0	± 1	± 2	± 3	± 4	...
$y=$	0	1	4	9	16	...

For (2) when

$x=$	1	3	...
$y=$	4	9	...

These graphs are drawn below to the same scale, taking



1 cm. as unit of abscissæ and $\frac{1}{2}$ cm. as unit of ordinates. The graphs intersect at points P and Q.

Measuring the abscissæ of P and Q, we obtain 3 and $-\frac{1}{2}$ as the two roots of the equation.

These three methods being general are applicable in all cases.

The third method is more convenient as it employs the graph of $y=x^2$, no matter what quadratic we wish to solve.

EXERCISE 93.

1. Draw to as large a scale as possible the graph of x^2 between $x=0$ and $x=4$ and read from it the approximate values of

(i) $(1{\cdot}4)^2$, (ii) $(2{\cdot}3)^2$, (iii) $(3{\cdot}2)^2$,
(iv) $\sqrt{3}$, (v) $\sqrt{7{\cdot}3}$, (vi) $\sqrt{13{\cdot}5}$.

2. Draw the graphs of

(i) $y=3x^2$, (ii) $y=4x^2$, (iii) $2y=x^2$, (iv) $y=\dfrac{x^2}{5}$.

3. Draw the graph of (i) $y^2=x$, (ii) $y^2=-x$ to the same scale and the same axes.

4. Draw neatly the graphs of $y=x^2$ and $y^2=x$ to the same scale and the same axes. Read carefully the co-ordinates of the common points.

5. Solve graphically each of the following equations:

(i) $x^2+x-6=0$. (ii) $x^2-4x+3=0$.
(iii) $3+7x+2x^2=0$.

6. Draw the graph of $y=x^2$ and use it in solving the following equations:

(i) $x^2-5x+6=0$, (ii) $2x^2+5x-3=0$,
(ii) $4x^2+4x-3=0$.

7. Draw the graph of the function $2x^2+5x$ and use it in finding the roots of $2x^2+5x-3=0$.

8. Draw the graphs of $y=x^2$ and $3y+4x-4=0$ to the same scale and the same axes and use them in finding out the roots of $3x^2+4x-4=0$.

9. Draw the graphs of $y=3x^2$ and $10x+8$ to the same scale and the same axes and use them in finding the roots of $3x^2-10x-8=0$.

10. A stone is projected vertically upwards. Its height h ft. above the point of projection, after t seconds is given by $h=96t-16t^2$.

Draw a graph showing the relation between h and t. From the graph, (i) find t when $h=128$, (ii) find h when $t=3$.

CHAPTER XXI

REMAINDER THEOREM AND ITS APPLICATION

1. Remainder Theorem. When an expression in x is divided by another expression of the form $x+p$, ordinarily we find the remainder by actual division. Now, we propose to find the remainder, in such cases, without actual division.

Example 1. Let $2x^2+3x+5$ be the dividend, $x+2$ the divisor, Q the quotient and R the remainder, then, by the identity,

$$\text{dividend} \equiv \text{divisor} \times \text{quotient} + \text{remainder},$$

$$(x+2)\times Q+R \equiv 2x^2+3x+5 \quad \ldots \quad \ldots \quad \text{(i)}$$

Since (i) is an identity and is true for *any value* of x, therefore it is true when $x=-2$.

Thus, $(-2+2)\times Q+R=2(-2)^2+3(-2)+5$

or $\quad 0\times Q+R=8-6+5$

or $\quad R=7.$

[Verify the result by actual division.]

Example 2. Let $2x^2+3x+5$ be the dividend, $x-2$ the divisor, Q the quotient and R the remainder,

then, $\quad (x-2)\times Q+R \equiv 2x^2+3x+5 \quad \ldots \quad \text{(ii)}$

As (ii) is true for *any value* of x, therefore it is true when $x=2$.

Thus, $(2-2)\times Q+R=2(2)^2+3(2)+5$

or $\quad 0\times Q+R=8+6+5$

or $\quad R=19.$

[Verify the result by actual division.]

Example 3. Let ax^3+bx^2+cx+d be the dividend, $x-p$ the divisor, Q the quotient and R the remainder,

then, $\quad (x-p)Q+R \equiv ax^3+bx^2+cx+d \quad \ldots \quad$ (iii)

Since (iii) is true for all values of x, therefore it is true when $x=p$.

Thus, $\quad (p-p)Q+R=a(p)^3+b(p)^2+c(p)+d,$

or $\quad 0+R=ap^3+bp^2+cp+d,$

or $\quad R=ap^3+bp^2+cp+d.$

[Verify the result by actual division.]

If the above dividend is to be divided by $x+p$,

then $\quad (x+p)Q+R \equiv ax^3+bx^2+cx+d \quad \ldots \quad$ (iv)

Since (iv) is true for all values of x, therefore it is true when $x=-p$.

Thus, $\quad (-p+p)Q+R=a(-p)^3+b(-p)^2+c(-p)+d.$

or $\quad 0+R=a(-p)^3+b(-p)^2+c(-p)+d.$

[Simplify this result and verify it by actual division.]

From these examples we deduce the following theorem :

Theorem. *When an expression in x is the dividend and $x \pm p$ the divisor ; the remainder R is obtained by substituting $\mp p$ for x in the expression.*

This is known as the **Remainder Theorem.**

Note. It is useful to note that in (i) if $x+2=0$ we get R, similarly, if in (ii), (iii) and (iv) we put respectively $x-2=0$, $x-p=0$ and $x+p=0$, we get R.

2. Factor Theorem.

Example 4. Prove that $x-1$, $x+1$, $x-2$ are the factors of $2x^4-5x^3+5x-2$.

(i) When $x-1$ is the divisor,

$$R=2(+1)^4-5(+1)^3+5(+1)-2$$
$$=2-5+5-2=0.$$

Since in this case the remainder is 0,

$\therefore \quad x-1$ is a factor of the given expression.

(ii) When $x+1$ is the divisor,

$$R=2(-1)^4-5(-1)^3+5(-1)-2$$
$$=2+5-5-2=0.$$

Since in this case the remainder is 0,

$\therefore \quad x+1$ is a factor of the given expression.

(iii) When $x-2$ is the divisor

$$R=2(+2)^4-5(+2)^3+5(+2)-2$$
$$=32-40+10-2=0.$$

Since in this case the remainder is 0,

$\therefore \quad x+2$ is a factor of the given expression.

We see from these examples that

if an expression in x is reduced to zero on substituting $(\mp p)$ for x, then $(x \pm p)$ is a factor of the expression.

This is known as the **Factor Theorem.**

In several cases, the two rules deduced from this theorem, as given below, will be found useful :—

Rule 1. $x-1$ *is a factor of an expression in x, if the sum of the co-efficients of terms containing x plus the term containing no x be equal to zero.*

Take for instance, $ax^4+bx^3+cx^2+dx+e$.

The remainder when it is divided by $x-1$ is

$$a.1^4+b.1^3+c.1^2+d.1+e=a+b+c+d+e$$
$$=\text{the sum of the co-efficients.}$$

If this is zero, then $x-1$ is a factor of $ax^4+bx^3+cx^2+dx+e$.

Rule 2. $x+1$ *is a factor of an expression, if the sums of the co-efficients of the alternate terms be equal, provided the terms are properly arranged and no term is missing.*

[Term containing no x is supposed to contain x^0.]

Take for instance, $ax^4+bx^3+cx^2+dx+e$.

The remainder when it is divided by $x+1$ is

$a.(-1)^4+b.(-1)^3+c.(-1)^2+d.(-1)+e=a-b+c-d+e.$

This will be zero, if $a+c+e=b+d$;

i.e. if the sum of the co-efficients of 1st, third and fifth terms = the sum of the co-efficients of 2nd and fourth terms.

Example 5. Resolve into factors $x^3-6x^2+12x-7$.

The sum of the co-efficients $=1-6+12-7=0$;

$\therefore$ by rule 1, $x-1$ is a factor of the expression.

Proceeding as in Type XII,

$$x^3-6x^2+12x-7=x^2(x-1)-5x(x-1)+7(x-1)$$
$$=(x-1)(x^2-5x+7).$$

Example 6. Resolve into factors $x^3+8x^2+19x+12$.

Here, the sum of the co-efficients of the 1st and third terms$=1+19=20$, and the sum of the co-efficients of the 2nd and the fourth terms $=8+12=20$;

$\therefore$ by rule 2, $x+1$ is a factor of the expression.

Proceeding as in Type XII,

$$x^3+8x^2+19x+12=x^2(x+1)+7x(x+1)+12(x+1)$$
$$=(x+1)(x^2+7x+12)$$

As $x^2+7x+12$ can be factorised into $(x+3)(x+4)$,

$\therefore$ $x^3+8x^2+19x+12=(x+1)(x+3)(x+4)$,

Example 7. Resolve into factors $x^5+x^3-3x^2-2x+3$.

When the missing term in the form of $0.x^4$ is added, the expression$=x^5+0.x^4+x^3-3x^2-2x+3$ and the sum of the co-efficients of the 1st, 3rd and 5th terms $=1+1-2=0$,

whereas the sum of the co-efficients of the 2nd, 4th and 6th terms $=0-3+3=0$.

As these sums are equal, $x+1$ is a factor of the expression.

Proceeding as in Type XII,

$$x^5+x^3-3x^2-2x+3=x^4(x+1)-x^3(x+1)+2x^2(x+1)$$
$$-5x(x+1)+3(x+1)$$
$$=(x+1)(x^4-x^3+2x^2-5x+3).$$

Again, as the sum of the co-efficients of $x^4-x^3+2x^2-5x+3$ is 0,

$\therefore$ $x-1$ is a factor of $x^4-x^3+2x^2-5x+3$.

$\therefore$ $x^4-x^3+2x^2-5x+3=x^3(x-1)+2x(x-1)-3(x-1)$
$$=(x-1)(x^3+2x-3).$$

Again, as the sum of the co-efficients of x^3+2x-3 is 0.

$\therefore$ $x-1$ is a factor of x^3+2x-3.

$\therefore$ $x^3+2x-3=x^2(x-1)+x(x-1)+3(x-1)$
$$=(x-1)(x^2+x+3).$$

Since x^2+x+3 cannot be factorised

$\therefore$ $x^5+x^3-3x^2-2x+3=(x+1)(x-1)(x-1)(x^2+x+3)$.

EXERCISE 94.

By the Remainder Theorem, find the remainder when

1. x^3-5x^2+6x-3 is divided by $x-1$.
2. $2x^3-3x^2+4x-1$ is divided by $x+2$.
3. $4x^3-2x^2+5x-3$ is divided by $x-a$.
4. $5x^3-x^2+3x-4$ is divided by $x+m$.
5. $2x^4+7x^3a-x^2a^2+11xa^3-6a^4$ is divided by $x+2a$.

Prove that:

6. $x-1$ is a factor of $x^4-3x^3+x^2+3x-2$.
7. $x+2$ and $x-3$ are the factors of $x^3-3x^2-4x+12$.
8. $x-2$ and $x+4$ are the factors of $2x^3+7x^2-10x-24$.
9. $(a+b)$ and $(b+c)$ are the factors of $(a+b+c)^3-a^3-b^3-c^3$.

[*Hint.* Substitute in the expression $-b$ for a in the first case and $-c$ for b in the second case.]

10. a is a factor of $(a+b+c)^3-(b+c-a)^3-(c+a-b)^3-(b+a-c)^3$. [*Hint.* $a=(a-0)$.]

Resolve into factors using the above two rules:

11. x^3+x^2+x-3.
12. $x^3-6x^2+11x-6$.
13. $x^3+9x^2+23x+15$.
14. $3x^3+x^2+x-5$.
15. $4x^3+3x^2+x+2$.
16. $2x^4+3x^3+2x^2-3x-4$.
17. $x^4+2x^3-5x^2+2$.
18. x^4-x^3+7x+5.
19. $x^4+x^3-7x^2-x+6$.
20. $2x^4-x^3-8x^2+11x-4$.

Example 8. For what value of m will $x-2$ be a factor of x^3-5x^2+mx+m?

$x-2$ will be a factor of the given expression

if $(2)^3-5(2)^2+m(2)+m=0$

or if $-12+3m=0$

or if $m=4$.

21. For what value of p is $x^2-(p+2)x+6$ exactly divisible by $x-p$?

22. Find the condition that ax^3+bx^2+cx+d may be exactly divisible by $x-1$.

23. Find the value of a, b for which x^2+x-6 may be a factor of x^3-ax^2-bx-6.

24. What number must be added to $5x^4-7x^3+6$ so that the result may be divisible by $x+3$?

25. If $x^2-4ax-19a^2-b$ is exactly divisible by $x-7a$, prove that $b=2a^2$.

26. The expression px^3+2x^2-3 and x^4-px+4 leave the same remainder when they are divided by $x-2$. Find the value of p.

Let us consider the relation between the algebraic sum of the co-efficients of a given expression and that of the co-efficients of its factors.

Taking the continued product of any number of binomial expressions, say $(2x+3)$, $(7x-4)$, $(3x-1)$, we get

$$42x^3+25x^2-49x+12\equiv(2x+3)(7x-4)(3x-1).$$

Since it is an identity, it is true for all values of x and therefore it is true when $x=1$.

Putting $x=1$ in the identity, we get the algebraic sum of the co-efficients of the expressions and the algebraic sum of the co-efficients of each of its factors as shown below:

$$42.(1)^3+25.(1)^2-49(1)+12=(2.1+3)(7.1-4)(3.1-1)$$

or $$42+25-49+12=(2+3)(7-4)(3-1).$$

Obviously $(42+25-49+12)$ or 30 is exactly divisible by each of $(2+3)$, $(7-4)$, $(3-1)$, *i.e.*, the algebraic sum of the co-efficients of the given expression is exactly divisible by the algebraic sum of the co-efficients of each of its factors.

Thus, we deduce the following rule :—

Rule 3. *The algebraic sum of the co-efficients of each factor of an expression is a factor of the algebraic sum of its co-efficients.*

Obviously the converse of this rule does not hold good for, in above example if we reverse the order of the co-efficients in the binomials we get $(3x+2)$, $(4x-7)$ and $(1-3x)$; the sum of the co-efficients of each of these is a factor of the sum of the co-efficients of the original expression but none of them is its factor.

Therefore rule 3 is a sure test of the rejection of impossible factors and not a sure test of the selection of the real factors. Thus its importance lies in the simplification of the process of selection, as illustrated in the next two examples.

Example 9. Factorise x^3-5x^2-5x-6.

As the expression is of the 3rd degree, it can have either three linear factors or one linear and one quadratic; *i.e.*, *it must have a linear factor.*

As the first term of the expression is x^3, the first term of the linear factor is x.

As the last term of the expression is -6, the last term of the linear factor may be one of

$$+1, \ -1, \ +2, \ -2, \ +3, \ -3, \ +6, \ -6.$$

Or the linear trial factors could be $(x+1)$, $(x-1)$, $(x+2)$, $(x-2)$, $(x+3)$, $(x-3)$, $(x+6)$ or $(x-6)$.

Since, in this case, neither rule 1 holds good nor rule 2, therefore $(x+1)$ or $(x-1)$ cannot be a factor of the expression,

Again, the sum of the co-efficients of the expression is

$$1-5-5-6=-15,$$

whereas the sum of the co-efficients of $x+2$ is 3 and it is a factor of 15, therefore $x+2$ may be a factor of the expression: the sum of the co-efficients of $x-2$ is -1 and it is a factor of 15, therefore $x-2$ may be a factor of the expression; the sum of the co-efficients of $x+3$, $x-3$, $x+6$ is 4, -2, 7 respectively and none of them is a factor of 15, therefore none of $x+3$, $x-3$, $x+6$ can be a factor of the expression; the sum of the co-efficients of $x-6$ is -5 and it is a factor of 15, thefore $x-6$ may be a factor of the expression.

Hence $x+2$, $x-2$, $x-6$ are the only binomials left for trial.

Substituting -2 or $+2$ for x in the given expression, we find that it does not vanish, but on substituting $+6$ for x, we see that it vanishes.

$\therefore$ $x-6$ is one of the factors.

The given expression can be put as

$$x^2(x-6)+x(x-6)+(x-6)=(x-6)(x^2+x+1).$$

As x^2+x+1 cannot be further factorised,

$\therefore$ $(x-6)(x^2+x+1)$ are the required factors.

Example 10. Factorise $3x^4+4x^3-57x^2-18x-40$.

Since, in this case, rules 1, 2 do not hold good,

$\therefore$ $(x-1)$ or $(x+1)$ is not a factor of this expression.

As the last term of the expression is -40, the last term of the linear factor (if any) would be 2, 4, 5, 8, 10 or 20.

Hence the possible trial factors are
$x+2$, $x-2$, $x+4$, $x-4$, $x+5$, $x-5$, $x+8$, $x-8$, $x+10$, $x-10$, $x+20$, $x-20$.

The sum of the co-efficients of the expression

$$=3+4-57-18-40=-108.$$

The sum of the co-efficients of the trial factors is respectively 3, -1, $+5$, -3, $+6$, -4, $+9$, -7, $+11$, -9, $+21$ and -19.

As 5, 9, -7, 11, 21 and -19 are not the factors of 108, we reject $x+4$, $x+8$, $x-8$, $x+10$, $x+20$ and $x-20$.

On substituting -2 or $+2$ for x in the given expression, we find that it does not vanish, therefore we reject $x+2$ and $x-2$ as well.

On substituting $+4$ for x in the given expression, we find that it vanishes.

$\therefore$ $x-4$ is one of its factors.

Hence $3x^4+4x^3-57x^2-18x-40$

$$=3x^3(x-4)+16x^2(x-4)+7x(x-4)+10(x-4)$$
$$=(x-4)(3x^3+16x^2+7x+10).$$

If we substitute -5 for x in $3x^3+16x^2+7x+10$, it vanishes.

$$\therefore\ 3x^3+16x^2+7x+10=3x^2(x+5)+x(x+5)+2(x+5)$$
$$=(x+5)(3x^2+x+2)$$

Since $3x^2+x+2$ cannot be further factorised

$\therefore$ $(x-4)(x+5)(3x^2+x+2)$ are the required factors.

Factorise :

27. $x^3+3x^2-10x-24$.

28. $x^3-5x^2-2x+24$.

29. $6x^3+x^2-19x+6$.

30. $x^3-19x-30$.

31. $x^3-39x+70$.

32. x^3-x^2+12.

33. $x^3-5x+12$.

34. $x^4-2x^3-20x^2+39x+36$

35. Prove that $(b+c)(c+a)(a+b)$ is a factor of $(a+b+c)^n-a^n-b^n-c^n$ and $(b-c)(c-a)(a-b)$ is a factor of $(b-c)^n+(c-a)^n+(a-b)^n$ where n is an odd number.

[*Hint.* Substitute $-c$ for b and put $b+c=0$.]

3. Divisibility of $x^n \pm a^n$ by $x \pm a$.

(i) When x^n-a^n is divided by $x-a$, the remainder is $a^n-a^n=0$,

$\therefore$ x^n-a^n is *always* divisible by $x-a$

(ii) When x^n-a^n is divided by $x+a$, the remainder is $(-a)^n-a^n$ and is equal to 0 or $-2a^n$, according as n is even or odd.

$\therefore$ x^n-a^n is divisible by $x+a$ only when n is *even*.

(iii) When x^n+a^n is divided by $x+a$, the remainder is $(-a)^n+a^n$ and is equal to 0 or $2a^n$, according as n is *odd* or *even*.

$\therefore$ x^n+a^n is divisible by $x+a$ only when n is *odd*.

(iv) When x^n+a^n is divided by $x-a$, the remainder is $a^n+a^n=2a^n$.

$\therefore$ x^n+a^n is *never* divisible by $x-a$.

Example 1. Shew that 10^n-1 is *always* divisible by 9.

Since x^n-a^n is always divisible by $x-a$,

$\therefore$ by putting $x=10$ and $a=1$, we find that

10^n-1^n is always divisible by $10-1$,

or 10^n-1 is always divisible by 9.

Example 2. Simplify $\dfrac{8x^3+27y^3}{2x+3y}$.

Since x^n+a^n is divisible by $x+a$ when n is *odd*,

$$\therefore \quad \frac{8x^3+27y^3}{2x+3y}=\frac{(2x)^3+(3y)^3}{2x+3y}=(2x)^2-(2x)(3y)+(3y)^2$$
$$=4x^2-6xy+9y^2.$$

Example 3. Simplify $\dfrac{x^5+32y^5}{x+2y}$.

Since x^n+a^n is divisible by $x+a$ when n is *odd*.

$$\therefore \quad \frac{x^5+32y^5}{x+2y}=\frac{(x)^5+(2y)^5}{x+2y}$$
$$=x^4-x^3(2y)+x^2(2y)^2-x(2y)^3+(2y)^4$$
$$=x^4-2x^3y+4x^2y^2-8xy^3+16y^4.$$

EXERCISE 95.

Shew that

1. 9^n-1 is *always* divisible by 8.
2. 10^n+1 is *never* divisible by 9.
3. 10^n-1 is divisible by 11 when n is *even*.
4. 10^n+1 is divisible by 11 when n is *odd*.
5. $4^{3n}-3^{3n}$ is *always* divisible by 37.
6. $8^{2m}-1$ is *always* divisible by 63.
7. 17^8-5^8 is divisible by 6 and 11.

Write down by *inspection* the quotient of :

8. $(8a^3-x^3)\div(2a-x)$.
9. $(a^3+64b^3)\div(a+4b)$.
10. $(64a^6-x^6)\div(2a+x)$.
11. $(27m^3-1)\div(3m-1)$.
12. $(x^4-256y^4)\div(x-4y)$.
13. $(1-16m^4)\div(1+2m)$.
14. $(32a^5+b^5)\div(2a+b)$.

*CHAPTER XXII

INTEGRAL FUNCTION, HOMOGENEITY, SYMMETRY, CYCLIC ORDER, INDETERMINATE CO-EFFICIENTS

1. Variables and constants. Let us consider the variation in the value of:

$$2p^2+3p+5 \quad \ldots \quad \ldots \quad \ldots \quad \text{(i)}$$

and

$$3x^2-4x+6 \quad \ldots \quad \ldots \quad \ldots \quad \text{(ii)}$$

If in expression (i) $p=1$, its value $=10$
if ,, ,, ,, $p=2$, ,, ,, $=19$
if ,, ,, ,, $p=3$, ,, ,, $=32$.

Thus, the value of expression (i) *varies as the value of* p *varies*—p is the **variable** in this expression.

Similarly, the value of expression (ii) *varies as the value of* x *varies*—x is the **variable.**

In each of these expressions, *only one letter occurs.*

Now, let us consider the expression $a^2x^2+abx+b^2$.

Here, three letters a, b, x occur. If the value of this expression is supposed to depend upon the value of a only, then a is the variable and the letters x, b are non-variables or **constants.** If its value is supposed to depend upon x or b, then x or b is the variable and the remaining letters are the constants.

It is usual to take x, y, z, as variables and a, b, c as constants.

2. Function. An algebraical expression which contains x as the variable is called a **function of** x, and is denoted as $f(x)$, $F(x)$, $\phi(x)$, &c. If it contains x and y both as the variables, it is called a function of x and y and is denoted as $f(x,y)$, $F(x,y)$, etc.

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* This chapter is meant for the brighter section of the class, as an extra assignment.

For instance, x^3+4x^2-5x+7 is a function of x and may be denoted by $f(x)$ and $3x^2+4xy+2y^2$ is a function of x and y and may be denoted by $f(x, y)$.

A function is said to be **integral** when none of the variables appears in any denominator, and it is said to be **fractional** if the variables ocour in its denominators. Thus x, y, z being the variables and a, b, c, d, &c., being the constants,

$$ax^2+by^2+cz^2+dxy+exz+fyz$$

and $\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}+\frac{xy}{ab}+\frac{xz}{bc}+\frac{yz}{ac}$ are integral functions,

and $\frac{a}{x^2}+bx+c$ and $ax^3+\frac{b}{y^3}+cz^3+dxyz$ are fractional functions.

A function is said to be **rational** if it contains no roots of the variables and **irrational** if it contains roots of the variables; for instance, $ax^2+2bxy+cy^2$ is rational and $ax+2b\sqrt{xy}+cy$ is irrational.

In this chapter we shall consider only the rational integral functions.

It is important to note that if in a particular discussion $f(x)$ stands for a particular expression, say, $3x^2-4x+5$, then $f(y)$ would stand for $3y^2-4y+5$,

$f(p)$	,,	,,	,,	$3p^2-4p+5$,
$f(-m)$	,,	,,	,,	$3(-m)^2-4(-m)+5$,
$f(1)$	,,	,,	,,	$3.1^2-4.1+5$,
$f(0)$	,,	,,	,,	$30.^2-4.0+5$,
$f(p+1)$	,,	,,	,,	$3(p+1)^2-4(p+1)+5$.

Example 1. If $f(x)=x^3-3x^2+3x-1$, find the value of (i) $f(-x)$, (ii) $f(p-1)$.

(i) Putting $-x$ for x in the given expression, we have $f(-x)=(-x)^3-3(-x)^2+3(-x)-1$.

$$=-x^3-3x^2-3x-1.$$

(ii) Putting $(p-1)$ for x in the given expression, we have

$$\begin{aligned} f(p-1) &= (p-1)^3-3(p-1)^2+3(p-1)-1 \\ &= p^3-3p^2+3p-1-3(p^2-2p+1)+3p-3-1 \\ &= p^3-3p^2+3p-1-3p^2+6p-3+3p-3-1 \\ &= p^3-6p^2+12p-8. \end{aligned}$$

3. **Degree of Function.** The degree or dimension of an integral function is the same as the degree of the highest term in it.

Thus, the degree of $2x^3-5x^6+4x-7x^2+8$ is 6, and the degree of $ax^2+abx^5+c^6x+9$ is 5, if x is the variable.

Functions of the second and third degree are also called **quadratic** and **cubic** functions respectively. Thus $ax^2+bxy+cy^2$ and $ax^3+by^3+cx^2y+dxy^2$ are respectively the quadratic and cubic functions of x, y.

Taking $a_0, a_1, a_2, a_3, \ldots\ldots\ldots$ as the co-efficients, the general expressions in x, of various degrees, are as follows:

1st degree a_0x+a_1 ... ... [2 terms.]
2nd degree $a_0x^2+a_1x+a_2$ ... ... [3 terms.]
3rd degree $a_0x^3+a_1x^2+a_2x+a_3$ ... [4 terms.]

... ...

... ...

... ...

and of the n^{th} degree is

$$a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots+a_{n-2}x^2+a_{n-1}x$$
$$+a_n \ldots [n+1 \text{ terms.}]$$

Similarly, the general expression in x, y of various degrees are as follows:

1st degree $a_0x+a_1y+a_2$ ... [3 terms.]
2nd degree $a_0x^2+a_1xy+a_2y^2+a_3x+a_4y+a_5$ ... [6 terms.]
3rd degree $a_0x^3+a_1x^2y+a_2xy^2+a_3y^3+a_4x^2+a_5xy+a_6y^2$
$+a_7x+a_8y+a_9$ ... .. [10 terms.]

Thus, a complete general expression of any degree should contain all the terms of the given degree, together with all the terms of lower degrees.

If we multiply a function of the 3rd degree, say $a_0x^3 + a_1x^2 + a_2x + a_3$ by a function of the 2nd degree, say $p_0x^2 + p_1x + p_2$, the product would evidently be of the 5th degree.

Similarly, if we multiply a function of the m^{th} degree by a function of the n^{th} degree, the product would be of the $(m+n)^{th}$ degree.

Again, if we divide a function of the 4th degree, say $x^4 + x^2y^2 + y^4$, by a function of the 2nd degree, say $x^2 + xy + y^2$, the quotient is evidently of the 2nd degree.

Similarly, if we divide a function of the m^{th} degree by a function of the n^{th} degree, the quotient would evidently be of the $(m-n)^{th}$ degree. These relations may be put in the form of the following law:

Law of Degree. The degree of the *product* of two integral functions is the *sum* of the degrees of the functions and the degree of the *quotient* of two integral functions is the *difference* of their degrees.

4. Method of Detached Co-efficients. When two integral functions of the *same variables* are to be multiplied together or one is to be divided by another, the process may be abbreviated, by taking the following steps, which are based upon the *law of degree*:

1. Supply the missing terms (if any), by affixing a zero to each of such terms as its co-efficient.

2. Arrange the integral functions in descending or ascending powers of the variable.

3. After removing the variable, do the process of multiplication or division as usual.

4. To the terms of the product or the quotient thus got, affix the variable with proper degree, bearing in mind the law of degree.

Example 2. Multiply x^3+3x-1 by x^2-4x+2.

The first expression when completed $=x^3+0x^2+3x-1$. As both the expressions are already arranged in descending powers of x, we take their co-efficients in the given order.

$$\begin{array}{rrrrrr}
1+0+3-1 & & & & & \\
1-4+2 & & & & & \\
\hline
1 & +0 & +3 & -1 & & \\
 & -4 & -0 & -12 & +4 & \\
 & & +2 & +0 & +6 & -2 \\
\hline
1 & -4 & +5 & -13 & +10 & -2
\end{array}$$

As the given expressions are of the 3rd and 2nd degrees respectively, their product must be of the 5th degree.

$\therefore$ the product $=x^5-4x^4+5x^3-13x^2+10x-2$.

Example 3. Divide x^4+9x^2+81 by x^2+3x+9.

The dividend when completed $=x^4+0x^3+9x^2+0x+81$. Taking the co-efficients in order, we have

$$\begin{array}{r|rrrrr|l}
1+3+9 & 1 & +0 & +9 & +0 & +81 & 1-3+9 \\
 & 1 & +3 & +9 & & & \\
\hline
 & & -3 & +0 & +0 & & \\
 & & -3 & -9 & -27 & & \\
\hline
 & & & +9 & +27 & +81 & \\
 & & & +9 & +27 & +81 & \\
\hline
 & & & * & * & * &
\end{array}$$

As the dividend and the divisor are of the 4th and 2nd degrees respectively, the quotient must be of the 2nd degree.

$\therefore$ quotient $=x^2-3x+9$.

The method employed in these examples is called the **method of detached co-efficients.**

EXERCISE 96.

1. Write down a general expression in x of the 8th degree.
2. Write down a general expression in x, y of the 4th degree.

Taking x, y, z as the variables, state which of the following expressions are :

(i) integral and rational,
(ii) integral and irrational,
(iii) fractional and rational,
(iv) fractional and irrational.

3. $a\sqrt{x}+\dfrac{b}{\sqrt{y}}+cz.$

4. $ax^2+\dfrac{b}{x}+\dfrac{1}{c}.$

5. $p\sqrt{x^3}+qx.$

6. $\left(x+\dfrac{1}{x}\right)\left(y+\dfrac{1}{y}\right)\left(z+\dfrac{1}{z}\right).$

7. $\left(x+\dfrac{1}{a}\right)^2+\left(y+\dfrac{1}{b}\right)^2+\left(z+\dfrac{1}{c}\right)^2.$

8. $\dfrac{ax-b\sqrt{x^3}}{c\sqrt{x}}.$

9. If $f(x)=3x^2-4x+5$, find the value of
(i) $f(1)$, (ii) $f(2)$, (iii) $f(-1)$, (iv) $f(0)$, (v) $2f(x)$, (vi) $3f(-k)$, (vii) $f(m+1)$.

10. If $f(x)=2x^3-x^2+3x+1$, find the value of
(i) $f(2)-f(1)$, (ii) $f(1)+f(0)$, (iii) $f(p)+f(-p)$.

11. If $f(x)=ax^2+bx+c$, find the value of $f(x+1)-f(x-1)$

12. If $f(x)=\dfrac{2x+1}{2x-1}$, find the value of $3f(2)-2f(0)$.

Multiply, using the method of detached co-efficients :

13. x^2-3x+2 by $2x^2+x-3$.
14. x^3-2x^2+x+3 by x^2-3x+4.
15. x^2-5x+3 by $2x^3-3x+1$.
16. $7x^4-2x^2+3$ by $2x^3-4x-3$.
17. $5x^3-2x-1$ by $3x^3+x^2+1$.
18. $5x^4-3x^3-2x^2+4x+1$ by x^3-2x^2+x-1.
19. $6x^4+x^3-1$ by $2x^4-3x^2+4$.
20. $\frac{3}{3}x^3-\frac{4}{5}x^2y+\frac{1}{2}y^3$ by $\frac{3}{5}x-2y$.
21. $\cdot 4x^2+\cdot 3xy-\cdot 2y^2$ by $\cdot 5x^2-\cdot 2xy+\cdot 5y^2$.
22. Expand, using detached co-efficients:
$(x^2+2x+1)(x^3-3x+1)(2x^3+x^2+3)$.
23. Multiply $(2x-3)(3x+2)(x-a)$ and find the value of a for which the co-efficient of x^2 vanishes.

Divide, using the method of detached co-efficients:

24. $x^5-2x-x^4+6x^2-4x^3$ by $2+x^3-4x$.

25. $11x^3+48-12x+30x^4-82x^2$ by $2x-4+3x^2$.

26. $1-5x+9x^2-6x^3+x^5$ by $1-2x+x^2$.

27. $8x^3-1$ by $4x^2+2x+1$.

28. x^4-1 by $x+1$.

29. Find the expression of the 4th degree which multiplied by x^3+x^2+x+1 gives x^7+x^6+x+1 as the product.

5. **Homogeneous expression.** An expression is said to be **homogeneous** when all its terms are of the same degree. Thus,

x^2-y^2+3xy is a homogeneous expression of the 2nd degree in x and y.

$x^2-y^2+z^2-2xy+3xz-yz$ is a homogeneous expression of the 2nd degree in x, y and z.

$ax^3+by^3+cz^3-dxyz$ is a homogeneous expression of the 3rd degree in x, y and z.

Complete Homogeneous Expression

$5x^3+4x^2y+3xy^2+y^3$ is a homogeneous expression of the 3rd degree in x and y and is **complete** in itself, for no term is missing; whereas $5x^3+3xy^2+y^3$ is also a homogeneous expression of the 3rd degree in x and y but is **incomplete**, for a term of the type x^2y is missing. It can be made complete by inserting $+0x^2y$ in it. The insertion of $+0x^2y$ does not change its value, for $0x^2y=0$.

Similarly, any incomplete expression can be put in the form as a complete expression by inserting the missing terms with zero as the co-efficient of each of such terms.

Type. Let us examine the forms of terms in a few complete homogeneous expressions in x and y:

a_0x+a_1y ... ... (i)

$a_0x^2+a_1xy+a_2y^2$ ... ... (ii)

$a_0x^3+a_1x^2y+a_2xy^2+a_3y^3$ ... ... (iii)

$a_0x^4+a_1x^3y+a_2x^2y^2+a_3xy^3+a_4y^4$ ... (iv)

$a_0x^5+a_1x^4y+a_2x^3y^2+a_3x^2y^3+a_4xy^4+a_5y^5$ ... (v)

and classify the terms in each, according to the forms or types.

In the first expression the terms are of *one* type only, *i.e.*, a_0x.

In the 2nd expression the terms are of *two* types, *i.e.*, a_0x^2 and a_1xy.

In the 3rd expression the terms are of *two* types, *i.e.*, a_0x^3 and a_1x^2y.

In the 4th expression the terms are of *three* types, *i.e.*, a_0x^4, a_1x^3y and $a_2x^2y^2$.

In the 5th expression the terms are of *three* types, *i.e.*, a_0x^5, a_1x^4y and $a_2x^3y^2$.

If in these expressions the terms of the same type be grouped together, we have

(a_0x+a_1y) ... ... (i)

$(a_0x^2+a_2y^2)+a_1xy$ ... ... (ii)

$(a_0x^3+a_3y^3)+(a_1x^2y+a_2xy^2)$ ... ... (iii)

$(a_0x^4+a_4y^4)+(a_1x^3y+a_3xy^3)+a_2x^2y^2$ ... (iv)

$(a_0x^5+a_5y^5)+(a_1x^4y+a_4xy^4)+(a_2x^3y^2+a_3x^2y^3)$... (v)

Thus, the best way of writing down a homogeneous expression is first to think out all the possible types and then to write down all the terms in each type.

Example 1. Write down a homogeneous expression of the 3rd degree in x, y and z.

The types of terms in the required expression are :

(i) ax^3, (ii) bx^2y, (iii) $cxyz$.

$\therefore$ the required expression is

$(a_0x^3+a_1y^3+a_2z^3)+(a_3x^2y+a_4x^2z+a_5y^2x+a_6y^2z+a_7z^2x+a_8z^2y)+a_9xyz$.

6. **Absolute Symmetry.** Just as in Geometry, a symmetrical figure is one in which all points on one side of the axis of symmetry have corresponding points on the other side, or if the points on one side be interchanged with the corresponding points on the other side, the figure remains unaltered, so in Algebra, a **symmetrical expression**, with respect to a pair of letters in it, is one which remains unaltered if those two letters be interchanged, *e.g.* $2x^2+3xy+2y^2$ is symmetrical with respect to x and y, for when x and y are interchanged, the resulting expression $2y^2+3yx+2x^2$ is equal to the original expression.

Such a symmetry is known as **absolute symmetry.**

The expression $2x^2+2y^2+2z^2+3xy+3xz+3yz$ is symmetrical with respect to (x, y), (y, z) and (z, x), for it remains unaltered when (i) x, y are interchanged, (ii) y, z are interchanged and (iii) when z, x are interchanged.

The expression $2x^2+2y^2+5z^2+3xy+7xz+7yz$ is symmetrical with respect to (x, y) but not with respect to (x, z) and (y, z), for if we interchange x and y, its value remains unaltered, but its value is altered when x and z or y and z are interchanged.

Ex. Think out the condition which will make the above expression symmetrical

(i) with respect to x, z;

(ii) with respect to y, z.

Let us consider the expressions:

$a_0x^5+a_1y^5$ ... ... (i)

$a_2x^4y+a_3xy^4$ ... ... (ii)

$a_4x^3y^2+a_5x^2y^3$ ... ... (iii)

Expression (i) will be symmetrical with respect to x, y if $a_0=a_1$.

Expression (ii) will be symmetrical with respect to x, y if $a_2=a_3$.

Expression (iii) will be symmetrical with respect to x, y if $a_4=a_5$.

Consequently, $(a_0x^5+a_1y^5)+(a_2x^4y+a_3xy^4)+(a_4x^3y^2+a_5x^2y^3)$ will be symmetrical, with respect to x, y if the terms of the type x^5 have equal co-efficients, the terms of the type x^4y have equal co-efficients and the terms of the type x^3y^2 have equal co-efficients.

Law. In a symmetrical expression all the terms of the same type must have equal co-efficients.

Let us consider the *sum*, the *difference*, the *product* and the *quotient* of symmetrical expressions.

Take the following 4 expressions, which are symmetrical with respect to x, y:

$x+y$ ... ... (i)

x^2+xy+y^2 ... ... (ii)

x^2-xy+y^2 ... ... (iii)

$x^4+x^2y^2+y^4$ ... ... (iv)

and apply the test of symmetry on the results we get by adding (i) and (ii), (i) and (iii), (i) and (iv), (ii) and (iii), (ii) and (iv), (iii) and (iv);

by subtracting (i) from (ii), (i) from (iii), (i) from (iv), (ii) from (iii), (ii) from (iv) and (iii) from (iv);

by multiplying (i) and (ii), (i) and (iii), (i) and (iv), (ii) and (iii), (ii) and (iv), (iii) and (iv);

and by dividing (iv) by (ii) and (iv) by (iii).

From the examination of the above results, we establish the following law:

Law. The sum, the difference, the product and the quotient of two symmetrical expressions are symmetrical.

7. Cyclic Order. In this figure, the letters a, b, c, are so arranged round the circumference of a circle that if we start from a and go round the circumference in the direction of the arrows, we meet with b and after b with c and after c with a, and so on. Such an arrangement of letters is said to be **cyclic.**

Let us consider the arrangement of terms in the following expressions:

$$a(b+c)+b(c+a)+c(a+b),$$
$$a^2(b-c)+b^2(c-a)+c^2(a-b).$$

In these expressions, if in the first terms a be replaced by b, b by c and c by a, we get the second terms; if in the second terms b be replaced by c, c by a and a by b, we get the third terms; if in the third terms c be replaced by a, a by b and b by c, we get the first terms. Such expressions are said to be written in **cyclic order.**

An expression is said to be **written in cyclic order** when each term in it can be derived from the preceding term by changing the first letter into the second, the second into the third and the third into the first.

Example 2. Write $a(c^2-b^2)+b(a^2-c^2)+c(b^2-a^2)$ in cyclic order.

The first factors of the terms, *i.e.*, a, b, c are already in cyclic order, but the second factors are not.

Arranging the second factors of the terms in cyclic order, we get $-a(b^2-c^2)-b(c^2-a^2)-c(a^2-b^2)$

$$=-[a(b^2-c^2)+b(c^2-a^2)+c(a^2-b^2)].$$

Example 3. Write down the cyclic expression in a, b, c whose first term is $\dfrac{a}{a(b-c)}$.

Putting b for a, c for b and a for c, we get

$$\frac{b}{b(c-a)} \text{ as the second term.}$$

Putting, in the second term, c for b, a for c and b for a, we get $\dfrac{c}{c(a-b)}$ as the third term.

$$\therefore \quad \text{the expression} = \frac{a}{a(b-c)}+\frac{b}{b(c-a)}+\frac{c}{c(a-b)}.$$

8. Sigma Notation. Whenever an expression consists of the sum of terms of the same type, it is quite easy to

derive all the terms from the first term, as illustrated in example 3. Such an expression can be put in an abbreviated form by prefixing the Greek letter Σ (Sigma) to the first term. The symbol Σ stands for *the sum of the terms of the same type.*

Thus, with variables a, b, c,

$\Sigma\, a$ means $a+b+c$,

$\Sigma\, a^2$ means $a^2+b^2+c^2$,

$\Sigma\, ab$ means $ab+bc+ca$,

$\Sigma\,(a^2+ab)$ means $(a^2+ab)+(b^2+bc)+(c^2+ca)$,

$3\,\Sigma\, ab$ means $3(ab+bc+ca)$,

and $2\,\Sigma\, a^2-3\,\Sigma\, ab$ means $2(a^2+b^2+c^2)-3(ab+bc+ca)$.

9. Cyclic Symmetry. Let us examine the nature of the expression $a^2(b-c)+b^2(c-a)+c^2(a-b)$. At first sight it looks as if it were absolutely symmetrical with respect to the letters (a, b), (b, c) and (c, a) but on interchanging a, b, we get the expression

$$b^2(a-c)+a^2(c-b)+c^2(b-a)$$

which $$=-a^2(b-c)-b^2(c-a)-c^2(a-b).$$

Thus, this expression differs from the original expression, in having each of its terms preceded by a negative sign instead of a positive sign ; hence the original expression *does not possess absolute symmetry.* However, it possesses a certain kind of symmetry which is known as the **cyclic symmetry**. Even such a symmetry is of great importance, as it helps us in deriving unknown results from one which is known, as illustrated in the next example.

Example 4. Simplify by the cyclic symmetry

$$(a+b-2c)(a+c-b)+(b+c-2a)(b+a-c)+(c+a-2b)(c+b-a).$$

By actual multiplication,

$(a+b-2c)(a+c-b)=a^2-b^2-2c^2-ac+3bc.$

By cyclic symmetry,

$(b+c-2a)(b+a-c)=b^2-c^2-2a^2-ba+3ca$

and $(c+a-2b)(c+b-a)=c^2-a^2-2b^2-cb+3ab)$

$\therefore$ the expression $=-2(a^2+b^2+c^2)+2(ab+ac+bc)$

$$=-2\,\Sigma a^2+2\,\Sigma\, ab.$$

EXERCISE 97.

Taking x, y, z as variables state which of the functions given below are

(i) homogeneous but not symmetrical,
(ii) symmetrical but not homogeneous,
(iii) homogeneous and symmetrical,
(iv) neither homogeneous nor symmetrical.

In cases (ii) and (iii), tell also whether the symmetry is absolute or cyclic:

1. $(x+y)^3$.
2. $(x-y)^3$.
3. $(x-y)(x+y-6)$.
4. $x^3-2x^2y+2xy^2+y^3$.
5. $x^3+ax^2y+bxy^2+y^3$.
6. $(x^2+y^2)(x^3-y^3)$.
7. $x^3+y^3+z^3-3xyz$.
8. $2(x^2+y^2)+5xy+a(x+y)$.
9. $x^3+3x^2y+3xy^2-y^3$.
10. $x^2+y^2+z^2-a(xy+yz+zx)$.
11. $x^2+y^2+z^2-2(x+y+z)$.
12. $xy(x-y)+yz(y-z)+zx(z-x)$.
13. Write down all the possible *types* of terms in a complete homogeneous expression of the 5th degree in x and y.
14. Write down all the possible types of terms in a complete homogeneous expression of the 4th degree in x, y and z.
15. Write down a homogeneous and symmetrical expression of the 4th degree in x, y and z, using a, b, c, &c. co-efficients.
16. Construct a homogeneous and symmetrical expression of the 2nd degree in x and y, which shall be equal to 16 when $x=1$ and $y=2$, and which shall be equal to 4 when $x=2$ and $y=-1$.
17. Construct a homogeneous and symmetrical expression of the 3rd degree in x and y, which shall be equal to 10 when $x=1$ and $y=1$, and equal to 36 when $x=2$ and $y=1$.

Write down all the types of terms in the following expressions, a, b, c being the variables and x being constant:

18. $\Sigma(x+a)(x+b)$.
19. $\Sigma(x-a)(b-a)(c-a)$.
20. $\Sigma(x+a)(x+b)(x-c)$.
21. $\Sigma\dfrac{x^2(a+b)}{ab}$.

22. $\Sigma \dfrac{bc}{(a-b)(a-c)}$. 23. $\Sigma \dfrac{1}{(a-b)(a-c)(x-a)}$.

Simplify the following expressions by the method of cyclic symmetry :

24. $(x+b)(x+c)+(x+c)(x+a)+(x+a)(x+b)$.

25. $(a+b-c)(b-c)+(b+c-a)(c-a)+(c+a-b)(a-b)$.

26. $(x+b+c)(x+b-c)+(x+c+a)(x+c-a)$
$+(x+a+b)(x+a-b)$.

27. $(a-b)(x+a)(x+b)+(b-c)(x+b)(x+c)$
$+(c-a)(x+c)(x+a)$.

28. $\Sigma(x+y)(x+z)$. 29. $\Sigma(x+y)(x-z)$.

30. $\Sigma xy(x-y)(x-z)$. 31. $\Sigma(a+b-c)^2$.

Under what conditions will the following expressions be symmetrical :

32. (i) $7x^2+ay^2+bz^2-3xy+cyz+dzx$,

(ii) $4x^3+py^3-2x^2y+qxy^2$,

(iii) $2x^4+my^4-nz^4+7x^2yz-py^2zx+qz^2xy$.

Without actual multiplication, state why the following relations are wrong :

33. $(a+b)(a^2-ab+b^2)=a^3-b^3$.

34. $a^3+b^3=(a-b)(a^2+ab+b^2)$.

35. $(a+b+c)(a^2+b^2+c^2-ab-bc-ac)$
$=(a^3+b^3+c^3-2ab-2bc-2ca)$.

36. $(x^2+y^2)(x+y)=x^3+2xy^2+3x^2y+y^3$.

37. $(x^3+y^3)(x+y)=x^4+x^3y+3xy^3+y^4$.

38. $(x+y+z)(x^2+y^2+z^2)=(x^3+y^3+z^3+xy+xz+zy)$.

39. $(x+y)(x^2+xy+y^2)=x^3+3xy+3xy^2+y^4$.

40. $(x+y+1)(x+y+z)=x^2+xy+xz+y^2$.

10. Principle of Indeterminate Co-efficients.

Let us compare the nature of relation in the following examples

$$x+3=8 \quad \ldots \quad \ldots \quad \ldots \quad \text{(i)}$$

$$2x-1=7 \quad \ldots \quad \ldots \quad \ldots \quad \text{(ii)}$$

$(2x-4)+(3x-2)=5x-6$ ... ... (iii)

$(x+1)(x-1)=x^2-1$ ... ... (iv)

$\dfrac{2x^2+3x+7}{x+1}=(2x+1)+6$ ... ... (v)

(i) is true for $x=5$ and for no other value of x,
(ii) is true for $x=4$ and for no other value of x,
(iii) is true for *any value* of x, say 0, 1, 2, 3, 4, 5, 6, &c.
(iv) is true for *any value* of x, say 0, 1, 2, 3, 4, 5, 6, &c.
(v) is true for *any value* of x, say 0, 1, 2, 3, 4, 5, 6, &c.

(i) and (ii) are equations, for they are true for definite values of x.

(iii), (iv) and (v) are identities, for they are true for any value of x.

Def. An identity is true for *any* value of the variable or variables.

Symbol $\equiv$ is used for '*identically equal to*'.

As the idea of identity implies that the two expressions *differ only in form*, it follows that the *co-efficient of any power of a variable in one expression is equal to the co-efficient of the same power of the variable in the other expression.* This is known as the **principle of indeterminate co-efficients** but really it should be called the **principle of undetermined co-efficients.**

Although the principle stated above is quite obvious, a formal proof is added below:

Proof. Let $a_0+a_1x+a_2x^2+a_3x^3+\ldots\equiv A_0+A_1x+A_2x^2+A_3x^3+\ldots$ be an identity and hence true for any value of x. Put $x=0$, then $a_0=A_0$.

Since $a_0=A_0$

$\therefore \quad a_1x+a_2x^2+a_3x^3+\ldots\equiv A_1x+A_2x^2+A_3x^3\ldots$

Dividing both sides by x, we get

$a_1+a_2x+a_3x^2+\ldots\equiv A_1+A_2x+A_3x^2+\ldots$

Put $x=0$, then $a_1=A_1$.

Similarly, $a_2=A_2$, $a_3=A_3$, &c.

Hence the *co-efficients of like powers of x are equal.*

The principle established here is of great importance.

Example 1. If $a_0x^3+a_1x^2+a_2x+a_3\equiv(x^2-3x+2)(x+3)$, find the values of a_0, a_1, a_2, a_3.

$$a_0x^3+a_1x^2+a_2x+a_3\equiv(x^2-3x+2)(x+3)$$
$$\equiv x^3-7x+6.$$

Equating the co-efficients of terms of the same degree, we get $a_0=1$, $a_1=0$, $a_2=-7$, $a_3=6$.

The co-efficient of the missing term is taken to be 0.

Example 2. If $2x^2+9x+6\equiv Ax(x+1)+Bx(x+2)$
$+C(x+1)(x+2)$,
find the values of A, B and C.

$$2x^2+9x+6\equiv A(x^2+x)+B(x^2+2x)+C(x^2+3x+2)$$
$$\equiv(A+B+C)x^2+(A+2B+3C)x+2C.$$

Equating the co-efficients of terms of equal powers, we get

$$A+B+C=2 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \text{(i)}$$
$$A+2B+3C=9 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \text{(ii)}$$
$$2C=6 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \text{(iii)}$$

From (iii), $C=3$.

Substituting the value of C in (i) and (ii), we get

$$A+B+3=2$$
or $\quad A+B=-1 \quad \ldots \quad \ldots \quad \ldots \quad$ (iv)

and $\quad A+2B+9=9$

or $\quad A+2B=0 \quad \ldots \quad \ldots \quad \ldots \quad$ (v)

From (iv) and (v), $A=-2$ and $B=1$.

$\therefore \quad A=-2$, $B=1$ and $C=3$.

Alternative Method

Instead of equating the co-efficients of terms of equal powers, it is sometimes more convenient to find the values of the constants, by assigning suitable values to x, as illustrated below:

In $2x^2+9x+6\equiv Ax(x+1)+Bx(x+2)+C(x+1)(x+2)$, put $x=0$; then $2\times0+9\times0+6=A\times0+B\times0+C\times1\times2$

or $\quad 6=2C$

$\therefore \quad C=3$.

Again, by putting -1 for x in the above identity, we get

$$2(-1)^2+9(-1)+6=A(-1)(0)+B(-1)(-1+2)$$
$$+C(0)(-1+2)$$

or $\quad 2-9+6=-B$

$\therefore \quad B=1.$

Again, by putting -2 for x in the above identity, we get

$$2(-2)^2+9(-2)+6=A(-2)(-2+1)+B(-2)(0)+$$
$$C(-2+1)(0)$$

or $\quad 8-18+6=2A$

$\therefore \quad A=-2.$

$\therefore \quad A=-2, \quad B=1$ and $C=3.$

Sometimes the first method is convenient and sometimes the second. It is only by practice that we can find out which method is to be employed in a particular case.

Example 3. Simplify $(a+b)^3+(b+c)^3+(c+a)^3-3(a+b)(b+c)(c+a)$ by the principles of homogeneity, symmetry and indeterminate co-efficients.

Since the given expression is symmetrical, homogeneous and of the 3rd degree in a, b, c, the result will also satisfy these conditions.

The types of terms in the result will be

(i) a^3, (ii) a^2b and (iii) abc.

$$\therefore \quad (a+b)^3+(b+c)^3+(c+a)^3-3(a+b)(b+c)(c+a)$$
$$\equiv p(a^3+b^3+c^3)+q(a^2b+a^2c+b^2c+b^2a+c^2a+c^2b)$$

$+r.abc$, where p, q, r are the co-efficients.

Putting $a=0$, $b=0$ and $c=1$, we get

$$(0)^3+(1)^3+(1)^3-3(0)(1)(1)\equiv p(1)+q(0)+r(0)$$

$\therefore \quad p=2.$

Putting $a=0$, $b=1$ and $c=1$, we get

$$(1)^3+(2)^3+(1)^3-3(1)(2)(1)\equiv p(2)+q(1+1)+r(0)$$
$$10-6=2p+2q$$
$$=2\times 2+2q$$
$$q=0.$$

Again, putting $a=2$, $b=1$ and $c=1$, we get

$$(3)^3+(2)^3+(3)^3-3(3)(2)(3)\equiv p(8+1+1)$$
$$+q(4+4+1+2+2+1)+r.\ 2.\ 1.\ 1.$$

or $27+8+27-54\equiv p\times 10+q\times 14+r\times 2.$

Substituting the values of p and q, we get

$$8\equiv 2\times 10+0\times 14+2r$$

or $r=-6.$

$\therefore$ $p=2,\ q=0$ and $r=-6.$

Hence the given expression $=2(a^3+b^3+c^3)-6abc.$

EXERCISE 98.

1. If $ax^2+bx+c\equiv 2(x+3)(x+1)-9$, find the values of a, b and c.

2. If $ax^3+bx^2+cx+d\equiv 3(x+1)(x-2)^2+3$, find the values of a, b, c and d.

3. If $lx^3+mx^2+nx+k\equiv(x+2)(x+3)(x-4)+3x+1$, find the values of l, m, n and k.

4. If $ax^4+bx^3+cx^2+dx+e\equiv(x+1)^4+(x-1)^4$, find the values of a, b, c, d and e.

5. If $a_0x^4+a_1x^3+a_2x^2+a_3x+a_4\equiv 2(x-2)(x+1)^3-10$, find the values of a_0, a_1, a_2, a_3 and a_4.

6. For what value of m will $x+3$ be a factor of x^2+9x+m?

7. If $x^3-6x^2+12x+k\equiv(x-a)^3$, find the values of k and a.

Find the numerical values of A, B, C and D in the following identities :

8. $9x-7\equiv A(x-1)+B(4x-3).$

9. $12x+5\equiv A(3x+4)-B(2x-1).$

10. $2(3x^2-13x+13)\equiv A(x-1)(x-2)+B(x-1)(x-3)$
$+C(x-2)(x-3).$

11. $3x^2+11x-7\equiv(Ax+B)(x-2)+C(x^2+3x-1).$

12. $2x^3-x^2+x-2\equiv A(x^2-1)(x+2)+B(x-1)(x^2-4)$
$+C(x+1)(x^2-4)+D(x^2-1)(x-2).$

13. Express $3x^2-4x+8$ in the form of
$$A(x-1)^2+B(x+2)+C.$$
14. Express $3x^3+9x^2+7x+2$ in the form of
$$A(x+1)^3+B(x+1)+C.$$
15. Express x^3-x^2-4x-6 in the form of $A(x+1)^3$
$$+B(x+1)^2+C(x+1)+D.$$
16. If $(x^2+y^2+z^2)(x+y+z)\equiv A\Sigma x^3+B\Sigma x^2y+C\Sigma xyz$, find the values of A, B and C.

Simplify :

17. $(x+y+z)^3-x^3-y^3-z^3$.

18. $(x+y+z)^3-(y+z-x)^3-(z+x-y)^3-(x+y-z)^3$.

19. $(y+z-2x)(y-z)^2+(z+x-2y)(z-x)^2$
$$+(x+y-2z)(x-y)^2+(y+z-2x)(z+x-2y)(x+y-2z).$$

Example 4. Find by the method of indeterminate co-efficients the square root of
$$16x^6-8x^5+x^4-16x^3+4x^2+4.$$

The square root must evidently be of the 3rd degree and its first term is $4x^3$.

Hence the form of the square root is $4x^3+ax^2+bx+c$.

Now, we have to find the values of a, b, c, so as to satisfy the identity $16x^6-8x^5+x^4-16x^3+4x^2+4\equiv(4x^3+ax^2+bx+c)^2$.

Using the method of detached co-efficients and avoiding the unnecessary terms, we get the corresponding co-efficients

$$\begin{array}{l}
\quad 4+\ a+\ b+\ c \\
\quad \underline{4+\ a+\ b+\ c} \\
16+4a+4b+4c \\
\qquad +4a+a^2+ab+\ \cdots \\
\qquad\qquad +4b+ab+\ \cdots \\
\qquad\qquad\qquad +4c+ \\
\hline
16+8a+(8b+a^2)+(2ab+8c)+\ \cdots
\end{array}$$

Equating the co-efficients of like powers, we have

(i) $8a=-8$ $\quad\therefore a=-1$.

(ii) $8b+a^2=1$ $\quad$ or $8b+(-1)^2=1$, $\quad\therefore b=0$.

(iii) $2ab+8c=-16$ $\quad$ or $2(-1)(0)+8c=-16$, $\quad\therefore c=-2$.

Substituting the values of a, b, c in $4x^3+ax^2+bx+c$, we get the required root $=4x^3-x^2-2$.

Find the square root of the following expressions by the method of indeterminate co-efficients ;

20. $9x^4-6x^3+13x^2-4x+4.$

21. $9x^6-12x^5+4x^4+24x^3-16x^2+16.$

22. $4x^6-12x^4+20x^3+9x^2-30x+25.$

23. Find the values of a and b for which $x^4+2ax^3-3x^2+2bx+4$ is a perfect square.

24. Find the values of p and q for which $9x^4+px^3+qx^2-10x+1$ is a perfect square.

Example 5. Find the cube root of $x^6+3x^5+9x^4+13x^3+18x^2+12x+8$ by the method of indeterminate co-efficients.

Since the expression is of the 6th degree, its cube root must evidently be of the 2nd degree and its first term $=\sqrt[3]{x^6}$ or x^2.

Hence the form of the cube root is x^2+ax+b.

Now, we have to find out the values of a and b, so as to satisfy the identity

$$x^6+3x^5+9x^4+13x^3+18x^2+12x+8\equiv(x^2+ax+b)^3.$$

Using the method of detached co-efficients and avoiding the unnecessary terms, we get the corresponding co-efficients.

$$\begin{array}{l}
1+\ a+b \\
\underline{1+\ a+b} \\
1+\ a+b \\
\quad +\ a+a^2+\dots \\
\qquad\quad \underline{+b\ +\dots} \\
1+2a+(a^2+2b)+\dots \\
\underline{1+\ a+b} \\
1+2a+(a^2+2b)+\dots \\
\quad +\ a+\qquad 2a^2+\dots \\
\qquad\quad \underline{+\qquad\ b\ +\dots} \\
1+3a+(3a^2+3b)+\dots
\end{array}$$

Equating the co-efficients of like powers, we have

(i) $3a=3$, $\quad\therefore\ a=1.$

(ii) $3a^2+3b=9$, or $3.1^2+3b=9$, $\quad\therefore\ b=2.$

Substituting the values of a and b in x^2+ax+b,

we get the required cube root $=x^2+x+2$.

Find the cube root of the following expressions by the method of indeterminate co-efficients :

25. $a^6-3a^5+9a^4-13a^3+18a^2-12a+8.$

26. $x^6-9x^5+21x^4+9x^3-42x^2-36x-8$.

27. $8x^6+12x^5-30x^4-35x^3+45x^2+27x-27$.

28. $27-54x+63x^2-44x^3+21x^4-6x^5+x^6$.

29. Find p so that $x^6+px^5+5x^3+px-1$ may be a perfect cube.

30. For what values of a and b will the expression $8x^6+12x^5-30x^4-5ax^3+5(a+2b)x^2+9(a-4b)x-27$ be a perfect cube ?

Example 6. Express $\dfrac{x+4}{(x+1)(x+2)}$ in the form of

$$\frac{A}{x+1}+\frac{B}{x+2}.$$

Putting $\dfrac{x+4}{(x+1)(x+2)}\equiv\dfrac{A}{x+1}+\dfrac{B}{x+2}$, we get

$$\begin{aligned} x+4 &\equiv A(x+2)+B(x+1) \\ &\equiv Ax+2A+Bx+B \\ &\equiv x(A+B)+(2A+B). \end{aligned}$$

Comparing the co-efficients, we have

$$A+B=1 \quad \ldots \quad \ldots \quad \ldots \quad \text{(i)}$$

$$\text{and } 2A+B=4 \quad \ldots \quad \ldots \quad \ldots \quad \text{(ii)}$$

From (i) and (ii), $A=3$ and $B=-2$.

Hence the required form is$=\dfrac{3}{x+1}-\dfrac{2}{x+2}$.

31. Express $\dfrac{2(1-x)}{(1+x)(x^2-x-6)}$ in the form of $\dfrac{A}{x+1}+\dfrac{B}{x-3}+\dfrac{C}{x+2}$.

32. Express $\dfrac{2(x+5)}{8x^3+1}$ in the form of $\dfrac{A}{2x+1}+\dfrac{Bx+C}{4x^2-2x+1}$.

33. Express $\dfrac{9}{(x-2)(x+1)^2}$ in the form of $\dfrac{A}{x-2}+\dfrac{Bx+C}{(x+1)^2}$.

34. If $\dfrac{2x^2-x-3}{(x-1)^2(x-2)}\equiv\dfrac{A}{x-2}+\dfrac{B}{x-1}+\dfrac{C}{(x-1)^2}$, find the values of A, B and C.

35. If $\dfrac{x^2+x+2}{(x^2+x+1)(x+1)^2}=\dfrac{A}{(x+1)}+\dfrac{B}{(x+1)^2}$ $+\dfrac{Cx+D}{x^2+x+1}$, find the values of A, B, C and D.

11. We now illustrate the combined application of the factor theorem, law of degree, homogeneity, symmetry and the principle of indeterminate co-efficients.

Example 1. Factorise x^3-5x^2-5x-6.

As the expression is of the 3rd degree, it can have either three linear factors or one linear and one quadratic, *i.e., it must have at least one linear factor.* [Law of degree.]

As the first term of the expression is x^3.

$\therefore$ the first term of the linear factor is x.

As the last term of the expression is -6.

$\therefore$ the last term of the linear factor may be one of $\pm 1, \pm 2, \pm 3, \pm 6$.

Substituting $\mp 1, \mp 2, \mp 3$, for x in the given expression, we find that it does not vanish, but on substituting $+6$ for x, we see that it vanishes.

$\therefore$ $x-6$ is one of the factors. [Factor theorem.]

The given expression can be put as

$$x^2(x-6)+x(x-6)+(x-6)$$
$$=(x-6)(x^2+x+1).$$

As x^2+x+1 cannot be further factorised,

$\therefore$ $(x-6)$, (x^2+x+1) are the required factors.

Example 2. Factorise x^3+x^2+4x-6.

As the expression is of the 3rd degree, it must have at least one linear factor.

As the first term of the expression is x^3, therefore the first term of the linear factor is x.

As the last term of the expression is -6, therefore the last term of the linear factor may be one of $\pm 1, \pm 2, \pm 3, \pm 6$.

On trial we find that the given expression vanishes when we substitute $+1$ for x,

$\therefore$ $(x-1)$ is one of the factors.

Now, the expression $=x^2(x-1)+2x(x-1)+6(x-1)$
$=(x-1)(x^2+2x+6)$.

As x^2+2x+6 cannot be further factorised,

$\therefore$ $(x-1)$, (x^2+2x+6) are the required factors.

Example 3. Factorise x^3-5x^2+2x+8.

As the expression is of the third degree, with x^3 as its first term and $+8$ as its last term, therefore the first term of the linear factor is x and the last term of the linear factor may be one of $\pm 1, \pm 2, \pm 4, \pm 8$.

On trial we find that the expression vanishes when $x=-1, +2$ and $+4$,

$\therefore$ $(x+1)$, $(x-2)$, $(x-4)$ are the three factors of the expression.

As these three linear factors form a cubic expression and the given expression is also cubic, therefore there will be no other factor.

EXERCISE 99.

Factorise:

1. x^3+x^2+x-3.
2. $x^3-6x^2+11x-6$.
3. $x^3+x^2-17x+15$.
4. $x^3+3x^2-10x-24$.
5. $x^3+10x^2+31x+30$.
6. x^3+5x^2-x-5.
7. $x^3-19x-30$.
8. x^3-x^2+12.
9. $x^4-2x^3-3x^2+2x+2$.
10. $x^4-2x^3-20x^2+39x+36$.
11. If n denote any odd number, prove that $(b+c)(c+a)(a+b)$ is a factor of $(a+b+c)^n-a^n-b^n-c^n$ and that $(b-c)(c-a)(a-b)$ is a factor of $(b-c)^n+(c-a)^n+(a-b)^n$.

12. If n denote any odd number which is not a multiple of 3, prove that $ab(a+b)(a^2+ab+b^2)$ is a factor of $(a+b)^n - a^n - b^n$.

Example 4. Factorise $(x+y+z)^3-x^3-y^3-z^3$.

As the given expression is symmetrical and of the 3rd degree in x, y, z, the possible forms of the linear factor are

(i) $(x+y+z)$, (ii) $(x-y)$, (iii) $(x+y)$, (iv) x.

Putting $-(y+z)$ for x in the given expression, we find that it does not vanish,

$\therefore$ $x+y+z$ is not a factor of the expression.

Putting y for x in the given expression, we find that it does not vanish,

$\therefore$ $x-y$ is not a factor of the expression.

Putting $-y$ for x in the given expression, we find that it vanishes,

$\therefore$ $x+y$ is a factor of the expression.

As the given expression is symmetrical and homogeneous, therefore $(y+z)$ and $(z+x)$ must *necessarily* be two other factors.

As the given expression is of the 3rd degree, it cannot have any other factor excepting a numerical one.

$\therefore$ $(x+y+z)^3-x^3-y^3-z^3 \equiv A(x+y)(y+z)(z+x)$... (i)

where A stands for the numerical factor.

If in (i) $x=0$, $y=1$ and $z=1$, then

$$(0+1+1)^3-0^3-1^3-1^3=A(0+1)(1+1)(1+0)$$

$$\therefore \qquad A=3.$$

$$\therefore \quad (x+y+z)^3-x^3-y^3-z^3 \equiv 3(x+y)(y+z)(z+x).$$

Example 5. Factorise $(xy+yz+zx)^2-x^2y^2-y^2z^2-z^2x^2$.

As the given expression is symmetrical and of the fourth degree in x, y, z, the possible forms of the linear factors, if any, are

(i) x, (ii) $x-y$, (iii) $x+y$, (iv) $x+y+z$.

Putting 0 for x, we find that the expression vanishes,

$\therefore$ x is one of the factors.

As the expression is symmetrical and homogeneous,

$\therefore$ y and z must be two other factors.

As the given expression is of the 4th degree, symmetrical and homogeneous, and the expression xyz formed by the factors x, y, z is of the 3rd degree, symmetrical and homogeneous, the remaining factor must be of the 1st degree, symmetrical and homogeneous, which cannot be other than $A\ (x+y+z)$ where A is the numerical co-efficient.

$\therefore (xy+yz+zx)^2-x^2y^2-y^2z^2-z^2x^2 \equiv Axyz(x+y+z)$...(i)

Putting in (i) $x=1$, $y=1$, $z=1$, we get $3A=6$ or $A=2$.

$\therefore \quad (xy+yz+zx)^2-x^2y^2-y^2z^2-z^2x^2=2xyz(x+y+z)$.

Example 6. Factorise $a^3(b^2-c^2)+b^3(c^2-a^2)+c^3(a^2-b^2)$.

As the given expression is symmetrical and of the 5th degree in a, b, c, the possible forms of the linear factors, if any, are

(i) a, (ii) $a-b$, (iii) $a+b$, (iv) $a+b+c$.

Putting 0 for a, we find that the expression does not vanish,

$\therefore$ a is not a factor of the expression.

Putting b for a, we find that the expression vanishes,

$\therefore$ $(a-b)$ is one of the factors.

As the expression is symmetrical and homogeneous,

$\therefore$ $(b-c)$ and $(c-a)$ must also be factors.

As the expression is of the 5th degree, symmetrical and homogeneous, and the expression $(a-b)(b-c)(c-a)$ formed by the factors is of the 3rd degree, there will be one more factor of the 2nd degree, in a, b, c which will be symmetrical, homogeneous and of the form $[A(a^2+b^2+c^2)+B(ab+bc+ca)]$.

$\therefore \quad a^3(b^2-c^2)+b^3(c^2-a^2)+c^3(a^2-b^2)$

$\equiv (a-b)(b-c)(c-a)\ \{A(a^2+b^2+c^2)+B(ab+bc+ca)\}$.

Comparing like terms, say a^3b^2, on both sides, we get $a^3b^2=-a^2b\times Bab=-Ba^3b^2$

$\therefore \quad B=-1$.

Again, comparing terms containing a^4b, we get

$$0 = -a^2b \times Aa^2 = -Aa^4b$$

$$\therefore \qquad A = 0.$$

$$\therefore \qquad a^3(a^2-b^2)+b^3(c^2-b^2)+c^3(a^2-b^2)$$

$$=(a-b)(b-c)(c-a)\{(0)(a^2+b^2+c^2)+(-1)(ab+bc+ca)\}$$

$$=-(a-b)(b-c)(c-a)(ab+bc+ca).$$

In the process of this example, the evaluation of the constants, by giving different values to the variables is tedious, therefore we have adopted the method of the *comparison of like terms.*

NOTE. In finding out the value of the constants, care should be taken not to give such values to the variables as to reduce the two sides to the form $0=0$.

13. Prove that $a^2(b-c)+b^2(c-a)+c^2(a-b) \equiv -(a-b)(b-c)(c-a)$.

14. Prove that $ab(a-b)+bc(b-c)+ca(c-a) \equiv -(a-b)(b-c)(c-a)$.

[Learn these two formulæ by heart.]

Factorise the following expressions using questions 11, 12, wherever necessary :

15. $a(b^2-c^2)+b(c^2-a^2)+c(a^2-b^2)$.

16. $(a-b)^3+(b-c)^3+(c-a)^3$.

17. $(a+b+c)(ab+ac+bc)-abc$.

18. $a^2(b+c)+b^2(c+a)+c^2(a+b)+2abc$.

19. $(a+b+c)^3-(b+c-a)^3-(c+a-b)^3-(a+b-c)^3$.

20. $a^3(b-c)+b^3(c-a)+c^3(a-b)$.

21. $a(b-c)^3+b(c-b)^3+c(a-b)^3$.

22. $a(b+c)^2+b(c+a)^2+c(a+b)^2-3abc$.

23. $(a+b)(a-b)^3+(b+c)(b-c)^3+(c+a)(c-a)^3$.

24. $(x-y)^5+(y-z)^5+(z-x)^5$.

25. $a^4(b-c)+b^4(c-a)+c^4(a-b)$.

26. $(x+y)^5-x^5-y^5$.

27. $(x+y+z)^5-x^5-y^5-z^5$.

28. $x^4(y^2-z^2)+y^4(z^2-x^2)+z^4(x^2-y^2)$.

29. $(x+y+z)^4-(x+y)^4-(y+z)^4-(z+x)^4+x^4+y^4+z^4$.

CHAPTER XXIII

MISCELLANEOUS ARTIFICES

In this chapter it is proposed to discuss

(i) a few important conditional identities with their application in factors and fractions,

(ii) harder types of factors, and

(iii) the application of the remainder theorem and the method of cross-multiplication in H.C.F. and L.C.M.

1. Conditional identities based on $a+b+c=0$.

Example 1. If $a+b+c=0$, shew that $(a+b)(b+c)(c+a)$
$$=-abc.$$

Since $a+b+c=0$, $\quad\therefore\quad a+b=-c,\ b+c=-a,\ c+a=-b.$

Multiplying these three results, we get

$$(a+b)(b+c)(c+a)=(-c)(-a)(-b)$$
$$=-abc.$$

Example 2. If $a+b+c=0$, shew that
$$a^2-bc=b^2-ca=c^2-ab.$$

Since $a+b+c=0$, $\quad\therefore\quad a=-b-c$

$$a^2-bc=a\times a-bc$$
$$=a(-b-c)-bc$$
$$=-ab-ac-bc=-(ab+bc+ca).$$

Similarly, $\quad b^2-ca=-(ab+bc+ca),$

and $\quad c^2-ab=-(ab+bc+ca).$

$\therefore\quad a^2-bc=b^2-ca=c^2-ab.$

Example 3. If $a+b+c=0$, shew that $a^3+b^3+c^3=3abc.$

Since $\quad a+b+c=0, \quad\therefore\quad a+b=-c$

$\therefore\quad (a+b)^3=(-c)^3$

$\therefore\quad a^3+b^3+3ab(a+b)=-c^3$

or $\quad a^3+b^3+c^3=-3ab(a+b)$

$$=-3ab(-c)=3abc.$$

Example 4. If $a+b+c=0$, shew that
$$a^5+b^5+c^5=-5abc(ab+ac+bc).$$

Since $\quad a+b+c=0, \quad\therefore\quad a+b=-c.$

$\therefore \quad (a+b)^5=(-c)^5$

$\therefore \quad a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5=-c^5$

$$
\begin{aligned}
a^5+b^5+c^5 &= -5ab(a^3+2a^2b+2ab^2+b^3)\\
&= -5ab\{(a+b)^3-ab(a+b)\}\\
&= -5ab\{(-c)^3-ab(-c)\}\\
&= -5ab(-c^3+abc)\\
&= -5abc(ab-c^2)\\
&= -5abc(ab-c\times c)\\
&= -5abc\{ab+c(a+b)\}\\
&= -5abc(ab+ac+bc).
\end{aligned}
$$

EXERCISE 100.

If $a+b+c=0$, shew that

1. $ab(a+b)=bc(b+c)=ca(c+a)$.
2. $a^2+b^2+c^2=-2(ab+ac+bc)$.
3. $a(b^2+c^2-a^2)=b(c^2+a^2-b^2)=c(a^2+b^2-c^2)=-2abc$.
4. $a^2+ab+b^2=b^2+bc+c^2=c^2+ca+a^2$.
5. $(a^4+b^4+c^4)=\frac{1}{2}(a^2+b^2+c^2)^2$.
6. $a^4+b^4+c^4=2(ab+bc+ca)^2$.
7. $\dfrac{a^5+b^5+c^5}{5}=\dfrac{a^3+b^3+c^3}{3}\cdot\dfrac{a^2+b^2+c^2}{2}$.
8. $a^5+b^5+c^5=5abc(c^2-ab)$.

Example 5. If $2s=a+b+c$, then $s^2+(s-a)^2+(s-b)^2+(s-c)^2=a^2+b^2+c^2$.

The left-hand side $=s^2+s^2-2sa+a^2+s^2-2sb+b^2+s^2-2sc+c^2$

$$
\begin{aligned}
&=4s^2-2s(a+b+c)+a^2+b^2+c^2\\
&=4s^2-2s.2s+a^2+b^2+c^2\\
&=4s^2-4s^2+a^2+b^2+c^2\\
&=a^2+b^2+c^2.
\end{aligned}
$$

If $2s=a+b+c$, shew that

9. $(s-a)^3+(s-b)^3+(s-c)^3+3abc=s^3$.
10. $s(s-b)(s-c)+s(s-c)(s-a)+s(s-a)(s-b)$ $=(s-a)(s-b)(s-c)+abc$.
11. $2(s-a)(s-b)(s-c)+a(s-b)(s-c)+b(s-c)(s-a)+c(s-a)(s-b)=abc$.
12. $(s-a)^3+(s-b)^3+(s-c)^3-3(s-a)(s-b)(s-c)$ $=\frac{1}{2}(a^3+b^3+c^3-3abc)$.

HARDER FACTORS

2. Type XIII. Factors of cyclic expressions by arrangement of terms.

Example 1. Factorise $a^3(b-c)+b^3(c-a)+c^3(a-b)$.

Arranging the terms, according to descending powers of a, we have

$$\begin{aligned}\text{the expression} &= a^3(b-c)-a(b^3-c^3)+(b^3c-c^3b)\\ &= a^3(b-c)-a(b^2+bc+c^2)(b-c)+bc(b+c)(b-c)\\ &= (b-c)\{a^3-ab^2-abc-ac^2+b^2c+bc^2\}.\end{aligned}$$

Arranging the 2nd factor, according to descending powers of b, we have

$$\begin{aligned}\text{the expression} &= (b-c)\{(b^2c-ab^2)+(bc^2-abc)-(ac^2-a^3)\}\\ &= (b-c)\{b^2(c-a)+bc(c-a)-a(c+a)(c-a)\}\\ &= (b-c)(c-a)\{b^2+bc-ac-a^2\}.\end{aligned}$$

Arranging the 3rd factor, according to descending powers of c, we have

$$\begin{aligned}\text{the expression} &= (b-c)(c-a)\{(bc-ac)+(b^2-a^2)\}\\ &= (b-c)(c-a)\{c(b-a)+(b+a)(b-a)\}\\ &= (b-c)(c-a)(b-a)(a+b+c)\\ &= -(a-b)(b-c)(c-a)(a+b+c).\end{aligned}$$

EXERCISE 101.

Resolve into factors :

1. $a^2(b-c)+b^2(c-a)+c^2(a-b)$.
2. $a^2(b+c)+b^2(c+a)+c^2(a+b)+2abc$.
3. $a^4(b-c)+b^4(c-a)+c^4(a-b)$.
4. $a^3(b^2-c^2)+b^3(c^2-a^2)+c^3(a^2-b^2)$.
5. $a^4(b^2-c^2)+b^4(c^2-a^2)+c^4(a^2-b^2)$.
6. $a^5(b-c)+b^5(c-a)+c^5(a-b)$.

Type XIV. Factors of expressions of the form ax^2+bx+c which involve the application of the equations

$$p+q=m$$
$$pq=n^2.$$

Example. 2 Factorise $20x^2+xy-30y^2$.

Here we have to find out two numbers, p and q such that

$$p+q=+1 \quad \dots \quad \dots \quad \text{(i)}$$

and

$$pq=20\times(-30)=-600 \quad \dots \quad \text{(ii)}$$

$$\because \quad (p-q)^2=(p+q)^2-4pq$$
$$=1^2-4(-600)$$
$$=2401$$

$$\therefore \quad p-q=\sqrt{2401}=49 \quad \dots \quad \text{(iii)}$$

Adding (i) and (iii), and subtracting (iii) form (i) we get

$$p=25 \text{ and } q=-24.$$

$$\therefore \quad 20x^2+xy-30y^2=20x^2+25xy-24xy-30y^2$$
$$=5x(4x+5y)-6y(4x+5y)$$
$$=(4x+5y)(5x-6y).$$

Factorise :

7. $18x^2-51xy+35y^2$.
8. $27a^2-33ab-20b^2$.
9. $55x^2-13xy-12y^2$.
10. $120x^4y^2+73x^3y^3-28x^2y^4$.
11. $44x^2-71xy+20y^2$.
12. $21x^2-58xy+21y^2$.
13. $28x^2-73xy-20y^2$.

Type XV. Factors of expressions of the form ax^2+bx+c, with literal co-efficients.

Example 3. Factorise $a^2-(x+y)a+(x+1)(y-1)$.

Here we have to find out two quantities whose product $=(x+1)(y-1)$, and whose sum $=-(x+y)$.

Obviously, $-(x+1)$ and $-(y-1)$ are the required quantities.

$$\therefore \quad a^2-(x+y)a+(x+1)(y-1)$$
$$=a^2-(x+1)a-(y-1)a+(x+1)(y-1)$$
$$=a\{a-(x+1)\}-(y-1)\{a-(x+1)\}$$
$$=(a-x-1)(a-y+1).$$

Resolve into factors :

14. $a^2+a(x+y)+(x-2)(2+y)$.
15. $(a-1)x^2-x-a$.
16. $x^2+x-(a+1)(a+2)$.
17. $x^2-(1+a+b)x+a(1+b)$.
18. $(b^2-1)x^2+(b^2+1)x+b$.
19. $a^2-(2b+1)a+b^2+b-6$.
20. $x^2+2px+p^2-1$.
21. $x^2+(a+b+c)x+ab+ac$.

Type XVI. Factors of cubic expressions with one term missing.

Example 4. Factorise $x^3-19x-30$.

$$\begin{aligned} x^3-19x-30 &= x^3-19x-38+8 \\ &= (x)^3+(2)^3-19(x+2) \\ &= (x+2)(x^2-2x+4)-19(x+2) \\ &= (x+2)(x^2-2x+4-19) \\ &= (x+2)(x^2-2x-15) \\ &= (x+2)(x-5)(x+3). \end{aligned}$$

Example 5. Factorise $2x^3-3x^2-4$.

$$\begin{aligned} 2x^3-3x^2-4 &= 2x^3-3x^2-16+12 \\ &= (2x^3-16)-3(x^2-4) \\ &= 2(x^3-8)-3(x+2)(x-2) \\ &= 2(x^2+2x+4)(x-2)-3(x+2)(x-2) \\ &= (x-2)\{2x^2+4x+8-3x-6\} \\ &= (x-2)(2x^2+x+2). \end{aligned}$$

Resolve into factors:

22. $x^3-13x+12$.
23. x^3-6x^2+32.
24. $x^3-3x-18$.
25. $x^3-5x+12$.
26. $x^3-3x^2y+20y^3$.
27. $8x^3+4x-3$.
28. $2a^3-a^2b-b^3$.
29. $27x^3+12x-5$.

Type XVII. Factors of expressions of the form $ax^4+bx^3+cx^2+bx+a$ or $ax^4+bx^3y+cx^2y^2+bxy^3+ay^4$.

Example 6. Factorise $8x^4-6x^3+7x^2-6x+8$.

The expression $=x^2\left(8x^2-6x+7-\dfrac{6}{x}+\dfrac{8}{x^2}\right)$

$$=x^2\left[8\left(x^2+\frac{1}{x^2}\right)-6\left(x+\frac{1}{x}\right)+7\right]$$
$$=x^2\left[8\left(x^2+2+\frac{1}{x^2}\right)-6\left(x+\frac{1}{x}\right)+7-16\right]$$
$$=x^2\left[8\left(x+\frac{1}{x}\right)^2-6\left(x+\frac{1}{x}\right)-9\right]$$

Putting A for $\left(x+\frac{1}{x}\right)$, the expression

$$=x^2[8A^2-6A-9]$$
$$=x^2[8A^2-12A+6A-9]$$
$$=x^2[4A(2A-3)+3(2A-3)]$$
$$=x^2(2A-3)(4A+3).$$

Substituting the value of A, we have

the expression $=x^2\left[2\left(x+\frac{1}{x}\right)-3\right]\left[4\left(x+\frac{1}{x}\right)+3\right]$

$$=x^2\left[\frac{2x^2-3x+2}{x}\right]\left[\frac{4x^2+3x+4}{x}\right]$$
$$=(2x^2-3x+2)(4x^2+3x+4).$$

NOTE. It may be noted that in such expressions, the co-efficients of terms equi-distant from the middle term are equal, irrespective of signs, and their special characteristic is that when they are equated to zero, the equations thus formed have reciprocal roots: hence such expressions are known as **Reciprocal Expressions.**

Resolve into factors :

30. $x^4-x^3-8x^2+x+1.$

31. $x^4+4x^3-10x^2+4x+1.$

32. $x^4-5x^3+6x^2-5x+1.$

33. $2x^4+3x^3+5x^2+3x+2.$

34. $3x^4-x^3y-8x^2y^2+xy^3+3y^4.$

35. $4x^4+8x^3y+3x^2y^2+8xy^3+4y^4.$

36. $6x^4-25x^3y+7x^2y^2+25xy^3+6y^4.$

Type XVIII. Factors of homogeneous expressions of the second degree involving three letters.

Example 7. Factorise $2a^2+3ab-9ac-2b^2+7bc-5c^2$.

If $c=0$, the expression is reduced to

$$2a^2+3ab-2b^2=(2a-b)(a+2b) \qquad \cdots \quad \text{(i)}$$

If $b=0$, the expression is reduced to

$$2a^2-9ac-5c^2=(2a+c)(a-5c) \qquad \cdots \quad \text{(ii)}$$

If $a=0$, the expression is reduced to

$$\begin{aligned} -2b^2+7bc-5c^2 &= (-2b+5c)(b-c) \\ &= (2b-5c)(-b+c) \qquad \cdots \quad \text{(iii)} \end{aligned}$$

Factors in (i), (ii) and (iii) are incomplete and complementary. Their proper association can be determined by comparing the co-efficients and signs of like terms.

Here combining (i) and (ii) and comparing the results with (iii), we have

$$\begin{aligned} &2a^2+3ab-9ac-2b^2+7bc-5c^2 \\ &\qquad =(2a-b+c)(a+2b-5c). \end{aligned}$$

Resolve into factors:

37. $2a^2-ab-6b^2+ac+19bc-15c^2$.
38. $2x^2-2y^2-3z^2+3xy-5xz+5yz$.
39. $x^2-xy+4zx+yz-2y^2+3z^2$.
40. $a^2+(x+y)a-2x^2+5xy-2y^2$.
41. $x(x+a)-2a^2+3b(a+x)+2b^2$.
42. $x^2-xy+4xz-2y^2-5yz+3z^2$.
43. $2x^2+5xy-3xz+5yz-3y^2-2z^2$.

Type XIX. Factors of expressions which satisfy the condition $a+b+c=0$.

Example 8. Factorise $(x-y)^3+(y-z)^3+(z-x)^3$.

Since $(x-y)+(y-z)+(z-x)=0$,

$$\therefore \ (x-y)^3+(y-z)^3+(z-x)^3=3(x-y)(y-z)(z-x).$$

[Example 3. Conditional identities, p. 436.]

Example 9. Factorise $(x-2y)^5+(2y-3z)^5+(3z-x)^5$.

Since $(x-2y)+(2y-3z)+(3z-x)=0$,

$\therefore \quad (x-2y)^5+(2y-3z)^5+(3z-x)^5$
$=-5(x-2y)(2y-3z)(3z-x)\ \{(x-2y)(2y-3z)+(x-2y)$
$(3z-x)+(2y-3z)(3z-x)\}$.

[Example 4. Conditional identities, p. 436.]

Resolve into factors:

44. $(2a-b)^3+(b-3c)^3+(3c-2a)^3$.
45. $(a-2b+1)^3+(a+b-1)^3+(b-2a)^3$.
46. $(a-1)^3+(a-2)^3-(2a-3)^3$.
47. $(a+b+1)^3+(a-b-1)^3-8a^3$.
48. $(2x+y)^3-(x-y)^3-(x+2y)^3$.
49. $a^3(b-c)^3+b^3(c-a)^3+c^3(a-b)^3$.
50. $(x-y)^5+(y-z)^5+(z-x)^5$.
51. $(x-1)^5+(x-2)^5-(2x-3)^5$.

Type XX. Miscellaneous.

Resolve into factors:

52. $a^2(b+c)+b^2(c+a)+c^2(a+b)+3abc$.
[*Hint.* $3abc=abc+abc+abc$.
Exp. $=[a^2(b+c)+abc]+$two similar terms.]
53. $a(b^2+c^2-a^2)+b(c^2+a^2-b^2)+c(a^2+b^2-c^2)+6abc$.
54. $a^2(b+c)+b^2(c+a)+c^2(a+b)+a^3+b^3+c^3$.
55. $2(a^2b^2+a^2c^2+b^2c^2)-a^4-b^4-c^4$.
[*Hint.* Exp. $=4a^2b^2-(a^4+b^4+c^4+2a^2b^2-2b^2c^2-2c^2a^2)$
$=(2ab)^2-(a^2+b^2-c^2)^2$.]
56. $(a+b)(a+c)(b+c)+abc$.
[*Hint.* Exp. $=[a^2+a(b+c)+bc](b+c)+abc$
$=a^2(b+c)+a(b+c)^2+bc(b+c)+abc$
$=[a^2(b+c)+abc]+[a(b+c)^2+bc(b+c)$.]
57. $x^4-5x^3+14x^2-20x+16$.
[*Hint.* Exp. $=(x^4+8x^2+16)-5x(x^2+4)+6x^2$.]
58. $x^4+8x^3+24x^2+32x-20$.

3. Fractions with denominators in cyclic order.

It is useful to learn by heart the following formulæ, after proper verification :

If a, b, c are the variables

(i) $\Sigma a = a+b+c$.

(ii) $\Sigma(a+b) = 2(a+b+c)$.

(iii) $\Sigma(a-b) = 0$.

(iv) $\Sigma(a^2-b^2) = 0$.

(v) $\Sigma a(b-c) = 0$.

(vi) $\Sigma a^2(b-c) = -(a-b)(b-c)(c-a)$.

(vii) $\Sigma ab(b-c) = -(a-b)(b-c)(c-a)$.

(viii) $\Sigma a(b^2-c^2) = (a-b)(b-c)(c-a)$.

(ix) $\Sigma(a-b)^3 = 3(a-b)(b-c)(c-a)$.

(x) $\Sigma a^3(b-c) = -(a-b)(b-c)(c-a)(a+b+c)$.

(xi) $\Sigma a^4(b-c) = -(a-b)(b-c)(c-a)$
$(a^2+b^2+c^2+ab+bc+ca)$.

(xii) $\Sigma a^3(b^2-c^2) = -(a-b)(b-c)(c-a)(ab+bc+ca)$.

Example 1.

Simplify $\dfrac{1}{(a-b)(a-c)} + \dfrac{1}{(b-c)(b-a)} + \dfrac{1}{(c-a)(c-b)}$.

Putting the denominators in cyclic order, we have the

$$\text{expressions} = -\frac{1}{(a-b)(c-a)} - \frac{1}{(b-c)(a-b)} - \frac{1}{(c-a)(b-c)}$$

$$= -\left\{\frac{1}{(a-b)(c-a)} + \frac{1}{(b-c)(a-b)} + \frac{1}{(c-a)(b-c)}\right\}$$

$$= -\frac{(b-c)+(c-a)+(a-b)}{(a-b)(b-c)(c-a)}$$

$$= -\frac{0}{(a-b)(b-c)(c-a)}$$

$$= 0.$$

Example 2.

Simplify $\dfrac{a^3}{(a-b)(a-c)} + \dfrac{b^3}{(b-a)(b-c)} + \dfrac{c^3}{(c-a)(c-b)}$.

Putting the denominators in cyclic order, we have the

$$\text{expression} = -\frac{a^3}{(a-b)(c-a)} - \frac{b^3}{(a-b)(b-c)} - \frac{c^3}{(c-a)(b-c)}$$

$$= -\left\{\frac{a^3}{(a-b)(c-a)} + \frac{b^3}{(a-b)(b-c)} + \frac{c^3}{(c-a)(b-c)}\right\}$$

$$= -\frac{a^3(b-c)+b^3(c-a)+c^3(a-b)}{(a-b)(b-c)(c-a)}$$

$$= -\frac{-(a-b)(b-c)(c-a)(a+b+c)}{(a-b)(b-c)(c-a)}$$

$$= +\frac{(a-b)(b-c)(c-a)(a+b+c)}{(a-b)(b-c)(c-a)}$$

$$= a+b+c.$$

EXERCISE 102.

Simplify :

1. $\dfrac{a}{(a-b)(a-c)} + \dfrac{b}{(b-c)(b-a)} + \dfrac{c}{(c-a)(c-b)}.$

2. $\dfrac{a^2}{(a-b)(a-c)}$ + two similar terms.

3. $\dfrac{bc}{(a-b)(a-c)}$ + two similar terms.

4. $\dfrac{1}{a(a-b)(a-c)} + \dfrac{1}{b(b-c)(b-a)} + \dfrac{1}{c(c-a)(c-b)}.$

5. $\dfrac{1}{bc(a-b)(a-c)} + \dfrac{1}{ca(b-c)(b-a)} + \dfrac{1}{ab(c-a)(c-b)}.$

6. $\Sigma \dfrac{a}{bc(a-b)(a-c)}.$

7. $\dfrac{x-a}{(a-b)(a-c)} + \dfrac{x-b}{(b-c)(b-a)} + \dfrac{x-c}{(c-a)(c-b)}.$

8. $\Sigma \dfrac{x^2-yz}{(x-y)(x-z)}.$

9. $\Sigma \dfrac{x^2+yz}{(x-y)(x-z)}.$

10. $\Sigma \dfrac{b^2c^2}{(a-b)(a-c)}.$

11. $\Sigma \dfrac{a^4}{(a-b)(a-c)}.$

12. $\Sigma \dfrac{bc(b+c)}{(a-b)(a-c)}.$

13. $\Sigma \dfrac{b^2+c^2}{(a^2-b^2)(a^2-c^2)}.$

14. $\Sigma \dfrac{bc}{a(a^2-b^2)(a^2-c^2)}.$

15. $\Sigma \dfrac{a(a+b)(a+c)}{(a-b)(a-c)}.$

16. $\Sigma \dfrac{(a+1)^2}{(a-b)(a-c)}$.

17. $\dfrac{b-c}{b+c}+\dfrac{c-a}{c+a}+\dfrac{a-b}{a+b}$.

18. $\dfrac{(x-b)(x-c)}{(a-b)(a-c)}+$ two similar terms.

19. $\dfrac{(a+b)(b+c)(c+a)}{abc}-\dfrac{a+b}{c}-\dfrac{b+c}{a}-\dfrac{c+a}{b}$.

20. $\dfrac{b^2+c^2-2a^2}{(a-b)(a-c)}+\dfrac{c^2+a^2-2b^2}{(b-c)(b-a)}+\dfrac{a^2+b^2-2c^2}{(c-a)(c-b)}$.

[*Hint.* $b^2+c^2-2a^2=(b^2-a^2)-(a^2-c^2)$. Decompose and factorise.]

21. $\dfrac{bc(x-a)^2}{(a-b)(a-c)}+\dfrac{ca(x-b)^2}{(b-c)(b-a)}+\dfrac{ab(x-c)^2}{(c-a)(c-b)}$.

Example 3. Simplify $\dfrac{a}{(a-b)(a-c)(x-a)}+\dfrac{b}{(b-a)(b-c)(x-b)}+\dfrac{c}{(c-a)(c-b)(x-c)}$.

Putting the denominators in cyclic order, we have

$$\text{the expression}=-\frac{a}{(a-b)(c-a)(x-a)}-\frac{b}{(a-b)(b-c)(x-b)}-\frac{c}{(c-a)(b-c)(x-c)}$$

$$=-\left\{\frac{a}{(a-b)(c-a)(x-a)}+\frac{b}{(a-b)(b-c)(x-b)}+\frac{c}{(c-a)(b-c)(x-c)}\right\}$$

$$=-\frac{a(b-c)(x-b)(x-c)+\text{two similar terms}}{(a-b)(b-c)(c-a)(x-a)(x-b)(x-c)}.$$

The numerator $=a(b-c)\{x^2-x(b+c)+bc\}+$ two similar terms

$$=x^2\Sigma a(b-c)-x\Sigma a(b^2-c^2)+abc\Sigma(b-c)$$
$$x^2\times 0-x\times(a-b)(b-c)(c-a)+abc\times 0$$
$$=-x(a-b)(b-c)(c-a).$$

$$\therefore \text{ the expression}=-\frac{-x(a-b)(b-c)(c-a)}{(a-b)(b-c)(c-a)(x-a)(x-b)(x-c)}$$

$$=\frac{x}{(x-a)(x-b)(x-c)}.$$

Simplify :

22. $$\frac{1}{(a-b)(a-c)(x-a)}+\frac{1}{(b-c)(b-a)(x-b)}+\frac{1}{(c-a)(c-b)(x-c)}.$$

23. $$\frac{a^2}{(a-b)(a-c)(x-a)}+\frac{b^2}{(b-c)(b-a)(x-b)}+\frac{c^2}{(c-a)(c-b)(x-c)}.$$

24. $$\frac{1}{(a-b)(a-c)(x+a)}+\frac{1}{(b-c)(b-a)(x+b)}+\frac{1}{(c-a)(c-b)(x+c)}$$

25. $$\frac{a^2}{(a-b)(a-c)(x+a)}+\frac{b^2}{(b-c)(b-a)(x+b)}+\frac{c^2}{(c-a)(c-b)(x+c)}.$$

26. $$\frac{1+a}{(a-b)(a-c)(x-a)}+\frac{1+b}{(b-c)(b-a)(x-b)}+\frac{1+c}{(c-a)(c-b)(x-b)}.$$

[*Hint.* Decompose the expression into $\frac{1}{(a-b)(a-c)(x-a)}+$ two similar terms $+\frac{a}{(a-b)(a-c)(x-a)}+$ two similar terms.]

27. $$\frac{1+pa+qa^2}{(a-b)(a-c)(x-a)}+\frac{1+pb+qb^2}{(b-c)(b-a)(x-b)}+\frac{1+pc+qc^2}{(c-a)(c-b)(x-c)}.$$

Example 4. Simplify $\frac{(a-b)^3+(b-c)^3+(c-a)^3}{a^3(b-c)^3+b^3(c-a)^3+c^3(a-b)^3}$.

The expression $=\frac{(a-b)^3+(b-c)^3+(c-a)^3}{(ab-ac)^3+(bc-ba)^3+(ca-cb)^3}$.

Since $(a-b)+(b-c)+(c-a)=0$ and $(ab-ac)+(bc-ba)+(ca-cb)=0$.

$$\therefore \text{ the expression} = \frac{3(a-b)(b-c)(c-a)}{3(ab-ac)(bc-ba)(ca-cb)}$$

$$= \frac{(a-b)(b-c)(c-a)}{abc(a-b)(b-c)(c-a)}$$

$$= \frac{1}{abc}.$$

Simplify:

28. $$\frac{(a^2-b^2)^3+(b^2-c^2)^3+(c^2-a^2)^3}{(a-b)^3+(b-c)^3+(c-a)^3}.$$

29. $$\frac{a^3(b^2-c^2)+b^3(c^2-a^2)+c^3(a^2-b^2)}{a^2(b-c)+b^2(c-a)+c^2(a-b).}.$$

30. $$\frac{27(a+b+c)^3-(a+2b)^3-(b+2c)^3-(c+2a)^3}{(a+3b+2c)(b+3c+2a)(c+3a+2b)}.$$

[*Hint.* Apply $(a+b+c)^3-a^3-b^3-c^3=3(a+b)(b+c)(c+a)$.]

31. $$\frac{(a+b)^3+(c+d)^3-(a+c)^3-(b+d)^3}{(a+b)^2+(c+d)^2-(a+c)^2-(b+d)^2}.$$

32. $$\frac{(a-b)(a+b)^3+(b-c)(b+c)^3+(c-a)(c+a)^3}{(a+b)(a-b)^3+(b+c)(b-c)^3+(c+a)(c-a)^3}.$$

33. $$\frac{x^2\left(\frac{1}{y}-\frac{1}{z}\right)+y^2\left(\frac{1}{z}-\frac{1}{x}\right)+z^2\left(\frac{1}{x}-\frac{1}{y}\right)}{x\left(\frac{1}{y}-\frac{1}{z}\right)+y\left(\frac{1}{z}-\frac{1}{x}\right)+z\left(\frac{1}{x}-\frac{1}{y}\right)}.$$

34. Shew that $\dfrac{a-b}{1+ab}+\dfrac{b-c}{1+bc}+\dfrac{c-a}{1+ca}$

$$=\frac{a-b}{1+ab}\times\frac{b-c}{1+bc}\times\frac{c-a}{1+ca}.$$

35. Shew that $\dfrac{a-b}{m+ab}+\dfrac{b-c}{m+bc}+\dfrac{c-a}{m+ca}$

$$=m.\frac{a-b}{m+ab}.\frac{b-c}{m+bc}.\frac{c-a}{m+ca}.$$

36. Prove that $\dfrac{1}{(a-b)^2}+\dfrac{1}{(b-c)^2}+\dfrac{1}{(c-a)^2}$

$$=\left\{\frac{1}{a-b}+\frac{1}{b-c}+\frac{1}{c-a}\right\}^2.$$

4. Conditional Evaluation.

Example 1. Reduce to the simplest terms.

$$\frac{x+y}{1-xy} \text{ and } \frac{x-y}{1+xy}$$

$$\text{where } x=\frac{a+b}{1-ab} \text{ and } y=\frac{a-b}{1+ab}.$$

$$x+y=\frac{a+b}{1-ab}+\frac{a-b}{1+ab}=\frac{(a+b)(1+ab)+(a-b)(1-ab)}{1-a^2b^2}$$

$$=\frac{2a(1+b^2)}{1-a^2b^2} \quad \ldots \quad \ldots \quad \text{(i)}$$

$$1-xy=1-\frac{a+b}{1-ab}\times\frac{a-b}{1+ab}=1-\frac{a^2-b^2}{1-a^2b^2}$$

$$=\frac{1-a^2b^2-a^2+b^2}{1-a^2b^2}$$

$$=\frac{(1-a^2)(1+b^2)}{1-a^2b^2} \quad \ldots \quad \ldots \quad \text{(ii)}$$

$$\therefore \quad \frac{x+y}{1-xy}=\frac{2a(1+b^2)}{1-a^2b^2}\div\frac{(1-a^2)(1+b^2)}{1-a^2b^2}$$

$$=\frac{2a(1+b^2)}{1-a^2b^2}\times\frac{1-a^2b^2}{(1-a^2)(1+b^2)}$$

$$=\frac{2a}{1-a^2}.$$

Similarly, $\dfrac{x-y}{1+xy}=\dfrac{2b}{1-b^2}.$

Example 2. Find the value of $\dfrac{x+a}{x-a}+\dfrac{x+b}{x-b}$ when $x=\dfrac{2ab}{a+b}.$

$$\text{The expression}=\frac{x-a+2a}{x-a}+\frac{x-b+2b}{x-b}$$

$$=1+\frac{2a}{x-a}+1+\frac{2b}{x-b}$$

$$=2+2\left\{\frac{a}{x-a}+\frac{b}{x-b}\right\}$$

$$=2+2.\frac{ax-ab+bx-ba}{(x-a)(x-b)}$$

$$=2+2.\frac{x(a+b)-2ab}{(x-a)(x-b)}$$

$$=2+2.\frac{0}{(x-a)(x-b)}$$

$\because \quad x(a+b)=2ab$, or $x(a+b)-2ab=0$

$\therefore \quad$ the expression $=2$.

EXERCISE 103.

Find the value of :

1. $\dfrac{1}{x-a}+\dfrac{1}{x-b}-\dfrac{1}{a}$ when $x=\dfrac{2ab}{a+b}$.

2. $\dfrac{a}{2am-2mx}+\dfrac{b}{2mb-2mx}$ when $x=\dfrac{a+b}{2}$.

3. $\dfrac{x^2-y^2+x}{y^2-x^2+y}$ when $x=\dfrac{a-b}{a+b}$ and $y=\dfrac{a+b}{a-b}$.

4. $\left(\dfrac{2x-a}{2x-b}\right)^2-\dfrac{a-x}{b-x}$ when $x=\dfrac{ab}{a+b}$.

5. $\left(\dfrac{x}{x+1}\right)^2+\left(\dfrac{x}{x-1}\right)^2$ when $x^2=\dfrac{a^2}{a^2-1}$.

6. $\dfrac{x+2a}{x-2a}+\dfrac{x+2b}{x-2b}$ when $x=\dfrac{4ab}{a+b}$.

7. $\dfrac{x+2a}{2b-x}+\dfrac{x-2a}{2b+x}+\dfrac{4ab}{x^2-4b^2}$ when $x=\dfrac{ab}{a+b}$.

5. CONDITIONAL IDENTITIES.

Example 1. If $a+b+c=0$, shew that

$$\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)\left(\frac{a+b}{c}+\frac{b+c}{a}+\frac{c+a}{b}\right)=9$$

Since $a+b+c=0$. $\quad\therefore\quad \left.\begin{aligned} a+b&=-c\\ b+c&=-a\\ c+a&=-b.\end{aligned}\right\}$

Substituting these values in the left-hand side, we get

$$\left\{\frac{a}{-a}+\frac{b}{-b}+\frac{c}{-c}\right\}\left\{\frac{-c}{c}+\frac{-a}{a}+\frac{-b}{b}\right\}$$

$=(-1-1-1)(-1-1-1)=9$.

Example 2. If $a+b+c=0$, shew that

$$\frac{1}{b^2+c^2-a^2}+\frac{1}{c^2+a^2-b^2}+\frac{1}{a^2+b^2-c^2}=0.$$

Since $a+b+c=0$, $\therefore\ a+b=-c$

$\therefore\ a^2+2ab+b^2=c^2$

$\therefore\ a^2+b^2-c^2=-2ab.$

Similarly, $b^2+c^2-a^2=-2bc,$

and $c^2+a^2-b^2=-2ac.$

Hence the expression $=\dfrac{1}{-2bc}+\dfrac{1}{-2ac}+\dfrac{1}{-2ab}$

$$=\frac{-a-b-c}{2abc}=\frac{-(a+b+c)}{2abc}$$

$$=\frac{0}{2abc}=0.$$

Example 3. If $2s=a+b+c$, shew that

$$\frac{1}{s-a}+\frac{1}{s-b}+\frac{1}{s-c}-\frac{1}{s}=\frac{abc}{s(s-a)(s-b)(s-c)}.$$

The left-hand side

$$=\left(\frac{1}{s-a}+\frac{1}{s-b}\right)+\left(\frac{1}{s-c}-\frac{1}{s}\right)$$

$$=\frac{(s-b)+(s-a)}{(s-a)(s-b)}+\frac{s-s+c}{s(s-c)}$$

$$=\frac{2s-a-b}{(s-a)(s-b)}+\frac{c}{s(s-c)}$$

$$=\frac{c}{(s-a)(s-b)}+\frac{c}{s(s-c)} \qquad [\because\ 2s-a-b=c.]$$

$$=c\left\{\frac{1}{(s-a)(s-b)}+\frac{1}{s(s-c)}\right\}$$

$$=c\left\{\frac{s(s-c)+(s-a)(s-b)}{s(s-a)(s-b)(s-c)}\right\}$$

$$=c\left\{\frac{s^2-sc+s^2-sa-sb+ab}{s(s-a)(s-b)(s-c)}\right\}$$

$$=c\left[\frac{2s^2-s(a+b+c)+ab}{s(s-a)(s-b)(s-c)}\right]$$

$$=c\ \frac{2s^2-2s^2+ab}{s(s-a)(s-b)(s-c)} \qquad [\because\ a+b+c=2s.]$$

$$=\frac{abc}{s(s-a)(s-b)(s-c)}.$$

Example 4. If $x+\frac{1}{y}=1$, $y+\frac{1}{z}=1$, prove that $z+\frac{1}{x}=1$ and $xyz+1=0$.

Since $y+\frac{1}{z}=1$, $\therefore$ $y=1-\frac{1}{z}=\frac{z-1}{z}$.

Substituting the value of y in $x+\frac{1}{y}=1$, we get

$$x+\frac{z}{z-1}=1$$

$$\therefore \qquad \frac{zx-x+z}{z-1}=1$$

or $\qquad xz-x+z=z-1$

or $\qquad xz+1=x.$

Dividing both sides by x, we get

$$z+\frac{1}{x}=1 \qquad \dots \qquad \dots \qquad \dots \qquad \text{(i)}$$

From (i) $z=1-\frac{1}{x}$

$$\therefore \qquad xyz=xy\left(1-\frac{1}{x}\right).$$

$$=xy-y$$

$$=y(x-1)$$

$$=y\times\left(-\frac{1}{y}\right) \qquad \left[\because\ x-1=-\frac{1}{y}\right]$$

$$=-1$$

$$\therefore \qquad xyz+1=0. \qquad \text{(ii)}$$

Example 5. If $xy+yz+zx=1$, shew that

$$\frac{x+y}{1-xy}+\frac{y+z}{1-yz}+\frac{z+x}{1-zx}=\frac{1}{xyz}.$$

Since $xy+yz+zx=1$, $\therefore$ $1-xy=z(x+y)$.

$$\therefore \quad \frac{x+y}{1-xy}=\frac{x+y}{z(x+y)}=\frac{1}{z} \qquad \dots \qquad \dots \qquad \dots \qquad \text{(i)}$$

Similarly, $$\frac{y+z}{1-yz}=\frac{1}{x} \qquad \dots \qquad \dots \qquad \dots \qquad \text{(ii)}$$

and $$\frac{z+x}{1-zx}=\frac{1}{y} \qquad \dots \qquad \dots \qquad \dots \qquad \text{(iii)}$$

$\therefore$ Adding (i), (ii) and (iii), we get

$$\text{the expression} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$$

$$= \frac{yz + xz + xy}{xyz}$$

$$= \frac{1}{xyz}. \qquad [\because yz + xz + xy = 1.]$$

EXERCISE 104.

If $a+b+c=0$, shew that :

1. $\dfrac{a^2+b^2-c^2}{2ab} + \dfrac{b^2+c^2-a^2}{2bc} + \dfrac{c^2+a^2-b^2}{2ca} = -3.$

2. $\dfrac{a^2}{2a^2+bc} + \dfrac{b^2}{2b^2+ca} + \dfrac{c^2}{2c^2+ab} = 1.$

[*Hint.* $2a^2+bc=a^2+a(-b-c)+bc$. Factorise it.]

If $2s=a+b+c$, shew that :

3. $\dfrac{a}{s-a} + \dfrac{b}{s-b} + \dfrac{c}{s-c} + 2 = \dfrac{abc}{(s-a)(s-b)(s-c)}.$

4. $1 - \left[\dfrac{b^2+c^2-a^2}{2bc}\right]^2 = \dfrac{4s(s-a)(s-b)(s-c)}{b^2c^2}.$

5. If $a=1-\dfrac{1}{b}$, $b=1-\dfrac{1}{c}$, prove that $c=1-\dfrac{1}{a}$.

6. If $x=b+c-a$, $y=c+a-b$ and $z=a+b-c$,

$$\text{then } \frac{x^3+y^3+z^3-3xyz}{a^3+b^3+c^3-3abc} = 4.$$

7. If $x=a(b-c)$, $y=b(c-a)$, $z=c(a-b)$,

$$\text{shew that } \left(\frac{x}{a}\right)^3 + \left(\frac{y}{b}\right)^3 + \left(\frac{z}{c}\right)^3 = \frac{3xyz}{abc}.$$

8. If $\dfrac{y+x}{y-x} + \dfrac{y+z}{y-z} = 2$, then $\dfrac{1}{x} + \dfrac{1}{z} = \dfrac{2}{y}$.

9. If $x+y=2z$, then $\dfrac{x}{x-z} + \dfrac{y}{y-z} = 2$.

10. If $b^2=ac$, $x=\frac{1}{2}(a+b)$, $y=\frac{1}{2}(b+c)$, then $\dfrac{a}{x} + \dfrac{c}{y} = 2$.

11. If $3(a^2+b^2+c^2)=(a+b+c)^2$, then $a=b=c$.

12. If $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{a+b+c}$, then $\frac{1}{a^5}+\frac{1}{b^5}+\frac{1}{c^5}=\frac{1}{(a+b+c)^5}$.

13. If $x=by+cz$, $y=cz+ax$ and $z=ax+by$,

$$\text{then } \frac{a}{1+a}+\frac{b}{1+b}+\frac{c}{1+c}=1.$$

6. H. C. F. and L. C. M. (*Continued.*)

We add here a few examples and exercises which involve the application of the *remainder theorem, the method of cross-multiplication*, and the *principles underlying the processes of H. C. F. and L. C. M.*

Example 1. For what value of x will the expressions $4x^3-8x^2+x+3$ and $4x^3-4x^2-5x+3$ vanish?

The H. C. F. of the given expressions, when worked out $=2x-3$. As both the expressions have $2x-3$ as their common factor, they will vanish when $2x-3=0$.

or $\qquad x=\frac{3}{2}$.

Example 2. The H. C. F. and the L. C. M. of two integral expressions, each of the second degree, are $x+3$ and x^3-7x+6 respectively. Find the expressions.

Let A and B be the two expressions and Q_1 and Q_2 be respectively the quotients, when they are divided by the H. C. F. $x+3$.

$\therefore \quad A=(x+3)(Q_1)$ and $B=(x+3)(Q_2)$.

Their L. C. M. $=(x+3)(Q_1)(Q_2)$.

But as the given L. C. M. $=x^3-7x+6$,

$\therefore \quad (x+3)(Q_1)(Q_2)=x^3-7x+6$

$$\therefore \qquad Q_1Q_2=\frac{x^3-7x+6}{x+3}=x^2-3x+2$$

$$=(x-1)(x-2).$$

Q_1, Q_2 could either be 1 and x^2-3x+2

or $\qquad (x-1)$ and $(x-2)$.

If we assume the first pair of values, we get expressions of the first and third degrees, which is contrary to the data.

$\therefore \quad Q_1$ and Q_2 are equal to $(x-1)$ and $(x-2)$.

$\therefore \quad$ the expressions are (i) $\quad (x+3)(x-1)=x^2+2x-3$

(ii) $\quad (x+3)(x-2)=x^2+x-6$.

Example 3. Shew that if ax^2+bx+c and $a'x^2+b'x+c'$ have a common factor of the form $x+p$, then

$$(ca'-c'a)^2=(bc'-b'c)(ab'-a'b).$$

As $x+p$ is a factor of ax^2+bx+c as well as $a'x^2+b'x+c'$,

$\therefore$ each will vanish if $x=-p$

$$\therefore \quad ap^2-bp+c=0 \quad \ldots \quad \ldots \quad \text{(i)}$$

$$a'p^2-b'p+c'=0 \quad \ldots \quad \ldots \quad \text{(ii)}$$

By the method of cross-multiplication, from (i) and (ii), we get

$$\frac{p^2}{-bc'+b'c}=\frac{p}{ca'-c'a}=\frac{1}{-ab'+a'b}$$

$$\therefore \quad \left(\frac{p}{ca'-c'a}\right)^2=\frac{p^2}{-bc'+b'c}\times\frac{1}{-ab'+a'b}$$

$$\therefore \quad \frac{p^2}{(ca'-c'a)^2}=\frac{p^2}{+(bc'-b'c)(ab'-a'b)}$$

$$\therefore \quad (ca'-c'a)^2=(bc'-b'c)(ab'-a'b).$$

Example 4. Shew that x^3+px^2+qx+1 and x^3+qx^2+px+1 cannot have a common factor, unless $p=q$ or $p+q+2=0$.

(i) Since the given expressions become identical if $p=q$, therefore each expression is a common factor, if $p=q$.

(ii) The common factor, if any, is obviously contained in the difference of these two expressions, *i.e.*, $(p-q)x^2+(q-p)x$, or is contained in $(p-q)x(x-1)$.

$(p-q)$ cannot be a common factor, since it is a constant.

and x is not a common factor, because each expression contains a term independent of x,

$\therefore$ $(x-1)$ can be the *only possible factor.*

Now, if $(x-1)$ be taken as the common factor, each expression must vanish if $x=1$, in either case

$$1+p+q+1=0$$

$$\text{or} \qquad p+q+2=0.$$

Example 5. If $x+c$ be the H. C. F. of x^2+ax+b and $x^2+a'x+b'$, prove that their L. C. M. will be

$$x^3+(a+a'-c)x^2+(aa'-c^2)x+(a-c)(a'-c)c.$$

As $(x+c)$ is the H. C. F. of the given expressions, therefore each is exactly divisible by $x+c$.

Dividing each by $x+c$, we get $(x+a-c)$ and $(x+a'-c)$ as the quotients.

$$\begin{aligned}\therefore\quad \text{L. C. M.} &= (x+c)(x+a-c)(x+a'-c)\\ &= x^3+x^2\{c+(a-c)+(a'-c)\}+x\{c(a-c)\\ &\qquad +c(a'-c)+(a-c)(a'-c)\}+c(a-c)(a'-c).\\ &= x^3+x^2(a+a'-c)+x(aa'-c^2)+(a-c)(a'-c)c.\end{aligned}$$

EXERCISE 105.

1. For what value of x will the expressions x^3-4x^2+2x+4 and $3x^3-7x^2+3x-2$ vanish?

2. Find the value of m, in order that x^2+mx+2 and x^2+5x+6 may have a common factor.

3. Find the expression which divides $x^4+x^3+4x^2+5x+8$ and $2x^4+4x^3+3x^2-2x-4$, leaving $2x-1$ and $4x+5$ as remainders respectively.

4. The H. C. F. and L. C. M. of two integral expressions, each of the second degree, are $x-4$ and $x^3-3x^2-10x+24$ respectively. Find the expressions.

5. If H be the H. C. F. and L the L. C. M. of two expressions A and B and if $H+L=A+B$, prove that $H^3+L^3=A^3+B^3$.

6. If $x+a$ be the H. C. F. of x^2+px+q and $x^2+p'x+q'$, shew that $a=\dfrac{q-q'}{p-p'}$.

7. Find the condition so that ax^3+bx+c and $a'x^3+b'x+c'$ may have a common factor of the form $x+m$.

8. For what value of a will the expressions $x^3-ax^2+19x-a-4$ and $x^3-(a+1)x^2+23x-a-7$ have a common factor?

9. Find the value of p so that x^2+x+p and x^3+x^2+x+1 may have a common linear factor.

10. Shew that ax^2+bx+c and cx^2+bx+a cannot have a common factor of the form $x+p$, unless $a+c=b$ or $a+b+c=0$.

11. If x^2+qx+r and x^3+px^2+qx+r have a common linear factor, then $(p-1)^2-q(p-1)+r=0$.

12. If x^2+ax+b and $x^2+a'x-b$ have a common linear factor, then $4b=a^2-a'^2$ and the factor is $x+\frac{1}{2}(a+a')$.

13. For what value of a, other than zero, will the expressions x^2+5x+a and x^3+3x+a have a common linear factor? *Hence*, find their H. C. F.

SECTIONAL REVISION VI

TEST PAPERS

PAPER 1

1. Exemplify the distinction (i) between an integral function and a fractional function; (ii) between a rational function and an irrational function.

Write down a general expression in x and y of the 3rd degree.

2. (i) Multiply, using the method of detached co-efficients, $5x^4-2x^2+x-4$ and $3x^3+x+2$.

(ii) Without actual multiplication find the value of a for which the co-efficient of x^2 in $(4x-1)(2x+3)(x-a)$ vanishes.

3. Under what conditions will the expression $9x^2+py^2+qz^2-4xy+ayz+bzx$ be symmetrical ?
4. If $x \equiv A(x-2)+B(x-1)$, find the values of A and B.
5. By the remainder theorem, find the remainder if $3x^4+2x^3-x^2+4x-1$ be divided by $x-2$.
6. Factorise $56x^2-59xy-231y^2$ by the application of $a+b=m$ and $ab=n^2$.
7. Find the equation whose roots are -4 and 5.

PAPER 2

1. (i) If $f(x)=4x^3-5x^2+3x-1$, find the value of $f(0)$, $f(1)$ and $f(-1)$.

   (ii) Divide by the method of detached co-efficients $3x^3-2x+2-x^2$ by $3x^2+2-4x$.

2. Write down a homogeneous expression of the 4th degree in x, y and z.
3. Write down all the types of terms in the expression $\sum \dfrac{x^2(a-b)}{ab}$, where a, b, c are variables and x is constant.

4. If $a_0x^4+a_1x^3+a_2x^2+a_3x+a_4 \equiv (x^2-2x+3)(x^2+x-1)$, find the values of a_0, a_1, a_2, a_3 and a_4.

5. Factorise $(2x-y)^5+(2y-z)^5+(z-y-2x)^5$.

6. Simplify $\Sigma \dfrac{a^2}{(a-b)(a-c)(x-a)}$ where a, b, c are variables and x is constant.

7. Find the equation whose roots are $+8$ and -7.

PAPER 3

1. Exemplify the distinction between absolute symmetry and cyclic symmetry.

2. Write down a cyclic expression in x, y, z whose first term is $\dfrac{x^2-y^2}{x(y-z)}$.

3. Without actual multiplication, state why

$$(a^2+b^2)(a+b)=a^3+2a^2b+3ab^2+b^3 \text{ is wrong.}$$

4. Express $\dfrac{x^2-2x-14}{(x-1)(x-2)(x+3)}$ in proper partial fractions with denominators of the first degree in x.

5. Factorise $(2x-3y+z)^3+(x+2y-z)^3+(y-3x)^3$.

6. Solve $x^2-6x-55=0$ by factors and also graphically by the *first method.*

PAPER 4

1. Write down $a^3(c-b)+b^3(a-c)+c^3(b-a)$ in cyclic order.

2. Express $\dfrac{5x^4-12x^3}{(x-1)(x-2)}$ in the form of

$$Ax^2+Bx+C+\frac{D}{x-1}+\frac{E}{x-2}.$$

3. A homogeneous and symmetrical expression of the 2nd degree in x, y and z has the value 14 when $x=-3$, $y=4$ and $z=3$ and the value 16 when $x=-2, y=1, z=-5$; find it.

4. Shew by the remainder theorem that x, $(x-1)$, $(x+1)$ are the factors of $x^4+2x^3-x^2-2x$.

5. Factorise $x^3-13x+12$ by the remainder theorem.

6. Use the method of cross-multiplication in solving the following equations :
$$\left.\begin{aligned} 3x-2y+\ \ z&=0\\ 2x+\ y-11z&=0\\ x+4y+\ 3z&=26.\end{aligned}\right\}$$

7. Solve $2x-1=\dfrac{15}{x}$ by factors and also graphically by the *second method.*

PAPER 5

1. Find the values of A, B and p, in order that
$$A\,(x-1)^2+B(x-p)^2\equiv x^2-8x+10.$$
2. Shew that $p+q+r+s$ is a factor of
$$(p+r)(p+s)(q+r)(q+s)-(pq-rs)^2.$$
3. Simplify by cyclic symmetry $\Sigma(b+c-2a)(b+c-a)$ where a, b, c are variables.
4. If $x+y+z=0$, shew that $\Sigma(x-y)^2=3\Sigma x^2$ where x, y, z are variables.
5. Factorise $10x^4-63x^3+72x^2+63x+10$.
6. Solve the equation $12x^2+16x-35=0$ by factors and verify the solution.
7. Find the square root of $x^6-6x^5+17x^4-26x^3+22x^2-8x+1$ by the method of indeterminate co-efficients.

PAPER 6

1. Determine the values of a and b for which
(i) $3x^4-7x^3+9x^2+ax+b$ and (ii) ax^4+bx+3
are exactly divisible by $(x-1)^2$.
2. Shew that $l+m+n$ is a factor of
$l^2(m+n)+m^2(n+l)+n^2(l+m)$ if $l^3+m^3+n^3=0$.
3. Shew that $5^{3n}-4^{3n}$ is always divisible by 61.
4. Simplify $\qquad \Sigma\dfrac{1}{a(a+b)(a+c)}$.
5. If $2s=a+b+c$, shew that
(i) $s^2+(s-a)^2+(s-b)^2+(s-c)^2=a^2+b^2+c^2$.
(ii) $s^3-(s-a)^3-(s-b)^3-(s-c)^3=3abc$.
(iii) $s^2+(s-b)(s-c)+(s-c)(s-a)+(s-a)(s-b)$
$=bc+ca+ab$.

6. Factorise $8x^3+4x+3$.

7. Solve the equation $15x^2+26x-21=0$ by completing the square and verify the solution.

PAPER 7

1. Using the principle of indeterminate co-efficients, prove
$$(x+y+z)^3 = x^3+y^3+z^3+3xy(x+y)+3yz(y+z) +3zx(z+x)+6xyz.$$

2. Factorise (i) $(a+b)^5-a^5-b^5$.
(ii) $\Sigma a^2(b^3-c^3)$.

3. Simplify $\Sigma \dfrac{a}{ab(a-b)(b-c)}$, where a, b, c are variables.

4. If $a+b+c=0$, shew that (i) $\Sigma a^3 = 3abc$,
(ii) $\Sigma a^5 = -5abc\Sigma ab$
where a, b, c are variables.

5. For what value of x will the expression $2x^3+10x^2+14x+6$ and $x^3+x^2+7x+39$ vanish ?

6. The H.C.F. and L.C.M. of two integral expressions, each of the second degree, are $x+2y$ and $6x^3+7x^2y-16xy^2-12y^3$ respectively ; find them.

7. Solve the equation $3x^2-17x+10=0$ by the formula.

PAPER 8

1. Factorise (i) $x^2+(a-1)x-(2a-3)(a-2)$.
(ii) $2a^2-ab-3ac+7bc-3b^2-2c^2$.

2. If $x-p$ be the H.C.F. of x^2+ax+b and x^2+mx+n, then their L. C. M. is $x^3+x^2(a+m+p)+x(am-p^2)-p(a+p)(m+p)$.

3. Find the value of p so that $x^2+(p-3)x-3p$ and $x^3+8x^2+17x+10$ may have a common linear factor.

4. Solve the equations :
$$\left.\begin{aligned} x+y+z&=0\\ (b+c)x+(c+a)y+(a+b)z&=0\\ bcx+cay+abz&=1. \end{aligned}\right\}$$

5. Simplify $\dfrac{(x-b)(x-c)}{(a-b)(a-c)}$ + two similar terms.

6. Express in partial fractions $\dfrac{7x^2+x+1}{x^3-1}$.

7. Solve the equation $2x^2-13x+15=0$ graphically by *the third method.*

MISCELLANEOUS EXERCISES

[Selected Questions]

1. (*a*) Write down *briefly* $53{\cdot}7 \times 53{\cdot}7 \times 53{\cdot}7 \times 1000000$.

(*b*) Find the value of $\sqrt{s(s-a)(s-b)(s-c)}$ when $a=41$, $b=40$, $c=9$ and $2s=a+b+c$.

2. (*a*) Multiply $3x^4-2x^3+x-4$ and $2x^3-5x^2+1$ by the method of detached co-efficients.

(*b*) Without doing full process, find the co-efficient of x^3 in the product of $4x^3-2x^2+3x-1$ and $2x^3+4x^2-x+3$.

3. Find the H.C.F. of x^5-x^3+8 and x^5-x^2+4.

4. (i) $2b=a+c$ and $\dfrac{2}{c}=\dfrac{1}{b}+\dfrac{1}{d}$, ; shew that $a:b=c:d$.

(ii) If a, b, c are in continued proportion, prove that $a^2+ab : b^2 :: b^2+bc : c^2$.

5. Resolve into factors:

(i) $a^2-b^2+ab(b-a)$.

(ii) $c(a^2+b^2-c^2)+b(c^2+a^2-b^2)+a(b^2+c^2-a^2)+6abc$.

(iii) $\left(a+\dfrac{1}{a}\right)^2-\left(b+\dfrac{1}{b}\right)^2$.

6. Simplify $\dfrac{(x+1)^2}{(x-y)(x-z)}+$ two similar terms.

7. Solve the equations (i) $x+y=8, y+z=10, z+x=12$.

(ii) $\dfrac{x-a}{b+c}+\dfrac{x-b}{c+a}+\dfrac{x-c}{a+b}=3$.

8. If $a=\dfrac{b}{1-b}$ and $b=\dfrac{c}{1-c}$, prove that $c=\dfrac{a}{1+2a}$.

9. Express $\dfrac{3x+11}{(x+2)(x+3)}$ in the form of $\dfrac{A}{x+2}+\dfrac{B}{x+3}$.

10. The perimeter of a rectangular field is 72 ft.; find its dimensions if it contains 315 sq. ft.

11. (*a*) Write down *briefly* $\dfrac{3{\cdot}4\times3{\cdot}4\times3{\cdot}4}{1{,}000{,}000}$.

(*b*) Find the value of $\sqrt{\dfrac{(s-a)(s-b)}{s(s-c)}}$ when $a=10$, $b=21$, $c=17$ and $s=\frac{1}{2}(a+b+c)$.

12. (*a*) Divide $8x^5+22x^4-11x^3-9x^2+7x-2$ by $4x^2+x-2$ by the method of detached co-efficients.

(*b*) For what value of m will $2x^3-5x^2+mx-21$ be exactly divisible by $x-3$?

13. Find the H.C.F. of $x^4+9x-20$ and $5x^4+9x^3-64$.

14. If $a:b::b:c$, prove that

(i) $a-2b+c=\dfrac{(a-b)^2}{a}=\dfrac{(b-c)^2}{c}$.

(ii) $(a+b+c):(a-b+c)=(a+b+c)^2:(a^2+b^2+c^2)$.

15. Resolve into factors:

(i) $a^2b^2-a^2-b^2+1$.

(ii) $x(y^2-a)-y(x^2-a)$.

(iii) $324x^4+1$, and *hence* find out the factors of 3240001.

16. Simplify $\dfrac{bc(x-a)^2}{(a-b)(a-c)}$ + two similar terms.

17. Solve the equations (i) $\dfrac{x-1}{2}+\dfrac{2x-7}{3}=x-2$.

(ii) $\sqrt{4x+5}+\sqrt{4x-11}=8$.

18. Eliminate h, k and l from the equations:

$h=k^a$, $l=k^b$, $h^bl^a=\sqrt[c]{k^2}$.

19. Express $\dfrac{11x+5}{(x-2)(x+1)^2}$ in the form of $\dfrac{A}{x-2}+\dfrac{Bx+C}{(x+1)^2}$.

20. The difference between the length and breadth of a rectangle is 18 ft. and its area is 448 sq. ft.; find

(i) its semi-perimeter,

(ii) its length and breadth.

21. Find the value of :

(i) $128^{\frac{4}{7}}$, $\sqrt[4]{81^3}$ and $\left(\frac{1}{27}\right)^{-\frac{4}{3}}$.

(ii) $\left\{(a+b)^{\frac{1}{2}}+(a-b)^{\frac{1}{2}}\right\}^2$.

22. Shew that (i) 7^n-1 is divisible by 6, and

(ii) $45^{2n+1}+1$ is divisible by 46,

if n is a positive integer.

23. Find the H. C. F. of $2x^5-11x^2-9$ and $4x^5+11x^4+81$, by the method of alternate destruction of the highest and the lowest terms.

24. If $x:a=y:b=z:c$, then

$$\frac{x^3}{a^2}+\frac{y^3}{b^2}+\frac{z^3}{c^2}=\frac{(x+y+z)^3}{(a+b+c)^2}.$$

25. Resolve into factors:

(i) $x^2-x-1332$.

(ii) $z(x+y)^2-y(z+x)^2$.

26. Simplify $\dfrac{3\sqrt{3}-2\sqrt{2}}{\sqrt{3}-\sqrt{2}}$.

27. Solve the equations (i) $\dfrac{2}{2x-5}+\dfrac{1}{x-3}=\dfrac{6}{3x-1}$.

(ii) $32^x=\dfrac{8^x}{16}$.

28. Eliminate t from the equations $x=t+\dfrac{1}{t}$, $y^2=t^2-\dfrac{1}{t^2}$.

29. Find the values of A, B, C in the following identity:

$$\frac{x^2}{(x-1)(x-2)(x-3)}\equiv\frac{A}{x-1}+\frac{B}{x-2}+\frac{C}{x-3}.$$

30. A certain sum was divided equally among a certain number of persons; had there been 3 persons more, each would have received 2 as. less, and had there been 6 persons fewer, each would have received 6 as. more; find the sum of money and the number of men.

31. (*a*) Find the value $\sqrt[5]{243^3}$ and $1296^{-\frac{3}{4}}$.

(*b*) If $x+\frac{1}{y}=1$, $y-\frac{1}{z}=1$, prove that $xyz=1$.

32. (*a*) Shew that $5^{3x}-4^{3x}$ is always divisible by 61.

(*b*) Divide $2x^3+5x^2-mx+4$ by x^2+2x-1, and find the value of m for which the given divisor will be a factor of the given dividend.

33. Find the H. C. F. of $8x^4+3x+10$ and $10x^4+3x^3+8$.

34. If $\frac{a^2+c^2}{ab+cd}=\frac{ab+cd}{b^2+d^2}$, shew that $\frac{a}{b}=\frac{c}{d}$.

35. Resolve into factors:

(i) $a(a+1)x^2+x-a(a-1)$.

(ii) $x^4+y^4+1+2x^2y^2-2x^2-2y^2$.

36. Simplify $\frac{x^3+6x^2+2x-15}{x^3+5x^2-2x-10}$.

37. Solve the equations:

(i) $\left.\begin{array}{l} 3^{x+1}+2^y=85 \\ 3^x-2^{y+2}=11. \end{array}\right\}$

(ii) $\left.\begin{array}{l} x+y+z=3x+2y+z=0 \\ 2x-3y+4z=24. \end{array}\right\}$

38. Prove that $\frac{n(n-1)(n-2)}{1.2.3}+\frac{n(n-1)}{1.2}=\frac{(n+1)n(n-1)}{1.2.3}$.

39. If $\frac{x^2+3x+7}{(x^2+x+1)(x+1)^2}\equiv\frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{Cx+D}{x^2+x+1}$, find the values of A, B, C and D.

40. A person bought 13 horses and 9 cows for Rs. 4,185 and at the same rates 8 horses and 6 cows for Rs. 2.640. Find the price of 9 horses and 5 cows.

41. (i) If $2s=a+b+c$, prove that

$$16s(s-a)(s-b)(s-c)=(a+b+c)(b+c-a)(c+a-b)(a+b-c).$$

(ii) If $a=AP^{p-1}$, $b=AQ^{q-1}$, $c=AR^{r-1}$, then

$$a^{q-r}.\ b^{r-p}.\ c^{p-q}=1.$$

42. (*a*) Simplify by the shortest possible method.

$$(5x-8)^3-(3x-8)^3-6x(5x-8)(3x-8).$$

(*b*) Without actual multiplication, work out the co-efficient of x^3 in the product of $x^4-5x^3-2x^2+7x-4$ and x^2-3x+1.

43. Find the H.C.F. of $3x^5-5x^3+2$ and $2x^5-5x^2+3$, by the method of the alternate destruction of the highest and the lowest terms.

44. If $\dfrac{1}{b+c}+\dfrac{1}{c+a}=\dfrac{2}{a+b}$, shew that $a^2+b^2=2c^2$.

45. Resolve into factors:

(i) x^3+64.

(ii) $(x+y+z)(xy+yz+zx)-xyz$.

46. Simplify $\dfrac{3^{2x+1}.\ 5^{2x-y}.\ 7^{2x+y+2}.\ 15^y}{15^x.21^{y+x+2}.35^x}$.

47. Solve the equations (i) $\dfrac{x-1}{x-2}+\dfrac{x-5}{x-6}=\dfrac{x-2}{x-3}+\dfrac{x-4}{x-5}$.

(ii) $y-\dfrac{1}{x}=x-\dfrac{1}{y}=\dfrac{3}{2}$.

48. Eliminate x, y, z from $y=az$, $z=bx$, $x=cy$.

49. Find the values of A, B, C in $(2x-3)^2\equiv Ax^2+2Bx-3C$.

50. A railway journey of 240 miles would take half an hour less if the speed of the train were increased by 2 miles an hour. Find the speed in miles per hour.

51. (i) If $2s=a+b+c$, shew that

$$(s-a)^3+(s-b)^3+(s-c)^3+3abc=s^3.$$

(ii) If $a^x=b^y=c^z$ and $ac=b^2$, then $\frac{1}{x}+\frac{1}{z}=\frac{2}{y}$.

52. (i) For what value of k is $36x^4-36x^3+kx^2-12x+4$ a perfect square?

(ii) Fill in the blanks in the following:

$$\frac{x}{5}=\frac{y}{3}=\frac{4x+3y}{\quad}=\frac{3x-2y}{\quad}.$$

53. Extract the square root of:

$$x^4+4x^2+\frac{1}{x^2}-2x-\frac{4}{x}+4.$$

54. If $a=\sqrt{\frac{x-1}{x+1}}$, express $\frac{1-a}{1+a}$ in terms of x.

55. Resolve into factors:

(i) x^4+64.

(ii) $x^2+(a+b+c)x+ab+ac$.

56. Simplify $\Sigma\frac{1}{(a-b)(a-c)(x-a)}$.

57. Solve the equations (i) $3^x+3^{-x}=9\frac{1}{9}$.

(ii) $a^{2x^m}=\{(a^x)^m)\}^x$.

58. Draw the graph of $\frac{3x-7}{6}$. From the graph find the value of the function when $x=3{\cdot}5$; also find for what value of x the function becomes equal to $1{\cdot}1$?

59. Determine the values of p, q, r in

$$(x-1)(x-2)(x-3)\equiv(x-4)^3+p(x-4)^2+q(x-4)+r.$$

60. A farmer bought an equal number of two kinds of sheep, one kind @ Rs. 6 each and the other @ Rs. 8 each; if he had spent his money equally on the two kinds, he would have had 3 sheep more than he had. How many of each kind did he buy?

61. (*a*) Write down *briefly* the value of

$$\frac{52 \cdot 9 \times 52 \cdot 9 \times 52 \cdot 9 \times 1000000}{125(3 \cdot 6)^3}.$$

(*b*) Find the value of $\dfrac{5^{x+1}}{(5^x)^{x-1}} \div \dfrac{25^{x+1}}{(5^{x-1})^{x+1}}.$

62. (i) If $a+b+c=15$ and $ab+ac+bc=71$, find the value of $a^2+b^2+c^2$.

(ii) If $a^2+b^2+c^2=155$ and $ab+ac+bc=143$, find the value of $a+b+c$.

63. The H.C.F. and L.C.M. of two integral expressions A and B, each of the second degree, are $x+3$ and x^3-7x+6; find A and B.

64. If $a:b::b:c$ shew that

$$(a+b+c)(a-b+c)=a^2+b^2+c^2.$$

65. Resolve into factors :

(i) $16x^8-1$.

(ii) $(5x+3y)^3-(3x+5y)^3-8(x-y)^3$.

66. Simplify $\dfrac{x^3-1}{x^2-1} \times \dfrac{x^2+3x+2}{x^4+x^2+1} \div \dfrac{x^2+x-2}{x^4+x}.$

67. Solve the equations (i) $\left.\begin{aligned} \frac{1}{x}+\frac{\cdot 1}{y}&=\cdot 25 \\ \frac{1}{y}+\frac{\cdot 1}{x}&=\cdot 52 \end{aligned}\right\}$

(ii) $\sqrt{3x+2}-\sqrt{3x-10}=2$.

68. If $x-\dfrac{1}{x}=k$, find the value of $x^3-\dfrac{1}{x^3}$.

69. Find the values of k, l, m in

$2x^2-11x+5 \equiv (kx+l)(x-3)+m(x^2+2x-5)$.

70. A number consists of two digits. When the number is divided by the sum of its digits, the quotient is 7. The sum of the reciprocals of the digits is 9 times the reciprocal of the product of the digits. Find the number.

71. If $x=b+c$, $y=c+a$ and $z=a+b$, find the value of

$$\frac{x^3+y^3+z^3-3xyz}{a^3+b^3+c^3-3abc}.$$

72. (*a*) For what value of x will both the expressions x^3-x^2-5x-3 and $x^3-4x^2-11x-6$ vanish?

(*b*) To make $9x^4-12x^3+10x^2-3x-3$ a perfect square,

(i) what should be added to it, (ii) what should be subtracted from it, (iii) what values should be given to x?

73. The H.C.F. of two expressions is x^2+3x+2 and their L.C.M. is $(x^2+3x+2)(x-3)(x+5)$, one expression is $x^3+8x^2+17x+10$; find the other.

74. (i) If $ax=y$ and $by=x$, shew that $\dfrac{1}{1+a}+\dfrac{1}{1+b}=1$.

(ii) $a:b=c:d=e:f$, prove that

$$(a^2+c^2+e^2)(b^2+d^2+f^2)=(ab+cd+ef)^2.$$

75. If $x+y+z=6$ and $xy+yz+zx=9$, shew that

$$\frac{1}{1-x}+\frac{1}{1-y}+\frac{1}{1-z}=0.$$

76. Simplify $\dfrac{(x^2)^{a+b}\times(x^2)^{b+c}\times(x^2)^{c+a}}{x^{4a}\times x^{4b}\times x^{4c}}$.

77. Solve the equations (i) $\dfrac{x+\sqrt{x-1}}{x-\sqrt{x-1}}=\dfrac{3}{7}$

(ii) $x^2-y^2=18$, $x-y=3$.

78. Shew that $N(N^2-1)$ will be divisible by 48, if N is an odd number.

79. If $f(x)=x^3-3x^2+3x-1$ find the values of $f(0)$, $f(1)$ and $f(x+1)-f(x-1)$.

80. A man, who went out for a walk between 5 and 6 and returned between 6 and 7, found that the hands of his watch had exactly changed places. When did he go out?

[*Hint.* Suppose he went out at x minutes past 5 and returned at y minutes past 6. From these suppositions, we get

$$x=30+\frac{y}{12} \text{ and } y=25+\frac{x}{12}.]$$

81. (*a*) If $x^2+y^2=1$, complete $\frac{x}{a}=\frac{y}{b}=\frac{1}{\ }$.

(*b*) Fill in the blank in $\frac{x^2+y^2}{a}=\frac{xy}{b}=\frac{x^2-y^2}{\ }$.

82. (*a*) Find the value of $x^3+y^3-z^3+3xyz$ when $x=3{\cdot}461$, $y=2{\cdot}314$, $z=5{\cdot}775$.

(*b*) Find the continued product of

$$x^2+x+1,\ x^2-x+1,\ x^4-x^2+1.$$

83. For what value of x will the expression $2x^4-7x^3+13x^2-9x-3$ be divisible by x^2-2x+3?

84. If $x+y=a$ and $xy=b$, express x^4+y^4 in terms of a and b.

85. Simplify $\frac{x^3+1}{x^2-5x+6}\times\frac{x^2-7x+10}{x^4+x^2+1}\div\frac{x^2-4x-5}{x^3+x^2+x}$.

86. Solve the equation $\frac{1}{x-1}+\frac{4}{x-4}=\frac{2}{x-2}+\frac{3}{x-3}$.

87. If $\frac{x}{a-b}=\frac{y}{b-c}=\frac{z}{c-a}$, prove that $x(a+b)+y(b+c)+z(c+a)=0$.

88. Construct a homogeneous and symmetrical expression of the second degree in x and y, which is equal to 128 when $x=-y=4$ and which is equal to 120 when $x=1$, $y=-5$.

89. Simplify $(a+b+c)^4-(b+c)^4-(c+a)^4-(a+b)^4+a^4+b^4+c^4$ by the principles of homogeneity, symmetry and indeterminate co-efficients.

90. An officer arranged his army of 3,900 soldiers in a hollow square 15 deep; find the number of soldiers in the front row.

91. (*a*) Simplify $\left(\frac{x^a}{x^b}\right)^{a+b} \times \left(\frac{x^b}{x^c}\right)^{b+c} \times \left(\frac{x^c}{x^a}\right)^{c+a}$.

(*b*) Shew that $7^{2n+1}+1$ is exactly divisible by 8 for all integral values of n, even or odd.

92. (*a*) If $x=b+c-a$, $y=c+a-b$, $z=a+b-c$, then

$$x^3+y^3+z^3-3xyz=4(a^3+b^3+c^3-3abc).$$

(*b*) Find the continued product of $(1+x)(1+x^2)(1+x^4)$, without actual multiplication.

93. (*a*) Shew that an integral expression is divisible by $x-1$ if the sum of its co-efficients is zero.

(*b*) Fill in the blanks

$(x-a)(x+b)(x+\qquad)\equiv x^3+x^2(\qquad)+x(\qquad)-abc.$

94. Find the L.C.M. of $3a^2-ab-2b^2$, $8a^3-27b^3$ and $6a^2-5ab-6b^2$.

95. Resolve into factors (i) $a^3-19ab^2+30b^3$,
(ii) $a^3-17x+26$.

96. Simplify $\dfrac{2^6\times16^{-2}\times256}{4^{-9}\times2^{23}}$.

97. Find the square root of $22-12\sqrt{2}$.

98. If $a+b+c=0$, shew that

$$a^2+ab+b^2=b^2+bc+c^2=c^2+ac+a^2.$$

99. Prove that $(a+b+c)^3-a^3-b^3-c^3\equiv3(a+b)(b+c)(c+a)$.

100. An officer can form his men into a hollow square 5 deep and also into a hollow square 6 deep, but the front in the latter case contains 4 men fewer than in the former. Find the number of men.

ANSWERS

[*Answers to the oral exercises have not been given intentionally.*]

EXERCISE 2. (Page 7.)

1. (i) 7^4, (ii) a^5, (iii) ap^3, (iv) ab^2, (v) $(\cdot 01)^3$, (vi) $\frac{1}{a^4}$, (vii) $3m^2+4n^2$, (viii) a^2n^3, (ix) p^4+p^3, (x) ax^3+by^3. 3. (i) 625, (ii) 49, (iii) 32, (iv) 81. 5. (i) 129, (ii) 359, (iii) 1113, (iv) 1114, (v) 14, (vi) 67. 6. 40, 65, 120, 40, 85. 8. (i) 5^7 or 78125, (ii) 7^6 or 117649, (iii) a^{13}, (iv) x^{13}. 9. (i) $6abcd$, (ii) $12x^4y^3$, (iii) $a^3b^3c^2$, (iv) $a^4b^2c^2$, (v) $15a^5b^7c^3$, (vi) $6p^5q^4$, (vii) $60m^4n^8p^5$. 11. (i) a^5, (ii) p^9, (iii) m^6, (iv) x^{12}. 12. (i) $2x^2y^4$, (ii) $5m^3n$, (iii) $2p^2q^2$, (iv) $2m^4p$, (v) $3l^2n^7$. (vi) $6x^3y^4z^2w^2$. 14. (i) 2^6 or 64, (ii) 3^6 or 729, (iii) a^{3m}, (iv) x^{3m}, (v) p^{4n}, (vi) y^{15}. 16. (i) 5^2 or 25, (ii) 4^3 or 64, (iii) x^6, (iv) $2x^3$, (v) $3y^4$, (vi) $5ab^2$. 17. (i) $4a^3$, (ii) $6x^4$, (iii) $5y^9$. 18. (i) $2a^2$, (ii) $3b^3$, (iii) $4c^5$. 19. 22. 20. 432.

EXERCISE 3. (Page 12.)

1. $11a^2$.
2. $9ab$.
3. $15p$.
4. $28xy$.
5. $6ab$.
6. $5abc$.
7. $5a^2$.
8. $4x^2$.
9. (i) $12a$, (ii) 12×13.
10. (i) $6x$, (ii) 6×5.
11. (i) $14x$, (ii) 14×4.
12. (i) $14y$, (ii) 14×7.
13. $2a+2b$.
14. $2x+3y$.
15. $2a+2b$.
16. $3x+2y$.
17. $3x+7$.
18. $2x+y+4$.
19. $a+b$.
20. $3x+7y+4$.
21. $2a$.
22. $3a+2b-2c$.
23. $2xy+3xz$.
24. $6a^2-3a$
25. x^3+x^2+4x.
26. $2p^2+p+3$.
27. a^4+4a^2+1.
28. $10+a-2a^2$.
29. a^3+2a^2+2a+1.
30. $7x^3-3x^2-2x+4$.
31. (i) $7x^4+x^3+5x^2+3x+4$, (ii) $4+3x+5x^2+x^3+7x^4$.
32. (i) $a^7+4a^6+5a^3+a^2+1$, (ii) $1+a^2+5a^3+4a^6+a^7$.
33. $5+3x+2x^2+2x^3+x^4$.
34. $3x^4-4x^3+8x^2+4x+9$.
35. $3.10^2+7.10+4$; no.
36. $5.10^3+7.10^2+4$; no.

EXERCISE 4. (Page 15.)

2. (i) $2a^2b^2$, $12a^3b^3$; (ii) $6ab$, $36a^3b^3$; (iii) $3ab$, $12a^2b$; (iv) $13a^2b^4$, $156a^3b^5$; (v) ab, $12a^3b^3c^2$; (vi) $2ab$, $72a^2b^2c^2$.

3. (i) 2.7^2, $2^3.7^3$ or 98, 2744; (ii) $3^2.5^2$, $3^3.5^3$ or 225, 3375; (iii) 144, 4320; (iv) 210, 4200.

4. (i) $\frac{3bc}{5a}$, (ii) $\frac{3xa}{y}$, (iii) $\frac{4y}{x}$, (iv) $\frac{1}{3acb}$, (v) $3xyz$. (vi) $\frac{3x}{ay^2}$. 5. The gaps are: (i) $35xy$, (ii) $12a^3b$, $2a$. (iii) a^3b^2, (iv) ab^2, (v) amn^3.

6. (i) $\frac{5x}{6ax}$, $\frac{8a}{6ax}$, $\frac{18c}{6ax}$; (ii) $\frac{9xy^2}{12x^2y^2}$, $\frac{2x^3}{12x^2y^2}$, $\frac{4y^3}{12x^2y^2}$; (iii) $\frac{6cx^2}{12abc}$, $\frac{4by^2}{12abc}$, $\frac{3az^2}{12abc}$; (iv) $\frac{60b^3x^2y}{60a^3b^3}$, $\frac{40ab^2x^3}{60a^3b^3}$, $\frac{45a^2by^3}{60a^3b^3}$, $\frac{48a^3xy^2}{60a^3b^3}$. 7. (i) $\frac{17x}{12}$, (ii) $\frac{3a+2b}{2x}$, (iii) $\frac{5}{6x}$, (iv) $\frac{x^2+y^2}{xy}$, (v) $\frac{x+y}{y}$, (vi) $\frac{1+3x}{x}$, (vii) $\frac{2}{3x}$, (viii) $\frac{a}{3b}$, (ix) $\frac{b-a}{b}$. (x) $\frac{xy-1}{xy}$, (xi) $\frac{z-x^2y^2}{xyz}$, (xii) $\frac{13x}{12}$, (xiii) $\frac{bcx+acy+abz}{abc}$, (xiv) $\frac{61x}{60}$, (xv) $\frac{13a}{12x}$, (xvi) $\frac{8ayz+5bxz+9cxy}{12xyz}$, (xvii) $\frac{15x^2-36xy+20y^2}{180}$, (xviii) $\frac{a^2+b^2+c^2}{abc}$, (xix) 0.

8. (i) $\frac{ab}{c^2}$, (ii) z^2, (iii) 1, (iv) abc, (v) $\frac{6c}{25a}$, (vi) $\frac{8x}{9z}$, 9. $\frac{1}{3}$.

10. No. 11. (i) $\frac{x}{x+y}$, (ii) $\frac{x+y}{y}$.

EXERCISE 5. (Page 18.)

6. a exceeds b by $a-b$. 7. $a+b=b+a$. 8. $a\times b=b\times a$.

9. $a^2\times a=a^3$. 10. $a\times\frac{1}{a}=1$. 11. $0\times a=0$. 12. $a\times 0=0$.

13. $2n$ is an even number. 14. $2n+1$ is an odd number.

15. The sum of the first n odd numbers is n^2.

16. $4n+n=5n$. **17.** $m\%$ of $n=\frac{m}{100}$ of n. **18.** If the length of a rectangle is l ft. and breadth b ft., its area $=l.b$ sq. ft.

19. $n-1$, n and $n+1$ are three consecutive numbers.

20. (a) $p+1$, (b) $p-1$. **21.** $m-1$, $m+1$. **22.** 120 sq. ft.

23. $707\frac{1}{7}$ sq. ft. **24.** Rs. 81. **26.** 104 degrees.

27. (i) 6 rt. angles, (ii) 12 rt. angles, (iii) 16 rt. angles, (iv) 20 rt. angles. **28.** (i) $\frac{4}{3}$ rt. angles, (ii) $\frac{3}{2}$ rt. angles.

29. $(16x+y)$ annas. **30.** (i) $\frac{x}{5}$ hours, (ii) $\frac{x}{y}$ hours.

31. (i) $\frac{m}{t}$ miles, (ii) $\frac{440m}{t}$ yards. **32.** $\frac{88m}{3}$ yards.

33. $\frac{60x}{s}$ days. **34.** 40 shillings, $\frac{8x}{5}$ shillings. **35.** $\frac{4x}{3}$ miles.

36. $\frac{ax}{15}$ yds. **37.** (i) Rs. $8y$, (ii) Rs. $\frac{my}{x}$. **38.** $\frac{7}{3}$ days.

$\frac{14x}{y}$ days. **39.** $\frac{xy}{x+m}$ days. **40.** $\frac{15(16x+y)}{74}$ srs. **41.** Rs. $\frac{n-m}{m}$.

42. $\frac{l+m+n}{a+b+c}$ miles per hour. **43.** $\frac{29x}{25}$. **44.** $\frac{100x}{y}$. **45.** (i) $\frac{100}{a}$, (ii) $\frac{100a}{b}$. **46.** Rs. $15r$. **47.** Rs. $\frac{pr}{20}$. **48.** Rs. $\frac{prt}{100}$.

49. $\frac{100}{x}$ years. **50.** $\frac{4x}{25}$ seeds. **51.** $\frac{27a}{4}$ annas. **52.** $\frac{19r}{20}$ rupees.

53. $\frac{4x}{25}$ as. **54.** $\frac{8(100-p)}{25}$ seers per rupee. **55.** $15l$ sq. ft.

56. Rs. $\frac{xy}{4}$. **57.** $\frac{A}{8}$ ft. **58.** $\frac{9xy}{20}$.

59. abc cubic inches, $2(ab+ac+bc)$ sq. inches.

60. $\frac{x}{lb}$ ft. **61.** $50x$ cubic inches. **62.** $\frac{32xy}{9}$.

63. $\frac{320ab}{9}$. **64.** $\frac{64h.w(l+b)}{27}$. **65.** hk.

66. $144t$ ft.

EXERCISE 8. (Page 31.)

3. 6 divisions in the negative direction.
4. 3 divisions in the positive direction. 5. 3 divisions in the positive direction. 6. 13 divisions in the positive direction.
7. 5 divisions in the negative direction. 8. 14 divisions in the negative direction. 9. 12 divisions in the positive direction.
10. 6 divisions in the negative direction.
11. 12 divisions in the negative direction.
12. + Rs. 60 − Rs. 25 − Rs. 15 + Rs. 12 − Rs. 16 + Rs. 24 + Rs. 40.
13. $-6x+8x-4x+2x+3x-2x-5x$.
14. $+8x-5x+11x-6x+13x-7x+15x$. 15. $-10a+14a-8a+16a$
16. + 850 ft. − 350 ft. + 260 ft. − 145 ft. 17. $-5^\circ, -2^\circ, +4^\circ, +8^\circ$.
18. (i) +6 hrs., (ii) +2 hrs., (iii) −2·5 hrs., (iv) +5 hrs. 20 mts. 19. (i) $+46^\circ$, (ii) -8°, (iii) 0°, (iv) -32°.
20. (i) +15 mls., (ii) −15 mls., (iii) −20 mls., (iv) −35 mls.

EXERCISE 9. (Page 36.)

11. $-2ab$. 12. $14xyz$. 13. $-4abx^2$. 14. $-18pq^2r$
15. $-a+4b-2c$. 16. $-3x-z$. 17. $-4p^2+2pq+9q^2$
18. $3ab+4bc+4ca$. 19. $2x^2yz+4xy^2z+2xyz^2$.
20. $12ax^2y-3a^2xy-11axy^2$. 21. (i) 20, (ii) 3, (iii) −14, (iv) −7, (v) $-\frac{1}{2}$, (vi) 0. 22. +2, +6, +2, −4, +1, −5.
23. +2, +8, −10, 0, −9, +1, +3, −6, +4, +9.
24. x^2+xy+y^2. 25. x^2+y^2 26. a^2-b^2.
27. $-4p^2+3pq+3q^2$. 28. $9a^2-4a-4$.
29. $-6x^2+4xy-5y^2$. 30. $-6x-2y-5z$.
31. $-3b^2+b^2c-4bc^2$. 32. $a^2x^2-2ax-7$. 33. $3a^2-8b^2$
34. $-2p^2-4pq+11q^2$. 35. $-2+x^2+4x^3$.
36. $-a+4c-19d+5e+6f$. 37. $x^2+y^2+3x-8xy+2yz-9$.
38. $\frac{13}{12}a^2+\frac{2}{3}ab+\frac{1}{6}b^2$. 39. $-x+12$. 40. $-a$.

EXERCISE 10. (Page 41.)

29. $2x+2z$. 30. $-3ab+7bc-ac$. 31. $-6y^2+4xy$.
32. $-x^2-2y^2-18$. 33. $2-m^2-m^3-m^4$. 34. $3x^2-4y^2-1$.
35. $-7a^2+3ab-12b^2$. 36. $-4x^2y^2-y^2z^2+8z^2x^2$.
37. $p^3-5p^2q+4q^3-6$. 38. $2x^2+\frac{1}{3}xy+2y^2-3z^2$.

39. a^2+b^2. 40. x^2+y^2-2xy. 41. $3a^2-2b^2-5$.
42. $3b^2-a^2-3$. 43. $-2a^2-b^2+8$. 44. $2a^2+3b^2-4$.
45. $4a^2-3b^2+2$. 46. $-2a^2+b^2+12$.
47. (i) -7 is greater than -12 by 5, (ii) 0 is greater than -4 by 4.

EXERCISE 11. (Page 44.)

5. $-p-3q$. 6. $2a-b$. 7. $2p+q$. 8. $-p+3q$.
9. $11k+2l$. 10. $5a-9b$. 11. $6l-9m+5n$.
12. $-2c$. 13. $-4p+3q+1$. 14. $p+2q+1$. 15. $4a-2b$.
16. $2a$. 17. $6y-3x-6z$. 18. $+1$. 19. -4. 20. 0. 21. 1.
22. $2b$. 23. $2y-2x-2z$. 24. 0. 25. 0. 26. $-2b$.
27. $5-(3-9)$. 28. (i) $-a^2+(m^2+b^2-n^2+b^2+c^2)$, $-a^2-(-m^2-b^2+n^2-b^2-c^2)$, (ii) $x^2+(-a^2+bc-b^2-ca+c^2-ab)$, $x^2-(a^2-bc+b^2+ca-c^2+ab)$. 29 (i) $(2x^4-5x^4)+(x^3-4x^3)+(3x^2+x^2)$, $-(5x^4-2x^4)-(4x^3-x^3)-(3x^2-x^2)$,
(ii) $(ax^3-dx^3)+(-bx^2-ex^2)+(cx-fx)$,
$-(dx^3-ax^3)-(ex^2+bx^2)-(fx-cx)$.
30. (i) -1. (ii) $+a$.

EXERCISE 13. (Page 53.)

4. (i) -1, (ii) $+1$, (iii) $-a^{83}$, (iv) a^{94}, (v) p^{43}, (vi) m^{54}, (vii) n^{71}.
5. 0. 6. 0. 7. $-49, 63, -21, 3, -3, -1$. 8. (i) $-59, 121, -11, 1$; (ii) $-42, -197, 3, 4$; (iii) $3, -117, -13, -9$.
9. (i) $648y^{12}$, (ii) $-4a^{14}b^7$. 10. $4x^4-18x^3y+10x^2y^2-xy^3+y^4$.

EXERCISE 14. (Page 57.)

2. x^2+5x+6. 3. $x^2+7x+10$. 4. $b^2+ab+bc+ac$.
5. $a^2+2ab+b^2$. 6. $6a^2+7ab+2b^2$. 7. $2x^2+5xy+3y^2$.
8. $a^3b-2a^2b^2+ab^3$. 9. $-x^6+20x^3-100$.
10. $-3x^3y+7x^2y^2-2xy^3$. 11. $-x^3-2x^2y-xy^2$.
12. $4x^4-25a^2b^2$. 13. $-9m^3n+9m^2n^2-2mn^3$.
14. $\frac{5}{4}x^2-\frac{121}{6}x+\frac{3}{2}$. 15. $-10a^3+\frac{2}{3}a^2-\frac{5}{4}a+\frac{1}{12}$. 16. $1-a^3$.
17. $1+a^3$. 18. $a^3+2a^2b+2ab^2+b^3$. 19. $a^3-2a^2b+2ab^2-b^3$.
20. a^6-b^6. 21. a^6+b^6. 22. $21a^4-36a^3b-a^2b^2+48ab^3-36b^4$.
23. $\frac{3}{4}x^3-\frac{2}{3}xy^2+\frac{2}{5}y^3$. 24. $12x^4+24x^3y-25x^2y^2+29xy^3-12y^4$.
25. $21x^5-34x^4y+18x^3y^2-18x^2y^3+17xy^4-4y^5$.

26. $x^4+x^2y^2+y^4$. 27. $x^4+x^3y+x^3z-3x^2yz+xy^3+xz^3-3xy^2z-3xyz^2+y^4+z^4+yz^3+y^3z$.

28. $x^4+x^3y-x^3z+3x^2yz+xy^3-xz^3-3xyz^2+3xy^2z+y^4+z^4-yz^3-y^3z$.

29. $2x^5-5x^4y+5x^3y^2-5x^2y^3+5xy^4-3y^5$.

30. $x^4-4x^3y+6x^2y^2-4xy^3+y^4-z^4$.

31. $a^5-5a^4+10a^3-10a^2+5a-1$. 32. $1-5a^2+4a^4$. 33. $1-a^5$.

34. $a^3+b^3+c^3-3abc$. 35. $x^3-y^3-z^3-3xyz$.

36. $y^3+x^3-1+3xy$. 37. $x^3y-x^3z+y^3z-y^3x+z^3x-z^3y$.

38. $x^8+4x^6-6x^4-28x^2+1$. 39. $-\frac{1}{12}a^4+\frac{1}{24}a^3b+\frac{61}{144}a^2b^2+\frac{1}{24}ab^3-\frac{1}{12}b^4$

40. $a^6-6a^5x+15a^4x^2-20a^3x^2+15a^2x^4-6ax^5+x^6$.

41. $-3x^9+14x^8+x^7-10x^6+6x^5+4x^4-6x^3+x^2+4x-1$.

42. $2a^{11}-a^9-a^8+6a^7-10a^6-2a^5+8a^4-5a^3-2a^2+5a-1$.

43. $1-a^4$. 44. $a^3+6a^2+11a+6$. 45. a^3+6a^2-a-30.

46. $x^3+x^2(a+b+c)+x(ab+ac+bc)+abc$.

47. $x^3+6x^2y+11xy^2+6y^3$. 48. $a^3+3a^2b+3ab^2+b^3$.

49. $a^3-3a^2b+3ab^2-b^3$. 50. x^6-y^6. 51. $a^8+a^4b^4+b^8$.

52. $a^{12}-b^{12}$. 53. $2a^2b^2+2b^2c^2+2c^2a^2-a^4-b^4-c^4$.

54. $a^2+b^2+c^2+2ab+2ac+2bc$. 55. $a^2+b^2+c^2-2ab+2ac-2bc$.

56. $4a^2+9b^2+16c^2-12ab+16ac-24bc$.

57. $a^3+b^3+c^3+3a^2b+3a^2c+3b^2a+3b^2c+3c^2a+3c^2b+6abc$.

58. $a^3-b^3+c^3-3a^2b+3a^2c+3b^2c-3bc^2+3ab^2+3ac^2-6abc$.

59. -1. 60. 2. 61. -4. 62. $-3y^2$. 63. -1. 64. 39.

65. -23. 66. $-2x^2$. 67. $-47x$. 68. $-3x$. 69. $-3y$.

70. $12x^2-4xy+4y^2$. 71. $3x^2-3y^2$. 72. $4x^4+8x^2a^2+4a^4$.

73. $-14x^2-5x+21$. 74. $\dfrac{x^2+79x+66}{12}$. 75. $9x^3-12x^2-9x+11$.

76. $2x^3+4x$. 77. $30x^4+89x^3-26x^2-28x+13$.

78. $x^{2m+2n}+x^{m+n}y^{m-n}-2y^{2m-2n}$.

79. $6x^4+11x^3+10x^2+7x+2$, 72072.

80. $ar-bq+pc$. 81. (i) $a^3+b^3+c^3-3abc$, (ii) 0, (iii) 0.

EXERCISE 15. (Page 64.)

13. $bd-xm$. 14. $\frac{1}{2}bv-\frac{1}{3}cz+\frac{1}{6}bczy$. 15. $ab+bc+ca$.

16. $5a^2x^3-4ax^2-7x$. 17. $-3a^2b^2+2x-abx^2$.

18. $-2a^3+3ab^2+6c^3$. 19. $40c^2-45b^2+24a^2$.

20. $6bc^2-8a^2c+9ab^2$. 21. $x+3$. 22. $x+5$. 23. $x-3$.

24. $x-7$. 25. $x-2$. 26. $5x+3$. 27. $3m-2$. 28. $a+3$.
29. $2a-3$. 30. $2p+1$. 31. $m-3$. 32. $3a+2$. 33. $1-x$.
34. a^2-2a+1 35. $x^2-3xy+y^2$. 36. a^2-ab-b^2.
37. $2a^2-a+3$. 38. a^2+3a+2. 39. $4a^2+3a-2$. 40. a^2-2a+3.
41. $-2x^2+3x-1$. 42. a^2-5a+1. 43. $a+1$. 44. $p^2-2pq+3q^2$.
45. x^2+x+1. 46. x^2-xy+y^2. 47. $x^4-x^3y+x^2y^2-xy^3+y^4$.
48. $x^4+x^3y+x^2y^2+xy^3+y^4$.
49. $x^7-x^6y+x^5y^2-x^4y^3+x^3y^4-x^2y^5+xy^6-y^7$.
50. $x^7-x^6+x^4-x^3+x-1$. 51. $a^2+b^2+c^2-ab-ac-bc$.
52. x^3-2x^2+2x-1. 53. $x^4+2x^3+3x^2+2x+1$.
54. $x^6-x^3y^3+y^6$. 55. $x^2+y^2-xy+x+y+1$.
56. $x^3-2x^2y+2xy^2-y^3$. 57. $x^8+x^6y^2-3x^2y^6-3y^8$.
58. $a^3+\frac{1}{3}a^2b+\frac{1}{9}ab^2+\frac{1}{27}b^3$. 59. $x^2-\frac{3}{4}x+1$. 60. $\frac{1}{2}x+\frac{1}{3}y-\frac{1}{6}z$.
61. $x^2+y^2+z^2+1$. 62. $a-b-c$. 63. $x^2+4y^2+9z^2-2xy+3xz+6yz$.
64. $a+2b+3c$. 65. $x^4+x^3y+x^2y^2+xy^3+y^4$.
66. $x^{2n}+x^ny^n+y^{2n}$. 67. $a+b-1+\dfrac{4}{1+a+b}$.
68. $a+2+\dfrac{5(a+1)}{a^2+5a+6}$. 69. $a^2+4a+1+\dfrac{2a+3}{a^2+3a+1}$.
70. $2a^2-ab+b^2-\dfrac{3b^4}{a^2+ab-2b^2}$.
71. Remainder $=x^4$, complete quotient $=1-x+x^2-x^3+\dfrac{x^4}{1+x}$.
72. Remainder $=135a^4$, complete quotient $=1-5a+15a^2-45a^3+\dfrac{135a^4}{1+3a}$.
73. Remainder $=-4a^5+8b^6$, complete quotient $=1+a-2a^3-4a^4+\dfrac{-4a^5+8a^6}{1-2a+2a^2}$.
78. $x^4-4x^3+6x^2-4x+1$. 79. x^2-3x+2. 80. -2.
81. -2. 82. $5a+4$. 83. $m=-4$. 84. -36. 85. $\dfrac{5}{m}$.
86. $a=-9$, $b=30$.

EXERCISE 17. (Page 76.)

1. 6. 2. 7 3. 3. 4. 4. 5. 28. 6. 18.
7. 40. 8. 48. 9. 18. 10. 10. 11. 18. 12. 15.
13. 13. 14. 17. 15. 13. 16. 11. 17. 7. 18. 9.
19. 9. 20. 11. 21. 6. 22. 6. 23. 5. 24. 4.
25. 3. 26. 3. 27. 6. 28. 9. 29. 9. 30. 7.
31. 1. 32. $\frac{12}{5}$. 33. 3. 34. $\frac{5}{2}$. 35 $\frac{10}{3}$. 36. $\frac{10}{3}$.
37. 6. 38. 6. 39. 4. 40. 3. 41. 9. 42. $\frac{3}{4}$.
43. $1\frac{5}{9}$. 44. $2\frac{1}{10}$ 45. $1\frac{2}{3}$. 46. 3. 47. 13. 48. 10.
49. 2. 50. 3. 51. 3. 52. $\frac{2}{3}$. 53. $2\frac{1}{3}$. 54. $4\frac{3}{7}$.
55. 2. 56. -3. 57. -11. 58. $-\frac{4}{3}$. 59. -4. 60. 1.
61. -7. 62. $-\frac{1}{2}$. 63. 2. 64. $2\frac{5}{6}$. 65. $\frac{3}{2}$. 66. $-\frac{2}{3}$.
67. $-\frac{10}{19}$. 68. 5. 69. 0. 70. $4\frac{2}{5}$. 71. $2\frac{4}{9}$. 72. 1. 73 2.
74. $2\frac{1}{4}$. 75. $\frac{2}{3}$. 76. 7. 77. -4. 78. $-3\frac{3}{11}$. 79. $1\frac{9}{13}$.
80. $4\frac{2}{3}$. 81. $-5\frac{7}{8}$. 82. $\frac{17}{18}$. 83. -3. 84. -2. 85. $-\frac{1}{3}$.
86. $\frac{2}{7}$. 87. $\frac{5}{13}$. 88. $-\frac{1}{19}$. 89. $2\frac{3}{7}$. 90. 10. 91. 1. 92. $8\frac{1}{3}$.
93. -1. 94. 1. 95. $-3\frac{1}{10}$. 96. $3\frac{16}{25}$. 97. 14. 98. $-\frac{1}{2}$.
99. $1\frac{2}{5}$. 100. (i) -1, (ii) any value of x, (iii) no value of x.

EXERCISE 18. (Page 83.)

1. 6. 2. 36. 3. 3. 4. 24. 5. 9. 6. $\frac{5}{6}$.
7. $\frac{3}{2}$ 8. $3\frac{3}{7}$. 9. $-\frac{16}{23}$ 10. $4\frac{1}{4}$ 11. 3. 12. $17\frac{1}{7}$. 13. 10.
14. -11. 15. 11. 16. 24. 17. 24. 18. $54\frac{2}{5}$. 19. $\frac{1}{2}$.
20. 2. 21. $3\frac{1}{2}$. 22. $1\frac{2}{5}$. 23. $-1\frac{31}{95}$. 24. $3\frac{3}{19}$. 25. 1.
26. $-\frac{5}{24}$. 27. 10. 28. $-\frac{28}{39}$. 29. $1\frac{3}{7}$. 30. $2\frac{3}{5}$. 31. $1\frac{13}{40}$.
32. $\frac{15}{184}$. 33. $-\frac{185}{73}$. 34. $-1{\cdot}67$. 35. 24.

EXERCISE 19. (Page 86.)

1. (i) $b=\dfrac{2A}{h}$, (ii) $h=\dfrac{2A}{b}$. 2. $r=\dfrac{C}{2\pi}$. 3. (i) $\dfrac{2A}{a+b}$,

(ii) $(a+b)=\dfrac{2A}{h}$. 4. (i) $r=\dfrac{A}{2\pi h}$, (ii) $h=\dfrac{A}{2r\pi}$.

5. $r=\sqrt{\dfrac{A}{\pi}}$. 6. $e=\sqrt{\dfrac{A}{6}}$. 7. (i) $x=180-2y$,

(ii) $y=\dfrac{180-x}{2}$. 8. (i) $r=100\left(\dfrac{y}{x}-1\right)$, (ii) 5.

9. (i) $r=\sqrt{\dfrac{V}{\pi h}}$, (ii) $h=\dfrac{V}{r^2\pi}$. 10. (i) $f=\dfrac{v-u}{t}$, (ii) $t=\dfrac{v-u}{f}$.

11. $t=\dfrac{\sqrt{S}}{4}$. 12. $a=\sqrt{c^2-b^2}$. 13. (i) $C=\frac{5}{9}(F-32)$, (ii) 10.

14. $h=\dfrac{3V}{\pi r^2}$, $r=\sqrt{\dfrac{3V}{\pi h}}$. 15. $f=\dfrac{2(s-ut)}{t^2}$.

16. $t=\dfrac{pv-2730}{10}$, $t=42$.

EXERCISE 21. (Page 94.)

1. 75°. 2. 102°. 3. 36°, 72°, 72°. 4. 25°, 5°.
5. 48 years, 24 years. 6. 45 years, 15 years. 7. 69°, 37°.
8. 50, 33. 9. 10. 10. 19. 11. 30. 12. 170.
13. 28. 14. 24. 15. 25. 16. Rs. 43, Rs. 53.
17. Rs. 81, Rs. 69. 18. 16, 21, 12. 19. Rs. 20, Rs. 27, Rs. 34.
20. Rs. 90, Rs. 120, Rs. 130. 21. Rs. 200, Rs. 160, Rs. 140.
22. 10. 23. 60. 24. 420. 25 60. 26. 36, 45.
27. $20\frac{4}{7}$ ft. 28. 120 miles 29. 56 gallons. 30. 7200.
31. 84. 32. 360. 33. 87. 34 420, 440. 35. 12, 28.
36. 566, 112. 37. 854, 210. 38 645, 214. 39. 9, 24.
40. 16, 36. 41. 21, 45. 42. 48. 43. 56, 28. 44. 54, 44.
45. 34, 8. 46. 28, 29, 30. 47. 17, 18, 19, 20. 48. 14, 16, 18.
49. 25, 27, 29. 50. 50, 51. 51. 110, 112. 52. 75, 77.
53. 102, 104 54. 115, 117. 55. 48. 56. 25. 57. 71.
58. 53. 59. 62. 60. 21. 61. 72. 62. 24.
63. 39. 64. $27\frac{6}{7}$ minutes past 8 P.M. 65. $12\frac{1}{2}$ days after the start of the first steamer ; 3000 miles from P.
66. $1\frac{10}{11}$ hours. 67. 3 miles. 68. 20 miles an hour.
69. $5\frac{5}{6}$ miles. 70. 48 miles. 71. A meets B after travelling $21\frac{2}{3}$ miles. 72. 30 miles per hour ; 240 miles.
73. After $10\frac{10}{11}$ minutes ; B overtakes A after travelling $10\frac{10}{11}$ miles.
74. (i) After $34\frac{10}{11}$ minutes, (ii) after $27\frac{3}{11}$ minutes, (iii) after $43\frac{7}{11}$ minutes. 75. (i) $10\frac{10}{11}$ minutes past 2, (ii) $21\frac{9}{11}$ minutes past 4. 76. (i) $54\frac{6}{11}$ minutes past 4, (ii) $43\frac{7}{11}$ minutes past 2. 77. (i) $43\frac{7}{11}$ minutes past 5, (ii) $49\frac{1}{11}$ minutes past 6. 78. 32 yds. and 48 yds. '9. Tea 14 as. per lb.

and coffee 10 as. per lb. **80.** 126 mangoes. **81.** 20 seers of the first kind, 30 seers of the second kind. **82.** Rs. 5400.
83. Rs. 1200, Rs. 2000. **84.** Rs. 2400. **85.** 3400 males, 3200 females. **86.** 40 goats of each kind. **87.** 44 men.
88. 255 ft. **89.** 49 ft., 40 ft. **90.** 16 ft., 10 ft.
91. 27 ft., 15 ft. **92.** Rs. 170, 310 eight-anna bits.
93. 60 four-anna bits, 20 two-anna bits. **94.** 20, 30, 5, 125.
95. 40, 31, 72, 12. **96.** 19, 13, 7, 42. **97.** 28 boys, Rs. 125.
98. 5 hours, 17 miles. **99.** 180 leaps. **100.** 40 leaps.

SECTIONAL REVISION I. (Page 110.)

Paper 1. **2.** $-35, -11, -5$. **3.** $x^2+xy-xz-yz$.
5. $-\frac{1}{3}$. **6.** 48.

Paper 2. **1.** $a^2+b^2+c^2$. **2.** 3. **3.** $\dfrac{6a-5b}{3}$.
4. a^6-b^6. **5.** $-4\frac{2}{9}$, **6.** 15 years, 36 years.

Paper 3. **1.** $6a-2b$. **3.** Quotient $=1+3x+4x^2+2x^3$, remainder $=-4x^4-4x^5$. **4.** $20a^2-5ab$. **5.** $\frac{1}{2}$.
6. $3\frac{6}{9}$ hours after the departure of B; $16\frac{8}{9}$ miles from the starting place.

Paper 4. **1.** $-a+b+4c$. **2.** x^2+3x-1. **3.** $x^{3n}-1$.
4. 1. **6.** $\frac{5}{12}$.

Paper 5. **2.** (i) $+1$, (ii) $+1$, (iii) $+1$, (iv) -1. **3.** Quotient $= 1-a+a^3-a^4$, remainder $=a^6$. **4.** 5. **5.** a^8+a^4+1.
6. $46\frac{2}{3}$ miles.

Paper 6. **1.** $2(ab+ac+2bc)$. **2.** $-56+68x-158x^2+107x^3-98x^4+47x^5-15x^6+x^7$. **3.** $11am^2$. **4.** x^3.
5. 2. **6.** 48 miles.

Paper 7. **1.** (i) 20. (ii) $9a^2+3a-5$. **2.** $8a^2+8b^2-22c^2-16ab-86ac+86bc$ or $22c^2-8a^2-8b^2+16ab+86ac-86bc$.
3. $18x^2+40xy$. **4.** $10a^2+2$. **5.** -1. **6.** Rs. 2300.

Paper 8. **2.** $2p, 2q$. **3.** $a^4-2a^3+3a^2+5a+11$.
5. $-7\frac{13}{17}$. **6.** 24.

Paper 9. 1. $11+7x-10x^2+x^3+2x^4-4x^5$. 2. 25, 121, 361, 841...

3. (i) $r=\sqrt{\dfrac{3v}{\pi h}}$. (ii) $h=\dfrac{3v}{\pi r^2}$. 4. $13x^2$. 5. $-\dfrac{5}{6}$.

6. 31 ft., 20 ft.

Paper 10. 1. (i) $\frac{9}{4}a^4b^6c^8d^2$. (ii) $\frac{4}{5}a^2b^5cd^2$.

2. $\dfrac{mn}{100}\%$ 3. (i) 30, (ii) 1. 4. $3x^3-2x^2-3x-6$.

5. (i) $3x-2y-2z$, (ii) $k=-a$. 6. 280 leaps.

EXERCISE 22. (Page 116.)

1. $4x^2+20xy+25y^2$. 2. $9a^2+42ab+49b^2$. 3. $25x^2+30x+9$.
4. $9+24p+16p^2$. 5. $9a^2b^2+24ab^3+16b^4$.
6. $64a^4+48a^3b+9a^2b^2$. 7. $81a^2x^2+72axy^2+16y^4$.
8. $m^4+2m^2n^2+n^4$. 9. $p^2x^2+2pqxy+q^2y^2$.
10. $p^6+2p^4q^2+p^2q^4$. 11. $4a^4b^2+12a^3b^3+9a^2b^4$.
12. $a^2-2ab+b^2$. 13. $56xy$. 14. $25n^2$. 15. $49q^2$.
16. $198a$. 17. 9. 18. $16p^2$ 19. 91204. 20. 164025. 21. 494209.
22. 649636. 23. 373321. 24. 1002001. 25. $5a^2+8ab+5b^2$.
26. $5x^2-5y^2$. 27. $p^2x^2+q^2y^2-q^2x^2-p^2y^2$.
28. $n^2+2mnx-2mxk-k^2$. 29. $9a^2-30ab+25b^2$.
30. $25m^2-30m+9$. 31. $36p^2-60p+25$.
32. $49x^2-14x+1$. 33. $16-56p+49p^2$.
34. $a^2+2ab+b^2$. 35. $16m^4-40m^2n^2+25n^4$.
36. $a^2x^2-2abxy+b^2y^2$. 37. $4a^4x^2-12a^2b^2xy+9b^4y^2$.
38. $49a^4-56a^2b^2+16b^4$. 39. $4a^6b^2-4a^3b^4+b^6$.
40. $49p^6q^4-112p^5q^5+64p^4q^6$. 41. $24ab$. 42. $16y^2$.
43. $16q^2$. 44. $154m$. 45. $36q^2$. 46. $25x^2$.
47. 39204. 48. 60025. 49. 156816.
50. 19044. 51. 247009. 52. 358801.
53. $4ab$. 54. $13a^2-24ab+13b^2$. 55. $4pqxy$.
56. $3m^2x^2-10mnxy-8n^2y^2$. 57. $17a^2b^2-12ab^3-4a^4$.
58. 369. 59. 185. 60. 289. 61. 305.

EXERCISE 23. (Page 119.)

1. $9x^2-25$. 2. $a^2m^2-b^2n^2$. 3. $a^2x^2-b^4$.
4. $p^2x^2-q^2y^2$. 5. $16a^2-25b^2$. 6. $9x^2y^2-49x^2z^2$.
7. a^4-b^4. 8. a^4-1. 9. a^8-b^8.
10. $x^2-y^2+2yz-z^2$. 11. $x^2-6xy+9y^2-16z^2$.
12. $x^4+x^2y^2+y^4$. 13. x^4+x^2+1. 14. x^8+x^4+1.

15. $3a^3-6ab+8ac+2bc-3c^2$. 16. $4a^3b+4ab^3$
17. $4ab-4ac+4ad$. 18. $24ab-32ac+40ad$. 19. 5445.
20. 7815. 21. 6460. 22. 8547.

EXERCISE 24. (Page 120.)

25. $4x^2+16x+15$. 26. $9m^2+9m-10$.
27. $1-x-6x^2$. 28. $9-3x-20x^2$.
29. $x^2-2xy+y^2+3zx-3zy+2z^2$.
30. $4x^2+4xy+y^2+4zx+2yz-3z^2$.
31. $4x^2+12xy+9y^2-8xz-12yz-5z^2$.

EXERCISE 25. (Page 122.)

1. $x^3+6x^2+11x+6$. 2. $x^3+9x^2+26x+24$.
3. $x^3+15x^2+74x+120$. 4. $x^3+2x^2-23x-60$.
5. $x^3+2x^2-43x-140$. 6. $x^3+3x^2-18x-40$.
7. $x^3-6x^2-7x+60$. 8. $x^3-5x^2-18x+72$.
9. $x^3-12x^2+47x-60$. 10. $x^3-12x^2+39x-28$.
11. $x^3-10x^2+31x-30$. 12. $x^3-18x^2+104x-192$.
13. $8x^3+36x^2+46x+15$. 14. $27x^3-63x^2+42x-8$.
15. $a^3x^3+11a^2x^2+36ax+36$. 16. $m^3x^3-8m^2x^2+11mx+20$.
17. $p^3x^3-14p^2x^2+61px-84$.
18. $(x+p)(x+q)(x+r)=x^3+(p+q+r)x^2+(pq+pr+qr)x+pqr$.
19. $(p+a)(p+2b)(p-3c)=(p)^3+(a+2b-3c)p^2+(2ab-3ac-6bc)p$ $+(-6abc)$. 20. $(m+3)(m-4)(m+5)=m^3+(3-4+5)m^2$ $+\{3\times(-4)+3\times5+(-4)5\times\}m-60$.

EXERCISE 26. (Page 123.)

1. $8x^3+12x^2y+6xy^2+y^3$. 2. $x^3+9x^2y+27xy^2+27y^3$.
3. $8a^3+36a^2b+54ab^2+27b^3$. 4. $a^3x^3+3a^2x^2by+3axb^2y^2+b^3y^3$.
5. $1+9x+27x^2+27x^3$. 6. $8+60x+150x^2+125x^3$.
7. $27x^6+54x^4+36x^2+8$. 8 $a^6+3a^4b^2+3b^4a^2+b^6$.
9. $27a^6+54a^4b^2+36a^2b^4+8b^6$.
10. $(4m+2n)^3=(4m)^3+3(4m)^2(2n)+3(4m)(2n)^2+(2n)^3$.
11. $(2p+3q)^3=(2p)^3+3(2p)^2(3q)+3(2p)(3q)^2+(3q)^3$.
12. $(5p+2q)^3=(5p)^3+3(5p)^2(2q)+3(5p)(2q)^2+(2q)^3$.
13. $(2a+3b)^3=(2a)^3+(3b)^3+3(6ab)(2a+3b)$.
14. $343a^3+294a^2b+84b^2a+8b^3$. 15. $125(x^3+3x^2y+3xy^2+y^3)$.
16. $2a^3+9a^2+15a+9$. 17. $24a^2+48a+26$.
18. 35. 19. 91. 20. 243. 21. 355. 22. -19. 23. -105.

26. $8x^3-12x^2+6x-1$. 27. $27-54x+36x^2-8x^3$.
28. $27a^3-108a^2b+144ab^2-64b^3$. 29. $64x^3-144x^2y+108xy^2-27y^3$.
30. $125x^3-300x^2y+240xy^2-64y^3$.
31. $64m^6-144m^4n+108m^2n^2-27n^3$.
32. $8a^3x^3-36a^2x^2by+54b^2y^2ax-27b^3y^3$.
33. $(2x-3y)^3=(2x)^3-3(2x)^2(3y)+3(2x)(3y)^2-(3y)^3$.
34. $(4m-5x)^3=(4m)^3-3(4m)^2(5x)+3(4m)(5x)^2-(5x)^3$.
35. $(4a-6b)^3=(4a)^3-3(4a)^2(6b)+3(4a)(6b)^2-(6b)^3$.
36. $(2p-3q)^3=(2p)^3-(3q)^3-3(2p)(3q)(2p-3q)$.
37. $27(m^3-12m^2n+48mn^2-64n^3)$. 38 $p^3+15p^2q+75pq^2+125q^3$.
39. $12x^2+24x+28$. 40. $96x^2-96x+152$.
41. 189. 42. 335 43. 448. 44. -3. 45. -7. 46. 988.

EXERCISE 27. (Page 126.)

1. x^3+27. 2. x^3+125. 3. x^3+343. 4. x^3+1331
5. $125a^3+27$ 6. $27m^3+125n^3$. 7. $x^3y^3+64z^3$. 8. $64x^3+729y^3$.
9. $a^6+b^3c^3$. 10. a^6+64b^6. 11. x^3+1. 12. 98.
13. $9(x^3+y^3)$. 14. $37(x^3-y^3)$. 15. 56. 16. x^3-8.
17. x^3-216. 18. x^3-512. 19. x^3-1728.
20. $27x^3-8$. 21. $125x^3-27y^3$. 22. $p^3q^3-8r^3$.
23. $a^6-b^3c^3$. 24. $19m^3-37n^3$. 25. $37(p^3+q^3)$.
26. -189. 27. $(a)^3+(3b)^3=(a+3b)\{(a)^2-(a)(3b)+(3b)^2\}$
28. $(2x)^3-(5y)^3=(2x-5y)\{(2x)^2+(2x)(5y)+(5y)^2\}$.
29. $(4p)^3+(3q)^3=(4p+3q)\{(4p)^2-(4p)(3q)+(3q)^2\}$.
30. $(7a)^3-(6b)^3=(7a-6b)\{49a^2+(7a)(6b)+36b^2\}$.

EXERCISE 28. (Page 128.)

1. $a^2+b^2+c^2+2ab-2ac-2bc$. 2. $a^2+b^2+c^2-2ab-2ac+2bc$.
3. $4a^2+b^2+c^2-4ab-4ac+2bc$. 4. $4a^2+b^2+c^2+4ab-4ac-2bc$.
5. $4a^2+b^2+9c^2-4ab+12ac-6bc$.
6. $25x^2+4y^2+9z^2-20xy+30xz-12yz$.
7. $4x^2+9y^2+1-12xy+4x-6y$.
8. $x^4+3x^2y^2+y^4-2x^3y-2xy^3$.
9. $p^2x^4+q^2x^2y^2+r^2y^4-2pqx^3y+2prx^2y^2-2qrxy^3$.
10. $4a^2+b^2+9c^2+d^2+4ab+12ac+4ad+6bc+2bd+6cd$.
11. $a^2+4b^2+9c^2+16d^2-4ab-6ac+8ad+12bc-16bd-24cd$.
12. $4p^2+9q^2+r^2+4s^2+t^2-12pq+4pr-8ps+4pt-6qr+12qs$
$-6qt-4rs+2rt-4st$.
13. $9x^2+16y^2+4z^2+w^2-24xy+12xz-6xw-16yz+8yw-4zw$.
14. $3a^2-6ab+8ac+2bc-3c^2$ 15. $2a^2+4ab+8ac+2bc-b^2-c^2$.

16. $4a^2+4b^2-2c^2+10ab-6ac+6bc$.
17. $2(a^2+b^2+c^2+d^2+2ac+2bd)$.
18. $2(a^2+b^2+c^2+d^2-2ac-2ad+2cd)$.
19. $2a^2-b^2-c^2-d^2-4ab+4ac-4ad+2bc-2bd+2cd$.
20. 361. 21. 841. 22. 20. 23. 52.

EXERCISE 29. (Page 130.)

1. 169. 2. 324. 3. 4. 4. 81. 5. 28. 6. 105.
7. 3. 8. ·61. 9. 3. 10. 3. 11. 12. 12. 23,527.
13. 11, 119. 14. p^4+4p^2+2. 15. 110. 16. 14. 17. m^3-3m.
18. a^3+3a. 19. $(3x+3y-z)^2$. 20. $(2x+5y+z)^2$.
21. $(x-y+z)^2$. 22. $(x-2y+3z)^2$. 25. 124. 26. 35, 984.

EXERCISE 30. (Page 134.)

1. $x^3+y^3-1+3xy$. 2. $x^3-y^3-1-3xy$.
3. $8x^3+y^3+1-6xy$. 4. $x^3+8y^3-z^3+6xyz$.
5. $x^3+8y^3-27z^3+18xyz$. 6. $8x^3-y^3-27z^3-18xyz$.
7. $9x^2+y^2+z^2+3xy+3xz-yz$. 8. $5x+3y-z$.
9. -572. 10. 63. 11. 740.

EXERCISE 31. (Page 136.)

1. $4a^2+2ax+x^2$. 2. $a^2-4ab+16b^2$.
3. $32a^5-16a^4x+8a^3x^2-4a^2x^3+2ax^4-x^5$.
4. $9m^2+3m+1$. 5. $x^3+4x^2y+16xy^2+64y^3$.
6. $1-2m+4m^2-8m^3$. 7. $16a^4-8a^3b+4a^2b^2-2ab^3+b^4$.
8. $m^{15}-m^{12}n^2+m^9n^4-m^6n^6+m^3n^8-n^{10}$.
9. $\frac{1}{9}a^2-\frac{1}{3}ab+b^2$. 10. $x^6-x^4yz+x^2y^2z^2-y^3z^3$.
11. $p+q-r$. 12. $a^2+b^2+c^2+ab-ac-2bc$,
13. $m^{15}+m^{12}n^2+m^9n^4+m^6n^6+m^3n^8+n^{10}$.
14. $x^8-x^7a+x^6a^2-x^5a^3+x^4a^4-x^3a^5+x^2a^6-xa^7+a^8$.
15. $x^7+x^6a+x^5a^2+x^4a^3+x^3a^4+x^2a^5+xa^6+a^7$.

EXERCISE 32. (Page 138.)

21. $(a+b)(c+d)$. 22. $(p+q)(p-q)$. 23. $(x-1)(7m+2)$.
24. $(x^2+x+1)(2a+3b)$. 25. $(ax+b)(3p+q+2r)$.
26. $(pq+r)(m^2+mn+n^2)$. 27. $(x-y)(a-b+c+d)$.
28. $x(x+7)(x^2+7x-25)$. 29. $5(p^2-qr)^2(3p^2-3qr-2)$.
30 [illegible]$(x-y)(3p+q)$. 31. $(x+y)\{(x+y)^2-3xy\}$

EXERCISE 33. (Page 140.)

1. $(x+y)(a+5)$.   2. $(x-y)(a-b)$.   3. $(x-y)(m+3)$.
4. $(x+2)(y+3)$.   5. $(x+b)(x-3a)$.   6. $(a+3)(a^2+9)$.
7. $(3p-5)(p^2+2)$.   8. $(2p-3q)(3p+2r)$.   9. $(p+1)(q+1)$.
10. $(a+1)^2(a-1)$.   11. $(2a-5)(a^2+1)$.   12. $(5a-2)(a^2+1)$.
13. $(x-y)(x-3)$.   14. $(2a+b)(3a-c)$.   15. $(2a-1)(a^3+2)$.
16. $(y-1)(y^2-a+1)$.   17. $(11x^2+7)(x+5)$.   18. $(x^2+1)(ax+b)$.
19. $(3y-2)(2x-3)$.   20. $(x-2a)(x-b)$.   21. $(x+1)(ax-1)$.
22. $(ax-1)(bx+y)$.   23. $(a+1)(a^2x-ay-z)$.
24. $(a+bx+cx^2)(x+1)(x-1)$.   25. $(kx^2+lx+m)(x+1)(x-1)$.
26. $(a-2b)(a-2b+3)$.   27. $(a+bc)(b+ca)$.
28. $(ax-by)(bx-ay)$.

EXERCISE 34. (Page 141.)

1. $(2a+7)^2$.   2. $(5p-3)^2$.   3. $(5a+4b)^2$.   4. $(7m-4n)^2$.
5. $(6a+5b)^2$.   6. $(6a-7b)^2$.   7. $(7a+3b)^2$.
8. $(5x-6y)^2$.   9. $(4x+7y)^2$.   10. $(9a-7b)^2$.
11. $(9x+5y)^2$.   12. $(3m-11n)^2$ 13. $(8x+y)^2$.
14. $(4p-6q)^2$.   15. $(3x+9y)^2$.   16. $(5m-7n)^2$.
17. $\left(x+\frac{1}{x}\right)^2$.   18. $\left(\frac{x}{y}-\frac{y}{x}\right)^2$ 19. $(ax+3by)^2$.
20. $(4ab-c^2)^2$.   21. $(2a+3b+c+d)^2$.   22. $(3p-q-r-s)^2$.

EXERCISE 35. (Page 142)

1. $(4x+3)(4x-3)$.   2. $(9a+5b)(9a-5b)$.   3. $(1+6x)(1-6x)$.
4. $(8a+1)(8a-1)$.   5. $(3x+10)(3x-10)$.   6. $(7+5a)(7-5a)$.
7. $(11+c)(11-c)$.   8. $(1+7ac)(1-7ac)$.   9. $(a^2+5)(a^2-5)$.
10. $(4x^3+7)(4x^3-7)$.   11. $(1+9x^2y)(1-9x^2y)$.
12. $(x^3+6y)(x^3-6y)$.   13. $(9x^2+1)(3x+1)(3x-1)$.
14. $x(4x^2+9)(2x+3)(2x-3)$.   15. $(1+4x^2)(1+2x)(1-2x)$.
16. $a^2(1+9a^2)(1+3a)(1-3a)$.   17. $x^3(6x^2+5a^2)(6x^2-5a^2)$.
18. $3x^5(8x^2+9y^2)(8x^2-9y^2)$.   19. $4a^5b^3(11a^6b^3+9)(11a^6b^3-9)$.
20. $(2x-3y)(2x-3y+2)(2x-3y-2)$.
21. $(a+5b+7c)(a+5b-7c)$.   22. $(x+2y-3z)(x-2y+3z)$.
23. $4ab$.   24. $(5x-y)(x-3y)$.   25. $(7x-y)(x-7y)$.
26. $(13a+b)(a+13b)$ 27. $(19m+13n)(23n-m)$.
28 $5n(12m-13n)$.   29. $12(5a-1)(a+2)$.   30. $4a(b-c)$.
31. $(10a+9b-7c)(15b-2a-25c)$.   32. $28a(3a-5)$.
33. $2ax(x^2+a^2)$.   34. 23935.   35. 714000.   36. ·476.

EXERCISE 36. (Page 143.)

13. $(a^2-2b^2)(a^4+2a^2b^2+4b^4)$.
14. $(3m^2+5n^2)(9m^4-15m^2n^2+25n^4)$.
15. $(a-b)(a^2+ab+b^2)(a^6+a^3b^3+b^6)$.
16. $(a^4-2b^2c^2)(a^8+2a^4b^2c^2+4b^4c^4)$.
17. $ab(7a^4+4b^4)(49a^8-28a^4b^4+16b^8)$.
18. $(3x+2y-z)(9x^2+4y^2+z^2+12xy+3xz+2yz)$.
19. $(2a-3b+3c)(4a^2+9b^2+9c^2-12ab-6ac+9bc)$.
20. $(a-b)(7a^2+13ab+7b^2)$.
21. $(a-2b+1)(a^2-4ab+4b^2-a+2b+1)$.
22. $(x-y-1)(x^2-2xy+y^2+x-y+1)$.
23. $2(b-2a)(151a^2-466ab+364b^2)$.
24. $(18x-5y)(156x^2+240xy+277y^2)$. 25. 2240. 26. 15500.
27. 31040. 28. $(a-b)(a+b)(a^2+ab+b^2)(a^2-ab+b^2)$.
29. $(a-b)(a+b)(a^2+b^2)(a^2+ab+b^2)(a^2-ab+b^2)(a^4-a^2b^2+b^4)$
30. $(2a^2+1)(2a^2-1)(4a^4-2a^2+1)(4a^4+2a^2+1)$.
31. $(3a+b)(3a-b)(9a^2-3ab+b^2)(9a^2+3ab+b^2)$.
32. $a(a-b)(a+b)(a^2+ab+b^2)(a^2-ab+b^2)$.
33. $3x^7(x+2y)(x-2y)(x^2-2xy+4y^2)(x^2+2xy+4y^2)$.

EXERCISE 37. (Page 146.)

1. $(x+6)(x+7)$. 2. $(a-13)(a-5)$. 3. $(x+13)(x+5)$.
4. $(p-9)(p-6)$. 5. $(x+12)(x+6)$. 6. $(x+13)(x+7)$.
7. $(x-13)(x-9)$. 8. $(x+8)(x+13)$. 9. $(a+13)(a-8)$.
10. $(x+9)(x+12)$. 11. $(a+12)(a-9)$. 12. $(x+12)(x+15)$.
13. $(p+13)(p-12)$. 14. $(a-15)(a+12)$. 15. $(a-13)(a+11)$.
16. $(a-16)(a+15)$. 17. $(x+15)(x-7)$. 18. $(a-30)(a+4)$.
19. $(1-15x)(1-4x)$. 20. $(1-51a)(1+2a)$. 41. $(p-10q)(p+2q)$
42. $(m-8n)(m-4n)$. 43. $(a+6b)(a-5b)$. 44. $(a-9b)(a+5b)$.
45. $(p-16q)(p+3q)$. 46. $(a+8b)(a+12b)$.
47. $(m+7n)(m-4n)$. 48. $(x-16y)(x+5y)$.
49. $(a-13b)(a+7b)$. 50. $(x-15y)(x+9y)$.
51. $(a-2)(a+2)(a^2+3)$. 52. $(x^2+7)(x+2)(x-2)$
53. $(x^2+7)(x+5)(x-5)$ 54. $(x^2-17)(x^2-8)$.
55. $(x^2+7y^2)(x+3y)(x-3y)$. 56. $(a-1)(a^2+a+1)(a^3+3)$
57. $(p-1)(p+3)(p^2+p+1)(p^2-3p+9)$. 58. $(x^3-12)(x^3-1$
59. $(m-2n)(m+2n)(m^2+4n^2)(m^4+5n^4)$.
60. $(p^2+q^2)(p^2-2q^2)(p^4-p^2q^2+q^4)(p^4+2p^2q^2+4q^4)$.
61. $(a^2-2a-6)(a^2-2a+5)$ 62. $(x^2+x-7)(x^2+x+5)$.
63. $(m^2-3m-6)(m^2-3m-2)$. 64. $(p^2-5p+9)(p^2-5p-4)$.

65. $(a-2)^2(a^2-4a+12)$.
66. $(x^2-6x-15)(x^2-6x+12)$.
67. $(m+3)(m+4)(m^2+7m-13)$.
68. $(p^2+4p+14)(p^2+4p+7)$.
69. $(x-3)(x-5)(x^2-8x-16)$.
70. $(m^2-7m-15)(m^2-7m+13)$.
71. $2(2x-y)(5x-3y)$.
72. $(7y-4x)(4x-y)$.
73. $(6a+19b)(a+9b)$.
74. $-2(5m+3n)(m+5n)$.
75. $10a(7b-a)$.
76. $-(3m+17n)(9m+7n)$.
77. $(3a-8b)(13b-4a)$.
78. $(a-4b)(a-b)^2(a+2b)$.
79. $7b^2(9a^2-11b^2)$.
80. $(a-6b)(a-4b)^2(a-3b)$.
81. $\frac{x+2}{x}$.
82. $\frac{x+1}{x+4}$.
83. $\frac{x-4}{x-2}$.
84. $x+3$.
85. $m=9$.
86. $p=-2$.
87. $c=20$.

EXERCISE 38. (Page 152.)

1. $(x+2)(2x+7)$.
2. $(x+5)(3x-1)$.
3. $(x+1)(5x+3)$.
4. $(2x+3)(3x-2)$.
5. $2(3x+1)(x-1)$.
6. $(2x-3)(4x-1)$.
7. $(4x+3)(2x-1)$.
8. $(4m+5)(3m-2)$.
9. $(a+1)(14a+15)$.
10. $(2p+5)(10p-3)$
11. $(p-1)(2p-3)$.
12. $(1+5m)(2-3m)$.
13. $(3a-4b)(3a-b)$.
14. $(2m-7n)(m+3n)$.
15. $(p-q)(4p+9q)$.
16. $(x-5y)(15x-2y)$.
17. $(2a+5b)(6a-b)$.
18. $(2x-7y)(5x-3y)$.
19. $\frac{x-2}{x-3}$.
20. $\frac{2x-5}{3x-5}$.
21. $(2x+2y+z)(x+y-2z)$.
22. $(a+b)^2(2a^2+ab+2b^2)$.
23. $(x-1)(x-3)(x+2)(x+4)$.
24. $(4a-5b-6)(2a+3b+8)$.
25. $7(40m-37n)(n-2m)$.
26. $(2x^2-5)(3x^2+4)$.
27. $(2x^2-3)(3x^2+1)$.
28. $(x^3-2)(5x^3+3)$.
29. $(2x^2+3y^2)(3x+y)(3x-y)$.
30. $(x-2)(2x-1)(x^2+2x+4)(4x^2+2x+1)$.
31. $(2x^2+y^2)(2x^2-y^2)(x^2+2y^2)(x^2-2y^2)$.
32. $(x-2y)(5x-3y)$.
33. $(x-3y)(5x+2y)$.
34. $(x-3)(7x-4)$.
35. $(x-4)(7x+3)$.
36. $xy(xy-5)(xy-4)$.
37. $a(2a-3b)(4a+5b)$.
38. $(a-9)(a+6)$.
39. $(x-5)(x+1)(x-2)^2$.
40. $(x^2+x-21)(x^2+x-5)$.
41. $(x+4)^2(x^2+8x+6)$.
42. $\left(\frac{x}{2}+5\right)\left(\frac{x}{2}-1\right)$.

EXERCISE 39. (Page 154.)

1. $(a-2b+3c)(a^2+4b^2+9c^2+2ab-3ac+6bc)$.
2. $(x-y-z)(x^2+y^2+z^2+xy+xz-yz)$.

3. $(a-b-1)(a^2+b^2+1+ab+a-b)$.
4. $(x+y+1)(x^2+y^2+1-xy-x-y)$.
5. $(a+b-6)(a^2+b^2+36-ab+6a+6b)$.
6. $(x-2y-4)(x^2+4y^2+16+2xy+4x-8y)$.
7. $(4a-3b+1)(16a^2+9b^2+1+12ab-4a+3b)$.
8. $(3x-5y-4)(9x^2+25y^2+16+15xy+12x-20y)$.
9. $(2a-3b-c)(4a^2+9b^2+c^2+6ab+2ac-3bc)$.
10. $\left(x-\frac{1}{x}-3\right)\left(x^2+3x+10-\frac{3}{x}+\frac{1}{x^2}\right)$.
11. $(a^2+2a-4)(a^4-2a^3+8a^2+8a+16)$.
12. $2y(3x^2+y^2+3z^2+3zx+3xy+3yz)$.
13. $2(c-b)(3a^2+b^2+c^2-3ab-3ac+bc)$.
14. $9(x+2)$. 15. $4(x+y+z)(x^2+y^2+z^2-xy-xz-yz)$.

EXERCISE 40. (Page 155.)

1. $(x^2+5x+7)(x^2+5x+3)$. 2. $(x^2+2x-7)(x^2+2x-4)$.
3. $(x^2-3x-16)(x^2-3x-12)$. 4. $(x+5)(x+6)(x^2+11x+8)$.
5. $(x^2-11x+12)(x^2-4x+12)$. 6. $(x^2-3x-16)^2$.
7. $(x-3)(2x+3)(2x^2-3x+7)$. 8. $(x+1)(3x-7)(3x^2-4x+3)$.
9. $8(x^2+x-7)(2x^2+2x-5)$. 10. $3(3x^2+x-1)(9x^2+3x+1)$.
11. $(x+2)^2(x^2+4x-6)$. 12. $(x^2+3x-13)(x^2+3x+5)$.
13. $4(2x-1)(x+4)(2x^2+7x+15)$. 14. $(9x^2-9x-17)(9x^2-9x-5)$.

EXERCISE 41. (Page 156.)

1. $(x+y+z)(x+y-z)$. 2. $(4x+2y-3z)(4x-2y+3z)$.
3. $(2x+3y-5z)(2x-3y+5z)$. 4. $(4x+6y+5z)(4x-6y+5z)$.
5. $(5x+7y-6z)(5x-7y-6z)$. 6. $(c+d+a-b)(c+d-a+b)$.
7. $(a-2b+3c-4d)(a-2b-3c+4d)$.
8. $(4a+3b-7c+5d)(4a-3b+7c+5d)$.
9. $(1-3b+2a-5c)(1-3b-2a+5c)$.
10. $(2a+3b-5c-6)(2a-3b+5c-6)$.
11. $(5c+7b-a-2)(5c-7b+a-2)$.
12. $(7a-4b-8c+1)(7a-4b+8c-1)$.
13. $(a^2+2a+3)(a^2-2a+3)$. 14. $(a^2+a+4)(a^2-a+4)$.
15. $(2a^2+2ab+3b^2)(2a^2-2ab+3b^2)$.
16. $(3a^2+5ab+4b^2)(3a^2-5ab+4b^2)$.
17. $(2a^2+5a-3)(2a^2-5a-3)$.
18. $(3a^2+3ab-4b^2)(3a^2-3ab-4b^2)$.
19. $(5a^2+7ab+3b^2)(5a^2-7ab+3b^2)$.
20. $(a^2+5ab-5b^2)(a^2-5ab-5b^2)$.

21. $(a^2+2ab-6b^2)(a^2-2ab-6b^2)$.
22. $(7a^2+2ab-4b^2)(7a^2-2ab-4b^2)$.
23. $(a^2+ab+b^2)(a^2-ab+b^2)$.
24. $(2a^2+10a+25)(2a^2-10a+25)$.
25. $(a^2+2a+2)(a^2-2a+2)$. 26. $(a^2+4a+8)(a^2-4a+8)$.
27. $9(a^2+2ab+2b^2)(a^2-2ab+2b^2)$.
28. $(8p^2+12pq+9q^2)(8p^2-12pq+9q^2)$.
29. $(a^2+a+1)(a^2-a+1)$. 30. $(a^2+a+1)(a^2-a+1)(a^4-a^2+1)$
31. $(a^2+ab+b^2)(a^2-ab+b^2)(a^4-a^2b^2+b^4)$.
32. $(5a^2+5ab+6b^2)(5a^2-5ab+6b^2)$.
33. $(3p-r)(3p+r-2q)$. 34. $(a-5c)(a+5c-6b)$.
35. $(5a-3c)(5a+3c-4b)$. 36. $(7a-3c)(7a-4b+3c)$.

EXERCISE 42. (Page 15

1. $(a+b)(a+b+1)$.
2. $(a-b)(a-b-1)$.
3. $(2a+3b)(2a+3b-4)$.
4. $(a-5b)(a-5b-3)$.
5. $(a+b-c)(a-b+c+1)$.
6. $(a-b-c)(a+b+c+1)$.
7. $(2a-b+3c)(2a+b-3c-3)$.
8. $(3a-4b-2c)(3a-4b+2c-5)$.
9. $(a-c)(a+b+c)$.
10. $(2a-3b)(2a+3b+4c)$.
11. $(3a-5b)(3a+5b-7c)$.
12. $(a+1)(a^2+1)$.
13. $(a+1)^2(a-1)$.
14. $(a-b)(a^2-ab+b^2)$.
15. $(a-2)(a^2+7a+4)$.
16. $(2a-3b)(4a+3b)(a+3b)$.
17. $(2a-3b)(4a+9b)(a+b)$.
18. $(3+4b)(9a^2-17ab+16b^2)$.
19. $(ax+by)(bx+ay)$.
20. $(a+b)(a^2+ab+b^2)$.
21. $(a-2b)(a+2b)(a^2+4b^2-5c^2)$.
22. $(2a+3b)(2a-3b)(4a^2+9b^2-7c^2)$.
23. $(a-b)^2(a^2+b^2)$.
24. $(a^2+2b^2)(a^2-5ab+2b^2)$.
25. $(a+b)(a-b)(p+3q)(p-q)$.

EXERCISE 43. (Page 160.)

1. $(a+5)(a+1)(2a-3)$.
2. $(a-3)(4a^2-3a-6)$.
3. $(a-6)(a+3)(2a-3)$.
4. $(a-1)(a-4)(2a+3)$.
5. $(a-b)(a^2+2ab+3b^2)$.
6. $(a-b)^2(a-2b)$.
7. $(a-4b)(a+2b)(a+5b)$.
8. $(2a-b)(a^2-ab+3b^2)$.
9. $(a-2b)(2a+b)(3a-4b)$.
10. $(a-b)(3a+2b)(4a+3b)$.
11. $(x+1)(2x-3)(x^2+1)$.
12. $(x-2)(2x+3)(x^2+1)$.
13. $(x-4)(x^3-2x^2-x+1)$.
14. $(x+4)(2x^3-x^2+3x-1)$.
15. $(x+2)(2x-1)(x^2+x-1)$.
16. $(x^2+x+1)(2x-1)(x+1)$.
17. $(x^2-x+1)(x^2+2x+3)$.
18. $(x-1)(x-3)(2x^2-x+1)$.
19. $(x-2)(x-3)(x^2-x+2)$.
20. $(x+3)(x-1)(x^2-2x+3)$.

EXERCISE 44. (Page 161.)

1. $\left(\frac{a}{3}+\frac{b}{2}\right)\left(\frac{a}{3}-\frac{b}{2}\right)\left(\frac{a^2}{9}+\frac{b^2}{4}\right)$.

2. $(1+a+b)(1-a-b)$. 3. $(a-b)(a^2+b^2+ab-1)$.

4. $(a+2b+c)(a-2b-c)$. 5. $\left(1-\frac{x}{y}\right)\left(1+\frac{x}{y}+\frac{x^2}{y^2}\right)$.

6. $(a^2+a+1)(a^2-a+1)(a^4-a^2+1)(a^8-a^4+1)$.

7. $(a-1)(a^2+a+1)(a+1)(a^2-a+1)(a^2+1)(a^4-a^2+1)$.

8. $(a+1)(a^2+6a+1)$. 9. $(x-a+b)(x-a-b)$.

10. $(a-3b-4)(a-3b+3)$. 11. $(2a+3)(8a+1)(34a^2+14a+5)$.

12. $(a^2+3a+12)(a^2+3a-7)$. 13. $4b^2$.

14. $4a^2b(a+b)$. 15. $4(6a-b)(a-6b)$.

16. $(a+b+c+d)(a+b+c-d)$. 17. $4abxy$.

18. $3a(a-7)(a+5)$. 19. $(ax+by)(x+y)$.

20. $(a-b)(a+b+c)$. 21. $(a^2+b^2)(x^2+y^2)$.

22. $(2a^2+3b^2)(3a+2b)$. 23. $(a+1)(a-1)(b+1)(b-1)$.

24. $(a^2c-1)(b^2c-1)$. 25. $3(1+2a-2b)(1-2a+2b)$.

26. $(6x-6y-1)^2$. 27. $\left(x+\frac{2}{x}\right)\left(x^2-2\right)$.

28. $(x+4)(x^2-4x+16)$. 29. $(x^2+4x+8)(x^2-4x+8)$.

30. $(x+y)(x-y)(x^2+y^2)(x^4+y^4)$.

31. $(x-y)(x^2+xy+y^2)(x^6+x^3y^3+y^6)$.

32. $(a^m-b^n)(a^{2m}+a^mb^n+b^{2n})$. 33. $3ab(3a+4b)(3a-4b)$.

34. $\left(\frac{x}{3}-\frac{y}{4}\right)\left(\frac{x^2}{9}+\frac{xy}{12}+\frac{y^2}{16}\right)$.

35. $(x-4y)(9x-5y.)$ 36. $(x+5y)(9x-4y)$.

37. $(1+9a-9b)(1-10a+10b)$. 38. $(1+a)(1+b)(1+c)$.

39. $(x-\cdot1)(x^2+\cdot1x+\cdot01)$.

40. $3(2x^2+6xy+9y^2)(2x^2-6xy+9y^2)$.

41. $-(3x-7)(7x+3)$. 42. $(2x-3)(4x+7)$.

43. $2(2x-1)(x-3)$. 44. $2x(3y-x)$. 45. $(3x^2+y^2)(x^2+3y^2)$.

46. $(3p^2-2pq+3q^2)(3p^2+2pq+3q^2)$.

47. $(ax+ay-x+y)(bx+by-x+y)$.

48. $(a+b)(a-b)\left(1+\frac{1}{ab}\right)\left(1-\frac{1}{ab}\right)$.

49. $(x+a)(ax+1)$. 50. $-(3a+13b)(6a+5b)$.

51. $(a-1)(a-4)(a^2-5a-7)$.

52. $(2x^2-19xy+8y^2)(2x^2+3xy-3y^2)$.

53 $(x-3y)(x^2-3xy+9y^2)$.

54. $\left(\frac{a}{2}+b+\frac{c}{3}\right)\left(\frac{a^2}{4}+b^2+\frac{c^2}{9}-\frac{ab}{2}-\frac{ac}{6}-\frac{bc}{3}\right)$.

55. $(x+py)\left(x-\frac{1}{p}y\right)$.

56. $(2x^2+14ax+49a^2)(2x^2-14ax+49a^2)$.
57. $(x^2+4x-3)(x^2-4x-3)$. 58. $2(b+c)(a+b)$.
59. $2a(a+b+c)$. 60. $2(b-d)(a+b+c+d)$.
61. $(a+b+c+d)(2a^2+2b^2+2c^2+2d^2+ab+bc+cd+da-2ac-2bd)$.
62. $(a+b+c-d)(a+b-c+d)(c+d+a-b)(c+d-a+b)$.
63. $(3s-a-b-c)(a^2+b^2+c^2-ab-ac-bc)$.
64. $(x+8)(x-1)(x^2+7x+30)$. 65. $(a^2+6a-18)(a^2+4a-18)$.
66. $(9x^2+33x+19)^2$.
67. $(a-c)(a+b+c)(a^2+c^2-2ac+ab+bc)$.
68. $xy(xy-6)(11xy+5)$. 69. $(ax+bx+1)(ax-bx-1)$.
70. $(ac+bd+ad-bc)(ac+bd-ad+bc)$.
71. $(ac+ad+bc-bd)(ac+ad-bc+bd)$.
72. $(x-a)(x+a)(x^2+a^2)(x^4+a^4-1)$.
73. $(x^2+y^2)(x^4-x^2y^2+y^4-1)$. 74. $(y-x)(y^2+xy+x^2-x-y)$.
75. $(1+x)(1-x^2+x^3)$. 76. $(x^2-yz)(z-y)$.
77. $(a+b)(a+b-2)$.
78. $(x-y)(2x^3+2y^3+2x^2y+2xy^2+x^2+xy+y^2)$.
79. $(5p-r)(5p+r-2q)$. 80. $(x+2)(2x^2+9x+1)$.
81. $(x+1)^2(x+4)$. 82. $(x-y)(x+y)^3$.
83. $(a^2+b^2)(px-qy+rz)$. 84. $x-7$, $x+3$. 85. x^2-4x+5.

EXERCISE 45 (Page 164.)

1. $-48x(x-5)$. 2. $12x(x^2+8)$.
3. $(y-x)(x+y+z)(x^2+y^2-2xy+yz+zx)$.
4. $x(11x-10y)(x^2+22y^2)$. 5. $1+a^4+a^8$.
6. $a^4+4a^3b+6a^2b^2+4ab^3+b^4+a^2+b^2+2ab+1$.
7. $(a^4-b^4)^2$. 8. $a^8+a^4b^4+b^8$. 9. $1-a^4(1+a)^4$.
10. $2a^2b^2+2a^2c^2+2b^2c^2-a^4-b^4-c^4$.
11. $x^4+y^4+z^4-2x^2y^2-2x^2z^2-2y^2z^2$.
12. $2a^2b^2+2a^2c^2+2a^2d^2+2b^2c^2+2b^2d^2+2c^2d^2+8abcd-a^4-b^4-c^4-d^4$. 13. $a^6-14a^4b^2+49a^2b^4-36b^6$
14. $4a^8-9a^6-4a^2+9$. 15. $a^8+16a^4b^4+256b^8$.
16. x^8-162x^4+6561. 17. x^8-32x^4+256.
18. $a+2b-c$. 19. $a^4-a^2b^2+b^4$.
20. a^4-b^4. 21. $a-1$. 22. $a-2$. 23. $a-2b$.

24. $a+b$. 25. $a^3+2a^2b+2ab^2+b^3$.
26. $a^2+b^2+c^2-2ab-ac+bc$.
27. $a^2+b^2+4c^2+2ab+2ac+2bc$. 28. $a^2+3a+13$.
29. $a^2-6ab+12b^2$. 30. a^2-2a-3.
31. $9a^2+b^2+4c^2+3ab+6ac-2bc$.
32. $25x^2+36y^2+z^2+30xy-5xz+6yz$. 33. $(a^2+ab+b^2)^2$.
34. $2x-3y+4z$. 35. $(a-b)^2+1$. 36. $6x^2-7x-3$.
37. $13x^2-23xy+13y^2$. 42. $\cdot 605$. 43. $\cdot 24$. 44. 266.
45. $113\frac{1}{9}$. 46. 12739. 47. 1569. 48. 1. 49. 10.
50. $\cdot 24$. 51. $3\cdot 030625$. 52. 132. 53. 0.
54. 330300. 55. 0. 56. 0. 57. $\cdot 000124$.
58. $\dfrac{x(x^2+3y^2)}{4}$. 59. $4(6a^2b^2-a^4-b^4)$. 60. 288.
61. 0. 62. $a-b$. 63. $a-3b$. 64. $a+2b$.
65. $a+2$. 66. $a-3$. 67. $a-1$. 68. $a+3$.
69. $3x^2y^3$ 70. $16a^3x^3(x+a)$. 71. $2ab$. 72. ab.
73. $a+2$. 74. $a-b-c$. 75. $x+4$. 76. $a-4b$.
77. $3a-4$. 78. $x-y-z$. 79. $a^2(7a+9)$. 80. $a-2$.
81. $a-2$. 82. $a+3$. 83. a^2+1. 84. $a-2$.
85. $a-3$. 86. $2x-1$. 87. $x-y-1$.
88. $a(a-1)^2(a+1)$. 89. $(a^3+8)(a-2)$.
90. $(a-1)^3(a^2+a+1)$. 91. $(a+1)(a+2)(a+3)$.
92. $(a-3b)(a-2b)(a+b)$. 93. $(x+a)(x+4a)(x+7a)(x+13a)$.
94. $(x+1)(x+2)(x+3)$. 95. a^6-1. 96. a^8-1.
97. $a^{12}-1$. 98. $(a^2-4b^2)^2$. 99. $36a^3b^2(a-b)^5$.
100. $72(a-b)^2(a+b)^3(a^2+ab+b^2)$.
101. $60a^3b^4(a-b)^3(a+b)(a^2+ab+b^2)$.
102. $36(a^2-b^2)^3(a^2+b^2)$. 103. $a^3b(a-b)^2(a^6-b^6)$.
104. $(x-1)^3(x+1)(x^2+1)$. 105. $12a^2(a^2-9b^2)^2$.
106. $64a^6-729b^6$. 107. $(2a-1)(a+2)(3a+1)$.
108. $(a+7)(2a-3)(3a+4)(a^3-2a^2-5)$. 109. x^6-a^6.
110. $(x^2-a^2)(x^2-b^2)(x^2-c^2)$.
111. $(1-4a^2)(1+4a+4a^2-16a^4)$. 112. $\dfrac{x+2}{x+5}$. 113. $\dfrac{4(a+b)}{5(a-b)}$.
114. $\dfrac{x+1}{x-5}$. 115. $\dfrac{x-3}{x+4}$. 116. $\dfrac{1-4x}{1-5x}$. 117. $\dfrac{1-2x^2}{1+3x^2}$.
118. $\dfrac{3x-5}{7x-1}$. 119. $3bc$. 120. $\dfrac{x-a}{x^2+a}$. 121. $\dfrac{2x-y}{x^2-1}$.
122. $\dfrac{a-b-c}{a+b-c}$. 123. $\dfrac{(x-1)^2}{x^2+1}$. 124. $\dfrac{(2a+1)(a+1)}{2a^2}$.

125. $\dfrac{x+a-b-c}{x+b-a-c}$. **126.** $\dfrac{1}{x^2+x+1}$. **127.** $\dfrac{x^2+x+1}{(x+1)(x^2+1)}$.

128. $\dfrac{x^2+1}{x^4+x^2+1}$. **129.** $\dfrac{x^4+1}{x^8+x^4+1}$. **130.** $\dfrac{3x^2+1}{4x(x^2+1)}$.

131. $\dfrac{x+5}{x+7}$. **132.** c. **133.** $\dfrac{x}{y(a-b)}$. **134.** $\dfrac{x-1}{x-3}$.

135. $\dfrac{x+2}{x-5}$. **136.** $\dfrac{ax+by}{a^3x^3-b^3y^3}$. **137.** 1. **138.** 1.

139. 1. **140.** 1. **141.** $x+b$. **142** 1.

SECTIONAL REVISION II. (Page 172.)

Paper 1. **1.** (i) $25x^2 \pm 40xy+16y^2$, (ii) $10(a^2+b^2)$. **2.** 34.
4. (i) $3(a+1)(a-1)$, (ii) $(1+2a+b)(1-2a-b)$,
(iii) $4(2x-3)(3x+4)$, (iv) $(x-5)(2x-3)(x-7)$.
5. $(2a-1)(a-3)(3a-2)$. 6. 1.

Paper 2. **1.** (i) (a) $49y^2$, (b) $132xy$; (ii) (a) 254,016, (b) 245,025.
2. 66. 3. (i) $3n-1$, (ii) $x^4+x^3y+x^2y^2+xy^3+y^4$.
4. (i) $9(5-19x-x^2)$. 5. (i) $x(x+7)(x-6)$, (ii) $4(2a-b)(3b-2a)$.
6. $(5a-3)(7a+2)(8a-1)$.

Paper 3. **1.** $p^2-4pq+4q^2-9r^2$ and 3255. **2.** 322.
3. $d^2=3c^2-3b^2+a^2$. 4. $(2x-3)(3x-2)$.
5. (i) $\dfrac{3(a-3)}{a-5}$, (ii) $(a-8)(a-9)^2$.
6. $(a-5b)(a-3b)(a+2b)(a+7b)$.

Paper 4. **1.** (i) $(p+2)(p-3)(p+4)=p^3+(2-3+4)p^2+(-6+8-12)p-24$, (ii) $x^3+4x^2-17x-60$.
2. 52. 4. (i) 6, (ii) 18, (iii) 23.
5. (i) $(2x^2+1)(2x^2-1)(2x^2+2x+1)(2x^2-2x+1)$,
(ii) $a^2b^2(3a-2b)(9a^2+6ab+4b^2)$,
(iii) $(7a+3b)(a+9b)$, (iv) $(x-2)(2x-1)(2x^2-5x+5)$.
6. $\dfrac{9(x-2y)^2}{8(x-3y)^2}$.

Paper 5. **1.** $(5m-2n)^3=(5m)^3-3(5m)^2(2n)+3(5m)(2n)^2-(2n)^3$.
2. (i) 341, (ii) 387, (iii) 4. 3. (ii) $\dfrac{ku}{k-u}$.
4. (i) $2x(x-3)(x^2+3)$, (ii) $x^2(x+6)(x-5)$,
(iii) $(x-y)(x+y-z)$.
5. (i) $n=\dfrac{rI}{E-RI}$ (ii) -33. 6. (i) 55, (ii) 1,

Paper 6. 1. $27+64a^3$ and 133. 2. p^3+3p. 3. (i) $4(x+2y)^2$, (ii) $(x+a)(x-a)(x-b)$. 4. (i) $16a^4$, (ii) $2m^3-9m^2+16m-14$.
5. (i) $a+2b$, (ii) $5a-3$. 6. 1.

Paper 7. 1. (i) $9x^2+4y^2+16z^2-12xy+24xz-16yz$, (ii) $2p^2-4q^2-4r^2-6pq+10pr-2rq$. 2. 50. 3. (i) 0.
4. (i) $(9a^2+12ab+8b^2)(9a^2-12ab+8b^2)$,
(ii) $(5x-5y-3)(7x-7y-4)$,
(iii) $(2x+3y)(2x-3y-3)$, (iv) $(4x^2+16x+11)^2$.
5. $(3a-1)(6a-5)(2a+3)$. 6. 1.

Paper 8. 1. $8m^3-n^3-27-18mn$. 2. 246,684,600,000.
3. (i) $18x(28x^2-10x+1)$, (ii) $(2x+1)(x+3)$.
(iii) $(2a+3b)(2a+3b-4)$, (iv) $(a+1)(a+2)(2a-1)$.
5. a^2-ab+b^2. 6. 10,000.

Paper 9. 1. (i) ·414, (ii) $x^7-x^6a+x^5a^2-x^4a^3+x^3a^4-x^2a^5+xa^6-a^7$
3. $6m^2-7m-3$. 4. $x-5$.
5. (i) $(a^2+2a+2)(a^2-2a+2)$, (ii) $(a+1)^2(a-1)$,
(iii) $(a^2-a+1)(a^2+a+1)(a^4-a^2+1)$. 6. 1.

Paper 10. 1. ·42. 2. (i) $2a^2b(a+b)$, (ii) $(x-2)(x-3)(x-5)$.
3. $x-a$. 4. $\dfrac{x^2-x+1}{x(x+9)}$. 5. $x^{12}-y^{12}$.

EXERCISE 46. (Page 180.)

1. a^2+3a+4.	2. $a+3$.	3. $a-4$.
4. $3a-1$.	5. $2a^2+3a+7$.	6. $3a+4$.
7. $2a^2(4a^2-12a+9)$.	8. $2a^2+a-1$.	9. $a^2-2ab+4b^2$.
10. a^2-a-1.	11. a^2-2.	12. a^2+3a+5.
13. a^2-3a+1.	14. a^2+5a+1.	15. a^2-3a+5.
16. $x-2$.	17. $x-3$.	18. $(x-1)(x-2)$.
19. $(x+y)(x+4y)$.	20. $3x-2y$.	21. $2x^2-x-3$.
22. x^2-3x+4.	23. x^2-x-2.	24. x^2+2x+3.
25. x^2-2x+1.	26. x^2+2x+3.	27. $3x^2-x+1$.
28. $\dfrac{x-2}{x^2-x-3}$.	29. $\dfrac{x-1}{x+2}$.	30. $\dfrac{x^2+x-12}{x^2+x+2}$.
31. $\dfrac{2x-3}{6x+5}$.	32. $\dfrac{x-1}{x+1}$.	33. $\dfrac{x^2+x-6}{x^2-1}$.
34. $\dfrac{3x^2+x+2}{2x^2+x+3}$.	35. $\dfrac{x^2+x-12}{x^2-x-12}$.	

EXERCISE 47. (Page 184.)

1. H.C.F. $= x^2-2x+3$ and L.C.M. $=(x-3)(x^3+x^2-3x+9)$.
2. H.C.F. $= x^2+3x+2$ and L.C.M. $=(x-5)(x^3-x^2-10x-8)$.
3. H.C.F. $= 2x^2-9$ and L.C.M. $=(2x-5)(6x^3+8x^2-27x-36)$.
4. H.C.F. $= 2x^2-3x+1$ and L.C.M. $=(x^2+2x-2)(2x^4-7x^3+11x^2-8x+2)$.
5. H.C.F. $= x^2+x-6$ and L.C.M. $=(x^2-3)(x^5+6x^2-49x+42)$.
6. H.C.F. $= 4x^2-3x+2$ and L.C.M. $=(2x^2-3)(16x^4-12x^3+20x^2-9x+6)$.
7. H.C.F. $= x^2-x+3$ and L.C.M. $=(x^2-x+1)(2x^4-x^3+6x^2+2x+3)$.
8. H.C.F. $= 3x^2-5ax-a^2$ and L.C.M. $=(5x-2a)(6x^4-25ax^3+26a^2x^2-a^4)$.
9. $15x^3-x^2-5x-1$.
10. $4x^4-x^3+4x-1$.
11. $1-x^4-x^6+x^7$.
12. $x^4-2x^3+10x^2-12x+24$.
13. $x^6+x^5+3x^4+11x^3+4x^2+12x+16$.
14. $x^7+x^6-x^5+2x^4-x^3-x^2+x-2$.
15. $(x^2+x-3)^2(x^2-x+3)(x^3-x+3)$.

EXERCISE 48. (Page 186.)

1. $\frac{4(x-1)}{4(x^2-1)}$, $\frac{3(x-1)}{4(x^2-1)}$, $\frac{4x}{4(x^2-1)}$.
2. $\frac{3(x-2)}{(x-3)(x-2)(x+1)}$, $\frac{4(x+1)}{(x-3)(x-2)(x+1)}$.
3. $\frac{a(a+b)(a^2+b^2)}{a^4-b^4}$, $\frac{b(a-b)(a^2+b^2)}{a^4-b^4}$, $\frac{ab(a^2+b^2)}{a^4-b^4}$, $\frac{b^2(a^2-b^2)}{a^4-b^4}$.
4. $\frac{x^2+xa+a^2}{x^4+x^2a^2+a^4}$, $\frac{x^2-xa+a^2}{x^4+x^2a^2+a^4}$, $\frac{a^2}{x^4+x^2a^2+a^4}$.
5. $\frac{a(a^2+ab+b^2)}{a^3-b^3}$, $\frac{a^2-b^2}{a^3-b^3}$, $\frac{ab}{a^3-b^3}$.
6. $\frac{x-c}{(x-a)(x-b)(x-c)}$, $\frac{x-b}{(x-a)(x-b)(x-c)}$, $\frac{x-a}{(x-a)(x-b)(x-c)}$.
7. $\frac{c-b-a}{(a+c-b)(b+c-a)(a+b-c)}$, $\frac{a-b-c}{(a+c-b)(b+c-a)(a+b-c)}$, $\frac{b-c-a}{(a+c-b)(b+c-a)(a+b-c)}$.
8. $\frac{a^2(a-b+c)}{ab(a-b-c)(a-b+c)}$, $\frac{b^2(a-b-c)}{ab(a-b-c)(a-b+c)}$, $\frac{abc}{ab(a-b-c)(a-b+c)}$.

9. $\dfrac{2(a+b)}{a^2-b^2}, \quad \dfrac{-3(a+b)}{a^2-b^2}, \quad \dfrac{4(a-b)}{a^2-b^2}$. 10. $\dfrac{(a+b)^2}{a^2-b^2}, \quad \dfrac{(a-b)^2}{a^2-b^2}, \quad \dfrac{-2ab}{a^2-b^2}$.

11. $\dfrac{(4a-b)(1+4ab)}{1-16a^2b^2}, \quad \dfrac{(4a+b)(1-4ab)}{1-16a^2b^2}, \quad \dfrac{4b(8a^2-1)}{1-16a^2b^2}$.

12. $\dfrac{3(x-1)}{(x-1)(x-2)(x-3)}, \quad \dfrac{4(x-2)}{(x-1)(x-2)(x-3)}, \quad \dfrac{-5(x-3)}{(x-1)(x-2)(x-3)}$.

EXERCISE 49. (Page 188.)

1. $\dfrac{5x}{(x-2)(x+3)}$. 2. $\dfrac{9x+1}{(x-3)(x+4)}$. 3. $\dfrac{4ax}{x^2-a^2}$.

4. $\left(\dfrac{x+y}{x-y}\right)^2$. 5. $\dfrac{2x}{(x+1)(x+2)}$. 6. $\dfrac{1}{1-x}$. 7. $\dfrac{1}{ab}$.

8. $\dfrac{1}{x-y}$. 9. $\dfrac{6xy}{8x^3+27y^3}$. 10. $\dfrac{1}{(a-b)(b-c)}$. 11. $\dfrac{2ab}{a^2-b^2}$.

12. $\dfrac{2a(a+5b)}{(a+b)^2(a-b)^2}$. 13. $\dfrac{2(x^2-7x+13)}{(x-2)(x-3)(x-4)(x-5)}$.

14. $\dfrac{1}{(x+5)(2x+3)}$. 15. $\dfrac{4}{(x-1)(x-5)}$. 16. 0.

17. $\dfrac{3}{(x+1)(x+2)(x+3)}$. 18. $\dfrac{1}{(x-1)(x+2)(x+3)}$.

19. $\dfrac{4}{(x+a)(x+13a)}$. 20. $3x$. 21. 7. 22. 1.

23. $\dfrac{-ab(2a+b)}{(a+b)(a-b)}$. 24. $\dfrac{2a^2-4a-3}{(a+1)(a-2)}$. 25. $4x^2$.

26. $\dfrac{x^2+2x+3}{(x^2-1)(x^2+1)}$. 27. $\dfrac{6(a^2-2)}{(a^2-1)(a^2-4)}$.

28. $\dfrac{6ab^2}{(a^2-b^2)(4a^2-b^2)}$. 29. $\dfrac{8a(a^2-7b^2)}{(a^2-b^2)(a^2-9b^2)}$.

30. $\dfrac{8(a^4-3)}{(a^4-1)(a^4-9)}$. 31. $\dfrac{1}{1-a^2}$. 32. $\dfrac{1}{1-9x^2}$.

33. $\dfrac{a^4+b^4}{a^4-b^4}$. 34. $\dfrac{8}{1-a^8}$. 35. $\dfrac{6}{a^6-1}$.

36. $\dfrac{-3b^2}{a(a+b)(a-b)}$. 37. $\dfrac{4x^7}{x^8-a^8}$ 38. $\dfrac{1}{a^8-1}$. 39. $\dfrac{-1}{x+a}$

40. $\dfrac{2(a+b)}{a^2+ab+b^2}$. 41. $\dfrac{1}{a^2+ab+b^2}$. 42. 0. 43. $\dfrac{4}{(n-1)(n+3)}$.

44. $\dfrac{6}{(x+2)(x+3)(x+4)(x+5)}$ 45. $\dfrac{-48}{x(x-2)(x+2)(x+4)}$.

46. $\dfrac{16(x+4)}{(x+1)(x+3)(x+5)(x+7)}$. 47. $\dfrac{2x(a^2-b^2)}{(x^2-a^2)(x^2-b^2)}$.

48. $\dfrac{10b^2(a^4-6b^4)}{(a^4-4b^4)(a^4-9b^4)}$.

EXERCISE 50. (Page 194.)

1. x. 2. 1. 3. 1. 4. $\dfrac{x+1}{x-1}$. 5. $\dfrac{2xy}{x^2+y^2}$. 6. 1.

7. $\dfrac{x^2-y^2}{x^4+x^2y^2+y^4}$. 8. $\dfrac{x+1}{x+3}$. 9. $\dfrac{x+a+x^2-ax}{x-a}$.

10. $\dfrac{x^4-y^4}{2xy}$. 11. 1. 12. $\dfrac{x-z}{1+xz}$. 13. $\dfrac{y-z}{1+yz}$. 14. 1.

15. a^2-2a+1. 16. $-\dfrac{1}{a}$. 17. $\dfrac{b(2a+b)}{a^2(a^2+b^2)}$. 18. $\dfrac{xy}{x+y}$.

19. $\dfrac{x^2}{x^4-x^2+1}$. 20. x^2. 21. $\dfrac{6(1-x)}{4-x}$. 22. $\dfrac{a}{a-b}$. 23. a.

24. $\dfrac{ab^3+a^2-ab-b^2}{ab^3+a^2-ab-b^2-b^3-a+b}$. 25. $1+x+x^2$. 26. $\dfrac{x(1+x+x^3)}{1+x^2}$.

27. $\dfrac{x^3}{y^2}$. 28. $\dfrac{x(2x^2-3x+6)}{6}$. 29. $-x$. 30. $\dfrac{2x^4y^5}{x^8+x^4y^4+y^8}$.

31. $1+\dfrac{4}{x+3}$. 32. $1+\dfrac{10}{x-3}$. 33. $2+\dfrac{7}{x-1}$.

34. $3-\dfrac{10}{x-2}$. 35. $3+\dfrac{4}{x-3}$. 36. $2-\dfrac{9}{2x-1}$.

37. $2-\dfrac{11}{2x+1}$. 38. $2-\dfrac{5}{3x+2}$. 39. $2-\dfrac{3}{3x-2}$.

40. $x^3-x^2a+xa^2-a^3+\dfrac{a^4-1}{x+a}$. 41. $2x+\dfrac{2x+1}{2x^2+2x+3}$.

51. 1·111.

EXERCISE 51. (Page 201.)

1. $x=\frac{6}{5}$, $y=-\frac{2}{3}$. 2. $x=1$, $y=-2$. 3. $x=3$, $y=2$.
4. $x=4$, $y=3$ 5. $x=y=1$. 6. $x=5$, $y=1$.
7. $x=7$, $y=4$. 8. $x=4$, $y=1$. 9. $x=\dfrac{b^2+ac}{a^2+b}$, $y=\dfrac{ab-c}{a^2+b}$.
10. $x=8$, $y=5$. 11. $x=1$, $y=3$.
12. $x=1$, $y=2$. 13. $x=5$, $y=6$. 14. $x=y=-1$.
15. $x=1$, $y=-\frac{1}{2}$. 16. $x=6$, $y=-\frac{2}{3}$. 17. $x=5$, $y=3$.
18. $x=3$, $y=2$. 19. $x=6$, $y=4$.

20. $x=\dfrac{c(c-b)}{a(a-b)}$, $y=\dfrac{c(a-c)}{b(a-b)}$.

21. $x=10, y=4$.
22. $x=19, y=7$.
23. $x=5, y=6$.
24. $x=4, y=2$.
25. $x=3, y=2$.
26. $x=14, y=15$.
27. $x=1, y=2$.
28. $x=5, y=3$.
29. $x=\cdot 02, y=2\cdot 9$.
30. $x=18, y=48$.
31. $x=4, y=2$.
32. $x=3, y=2$.
33. $x=3, y=2$.
34. $x=8, y=5$.
35. $x=3, y=2$.
36. $x=40, y=60$.
37. $x=\frac{4}{3}, y=\frac{5}{2}$.
38. $x=5, y=2$.
39. $x=2, y=5$.
40. $x=\dfrac{ac-bd-c^2}{a^2-ac-ab+b^2}, y=\dfrac{-ac+bc+ad}{a^2-ac-ab+b^2}$.
41. $x=1, y=1$.
42. $x=3, y=4$.
43. $x=4, y=10$.
44. $x=3, y=4$.
45. $x=\frac{1}{5}, y=\frac{1}{7}$.
46. $x=\frac{1}{2}, y=1$.
47. $x=\dfrac{mp-nq}{ap}, y=\dfrac{mp-nq}{aq}$.
48. $x=2, y=0$.
49. $x=5, y=3$.
50. $x=-3, y=4$.
51. $x=-1, y=2$.
52. $x=10, y=15$.
53. $x=4, y=6$.
54. $x=\dfrac{am-bn}{a^2-b^2}, y=\dfrac{an-bm}{a^2-b^2}$.
55. $x=7, y=3$.

EXERCISE 52. (Page 208.)

1. $x=6, y=11, z=6$.
2. $x=1, y=2, z=3$.
3. $x=-11, y=-8, z=-6$.
4. $x=2, y=3, z=1$.
5. $x=3, y=2, z=5$.
6. $x=2, y=3, z=4$.
7. $x=1, y=\frac{1}{2}, z=\frac{1}{3}$.
8. $x=\frac{1}{3}, y=\frac{1}{4}, z=\frac{1}{5}$.
9. $x=11, y=9, z=7$.
10. $x=a+c-b, y=a+b-c,$ $z=b+c-a$.
11. $x=\frac{2}{3}, y=-2, z=\frac{2}{5}$.
12. $x=\dfrac{2}{a+c-b}, y=\dfrac{2}{a+b-c}, z=\dfrac{2}{b+c-a}$.
13. $x=14, y=12, z=10$.
14. $x=4, y=3, z=2$.
15. $x=\dfrac{b^2+c^2-a^2}{2bc}, y=\dfrac{c^2+a^2-b^2}{2ca}, z=\dfrac{a^2+b^2-c^2}{2ab}$.
16. $x=y=z=5$.
17. $x=y=z=1$.
18. $x=2, y=4, z=6$.
19. $x=\dfrac{2}{a+c-b}, y=\dfrac{2}{a+b-c}, z=\dfrac{2}{b+c-a}$.
20. $x=abc, y=ab+bc+ca, z=a+b+c$.
21. $x=-(ab+bc+ca), y=a+b+c, z=1$.
22. $x=5, y=4, z=6$.
23. $x=4, y=5, z=6$.
24. $x=7, y=5, z=3$.
25. $x=7, y=4$.
26. $x=12, y=5$.
27. $x=15, y=8$.
28. $x=12, y=7$.
29. $x=8, y=7$.

30. $x=11$, $y=7$ 31. $x=\frac{a+\sqrt{a^2-4b^2}}{2}$, $y=\frac{a-\sqrt{a^2-4b^2}}{2}$.

32. $x=\frac{\sqrt{m^2+4n^2}+m}{2}$, $y=\frac{\sqrt{m^2+4n^2}-m}{2}$.

33. $x=3$, $y=4$, or $x=4$, $y=3$. 34. $x=2$, $y=5$.

EXERCISE 53. (Page 213.)

1. 23, 16. 2. 15, 13. 3. 21, 12. 4 18, 15.
5. ·32, ·24. 6. 12″, 15″ 7 15, 22, 40. 8. 18, 12.
9. 20, 12, 8. 10. Rs. 140, Rs. 180.
11. $x=160$, $y=180$, $z=210$. 12. Rs. 70, Rs. 50.
13. Rs. 3 as. 8, Rs. 5 as. 8. 14. Rs. 7, Rs. 4.
15. $\frac{7}{9}$. 16. $\frac{2}{3}$. 17. $\frac{8}{15}$. 18. $\frac{5}{11}$. 19. $\frac{6}{12}$.
20. Rs. 350, Rs. 460. 21. Rs. 22, Rs. 24.
22. Rs. 535, Rs. 315. 23. 79. 24. 15 shillings ; 18 pence.
25 Tea 3*s*. 6*d*. per lb. ; butter 2*s*. 6*d*. per lb. 26. 63.
27. 54. 28. 426. 29. 648. 30. 253.
31 40 miles. 32. 8½ hours. 33. 4 miles, 3½ miles per hour.
34. 3½ miles, 1½ miles. 35. 22 miles, 132 miles.
36. 3 miles, 1 mile an hour 37. 8 miles and 3 miles an hour.
38. 288 sq. ft. 39. $x=40°$, $y=35°$. 40. 90 lbs., 30 lbs.
41. 4½*s*., 7½*s*., 10*s*. 42. 2½ mds., 3½ mds. ; limit of concession 20 seers. 43. Rs. 55. 44. 48 ft., 36 ft.
45. 20 ft., 8 ft. 46. 80 ft., 48 ft. 47. 4″.
48. 36 ft., 25 ft., 11 ft.

SECTIONAL REVISION III. (Page 221.)

Paper 1 2. (i) $\frac{1}{a-b}$. (ii) $2(a+b)$. 3. $x=4$, $y=6$.
4. 6, 7. 5. 52.

Paper 2. 1. $3x-2$. 2. (i) $x^2-xa+a^2-\frac{a^3+1}{x+a}$.
(ii) $\frac{2}{x^2(x+2)(x-2)}$. 3. $x=3$, $y=5$. 4. *A* 32 years, *B* 20 years. 5. Rs. $6n+44$.

Paper 3. 1. x^2-2x+1. 2. $\frac{17x}{(x-3)(x-7)(x+4)}$. 3. $x=3$, $y=2$.
4. 87. 5. $(2x+3)(3x-8)$.

Paper 4. 1. $\frac{2x^2+7x+3}{x^2+3x+2}$. 2. $\frac{8x^2}{1-x^8}$. 3. $x=4$, $y=5$, $z=3$.

4. 15 miles per hour, 90 miles. 5. 27.

Paper 5. 2. $\dfrac{34xy}{49x^2-y^2}$. 3. $x=4, y=3, z=2$. 4. 25 miles.
5. (i) $(x^2-3x-6)(x^2-3x+4)$, (ii) $(9x^2+12x+8)(9x^2-12x+8)$.

Paper 6. 2. $\dfrac{1}{x-1}$. 3. $x=3, y=4$. 4. Rs. 480, Rs. 960.
5. 143.

Paper 7. 1. $3x^2+8xy+4y^2$. 2. $\dfrac{2}{x}$. 4. Tea 2*s*. 6*d*. per lb., coffee 1*s*. 4*d*. per lb. 5. -5.

Paper 8. 1. H.C.F. $=x^2+1$, L.C.M. $=x^3(x^2+1)(x^2-1)(x-2)$.
2. $\dfrac{(x+1)^2}{x+2}$. 3. $x=4, y=5$ 4. 2 yards. 5. $(a-3)(2a^2-a+1)$.

Paper 9. 1. $(x^2+8ax-2a^2)(3x-7a)(7x-4a)$. 2. 2.
3. $x=2, y=3, z=4$. 4. 48 persons; Rs. 2 8 as. each.
5. $+11$.

Paper 10. 1. a^3+4a^2-3. 2. $\dfrac{b(2a+b)}{a^2(a^2+b^2)}$. 3. $x=5, y=4$; or $x=4, y=5$.
4. $x=47, y=27$. 5. $(a+b)(a^2-ab+b^2)(p-q)(p-2q)$.

EXERCISE 54. (Page 227.)

1. 10, 26, 17, 25.
2. (i) 5, (ii) 10, (iii) 13, (iv) 17.
5. 5.
6. 42.

EXERCISE 56. (Page 239.)

5. (i) $y=\dfrac{x}{2}-\dfrac{5}{2}$. (ii) $\dfrac{x}{5}+\dfrac{y}{-\frac{5}{2}}=1$. (iii) $x-2y-5=0$.

6. (i) $y=-2x+8$. (ii) $\dfrac{x}{4}+\dfrac{y}{8}=1$. (iii) $2x+y-8=0$.

7. (i) $y=\frac{3}{4}x-\frac{5}{2}$. (ii) $\dfrac{x}{\frac{10}{3}}+\dfrac{y}{-\frac{5}{2}}=1$. (iii) $3x-4y-10=0$.

8. (i) $y=\frac{1}{8}x-\frac{3}{8}$. (ii) $\dfrac{x}{3}+\dfrac{y}{-\frac{3}{8}}=1$. (iii) $x-8y-3=0$.

9. (i) $y=\frac{4}{3}x+4$. (ii) $\dfrac{x}{-3}+\dfrac{y}{4}=1$. (iii) $4x-3y+12=0$.

10. (i) $y=-\frac{16}{25}x+\frac{4}{5}$. (ii) $\dfrac{x}{\frac{5}{4}}+\dfrac{y}{\frac{4}{5}}=1$. (iii) $16x+25y-20=0$.

11. $y=0x+3$.

12. $0y = -x + 5$. 13. $y = 0x + 0$. 14. $0y = x + 0$.
34. 34·225. 35. $x = 16$. 36. 11, −4·3, 6·5.
43. $3x - 2y - 2 = 0$. 44. $x - y + 8 = 0$. 45. $4x = 5y + 20$.
46. $4x + 3y = 0$. 47. $m = -\frac{1}{2}$, $c = \frac{13}{2}$. 48. $y + x = 0$.
49. No. 50. $3x - 2y = 1$, $2x - y = 2$, (3, 4).

EXERCISE 57. (Page 242.)

1. $x = 3, y = 4$. 2. $x = 5, y = 4$. 3. $x = 4, y = 5$.
4. $x = 4, y = 5$. 5. $x = 2, y = 3$. 6. $x = 3, y = 2$.
7. $x = -5, y = -2$. 8. $x = 2, y = -5$. 9. $x = 4, y = 3$.
10. $x = 4, y = -3$. 11. $x = -3, y = 4$. 12. $x = 3, y = 4$.
13. $x = 5, y = 6$. 14. $x = 4, y = 5$. 15. $x = 2, y = 3$.
16. (−5, −5), (−5, 9), (2, 2), length of sides $7\sqrt{2}$, $7\sqrt{2}$, 14.

EXERCISE 58. (Page 248.)

1. $y = \frac{3}{2}x$. 4. 72·6. 5. (i) 53·13 gallons. (iii) 10·6 cu. ft.
7. 11·7 ft. per second, 36·7 ft. per second, 16·4 miles per hour. 23·9 miles. 8. 102·7, 10-20 A.M. and 7-20 P.M.
9. 22nd June to 6th July. 10. 13, 14·4, 20·6. 11. 27″, 22″ (Approx.)
12. Rs. 26 as. 12, Rs. 33. 13. 38·6°. 14. 324 ft., 81 ft.

EXERCISE 59. (Page 256.)

6. (i) The latter is greater ; (ii) The latter is greater.
7. −57. 11. 23. 12. $\dfrac{-ab}{a+b}$. 13. $\dfrac{-ab(a+b)}{(a^2+ab+b^2)}$.
14. 28, 32. 15. 15, 20. 16. $49\frac{1}{2}$, $38\frac{1}{2}$. 17. 21, 35.
18. $\dfrac{bx}{a}$. 19. $\dfrac{7}{48}$. 20. $1\dfrac{9}{29}$. 21. $\dfrac{3}{4}$. 22. $\pm\dfrac{5}{4}$.
23. $\dfrac{6}{5}, -\dfrac{2}{11}$. 24. The former is greater. 25. The former is greater.
26. 1. 27. 3, 12. 28. 45, 36. 29. 9 : 10. 30. $\dfrac{pq}{p+q}$.
31. $\sqrt{2} : \sqrt{5}$. 32. 28000, 8400. 33. $-4\frac{1}{2}$. 34. $1\frac{1}{12}$.
35. $-1\frac{1}{2}$. 36. $-\dfrac{2}{7a}$. 37. $\dfrac{1}{10}$. 38. 2.

EXERCISE 60. (Page 263.)

1. 21. 2. 20. 3. 40. 4. $\frac{1}{6}$. 5. $4a^2b^2$. 6. 12.
7. 35. 8. 24. 9. 132. 10. 3. 11. xy.

12. $(a+b)^2(a-b)^2$. **13.** 8. **14.** $12\frac{4}{5}$. **15.** $2\frac{2}{5}$.

16. $\dfrac{(x+y)^2}{x-y}$. **17.** 3. **18.** 5. **19.** 6. **21.** 4, 36

22. 22. **24.** 12, 21, 24, 42, 36, 63, 48, 84.

25. Copper 60 lbs., zinc 120 lbs., lead 160 lbs., tin 200 lbs.

26. 13 : 35. **27.** 3 : 4.

EXERCISE 61. (Page 268.)

25. 2. **26.** 2.

EXERCISE 62. (Page 273.)

11. $\dfrac{x+y}{11}=\dfrac{x-y}{5}=\dfrac{15x-12y}{84}$. **12.** $\dfrac{x+y}{10}=\dfrac{x-y}{4}=\dfrac{3x+5y}{36}$.

EXERCISE 64. (Page 278.)

1. $\frac{1}{2}$. **2.** $\frac{25}{72}$. **3.** $4\frac{1}{5}$. **4.** $\frac{1}{30}$. **5.** $\dfrac{1}{ab}$

6. 3. **7.** 7. **8.** 6. **9.** 12. **10.** $\frac{5}{13}$.

11. $-7\frac{6}{11}$. **12.** $\dfrac{2ab}{a+b}$. **13.** $\dfrac{-2ab}{a^2+b^2}$. **14.** 14. **15.** $\dfrac{-1}{a+b}$.

16. 1. **17.** $-2\frac{1}{2}$. **18.** -6. **19.** $-1\frac{1}{2}$. **20.** 4.

21. 6. **22.** $-5\frac{1}{3}$. **23.** $\dfrac{a^2+b^2}{a+b}$. **24.** $\frac{1}{12}$. **25.** 7.

26. 7. **27.** 13. **28.** 0. **29.** $-1\frac{1}{2}$. **30.** $-1\frac{3}{5}$.

31. $2\frac{21}{50}$. **32.** $-3\frac{11}{38}$. **33.** $\dfrac{-2ab}{a+b}$. **34.** $\dfrac{2(a+b)}{a+2b}$.

35. $\dfrac{ab(a+b-2c)}{a^2-ac-bc+b^2}$. **36.** $\dfrac{ab}{c-a-b}$. **37.** $2\frac{3}{4}$. **38.** $3\frac{4}{7}$.

39. $-\frac{2}{13}$. **40.** $-\frac{49}{114}$. **41.** $\frac{3}{4}$. **42.** -1. **43.** $-\frac{50}{263}$.

44. $\dfrac{a^2-bc}{b+c-2a}$. **45.** $-2\frac{1}{2}$. **46.** $3a$. **47.** $\dfrac{-7}{26}$.

EXERCISE 65. (Page 283.)

1. 3. **2.** 2. **3.** 3. **4.** $\frac{1}{2}$. **5.** 1. **6.** $\dfrac{ac-bd}{ad-bc}$.

7. $-\frac{7}{12}$. **8.** $-\frac{5}{16}$. **9.** $-\frac{1}{20}$. **10.** $-1\frac{1}{3}$. **11.** 0.

12. $\dfrac{ac^2-b^3}{b^2+ab-c^2-a^2}$. 13. 3. 14. 5. 15. 6. 16. 7.
17. 8. 18. 7. 19. $-\frac{13}{18}$. 20. $-\frac{44}{105}$. 21. $\frac{2}{19}$.
22. $\dfrac{-b^3}{(a-b)^2}$. 23. 4. 24. 4. 25. $-\frac{3}{5}$. 26. $1\frac{4}{23}$.
27. $2\frac{1}{5}$. 28. $1\frac{1}{3}$. 29. $-1\frac{2}{5}$. 30. -4. 31. $\frac{7}{9}$.
32. $\frac{2}{5}$. 33. $-1\frac{2}{3}$. 34. $-\frac{10}{11}$. 35. $\frac{8}{17}$. 36. $-\frac{8}{7}$.
37. $-2\frac{3}{5}$. 38. $-1\frac{4}{7}$. 39. $1\frac{5}{7}$. 40. $\frac{1}{5}$. 41. $\dfrac{ab}{a+b}$.
42. $-\frac{3}{4}$. 43. -4. 44. $\frac{1}{2}$. 45. $8\frac{1}{2}$. 46. 8.
47. $2\frac{1}{2}$. 48. $-\frac{5}{4}$. 49. 4. 50. $4\frac{1}{2}$. 51. $-\frac{5}{6}$.
52. $-3\frac{1}{2}$. 53. 1.

EXERCISE 66. (Page 289.)

1. $\frac{8}{29}$. 2. $\frac{13}{22}$. 3. $\frac{13}{37}$. 4. $\frac{1}{5}$. 5. $-\frac{5}{74}$.
6. $\frac{9}{38}$. 7. $\dfrac{q^3-pr^2}{p^2-q^2-pq+r^2}$. 8. $\dfrac{1}{ab+3a-b-2}$.
9. $-\frac{23}{7}$. 10. $-4\frac{1}{3}$. 11. $-3\frac{5}{7}$. 12. $-4\frac{6}{7}$.
13. $-3\frac{10}{11}$. 14. -1. 15. $-2\frac{2}{5}$. 16. $-3\frac{3}{4}$.
17. $4\frac{4}{9}$. 18. $1\frac{7}{8}$. 19. $\frac{20}{21}$. 20. -4.
21. $\dfrac{2bc}{a(b+c)}$. 22. $\dfrac{b-c}{2a}$. 23. $\frac{3}{7}$. 24. -1.
25. $\frac{8}{3}$. 26. $-\frac{1}{4}$. 27. $2\frac{1}{2}$. 28. $1\frac{3}{5}$.
29. 1. 30. 4.

EXERCISE 67. (Page 295.)

1. $a+b$. 2. $a+b$. 3. $\dfrac{2ab}{a+b}$. 4. $\dfrac{a+b}{2}$.
5. $\dfrac{a^2+4ab+b^2}{2(a+b)}$. 6. $\dfrac{6a}{7}$. 7. $\dfrac{1}{ab}$. 8. 1. 9. ab.
10. $\dfrac{a+b+c}{3}$. 11. a or b. 12. $a+b+1$. 13. $ab+bc+ca$.
14. $\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}$. 15. $a(a+b+c)$. 16. $a^3+b^3+c^3$. 17. $\dfrac{a+b}{2}$.
18. $-\dfrac{a+b}{2}$. 19. $\dfrac{2}{a+b}$. 20. $\dfrac{b(a+c-b)}{a}$. 21. $\dfrac{2c-b-a}{2ab-ac-bc}$.
22. 1. 23. $\dfrac{a+b}{2}$. 24. $\dfrac{2ab}{b-a}$. 25. $\dfrac{br-cq}{cp-ar}$.
26. $-b$. 27. 25. 28. $-\dfrac{a+b}{2}$. 29. $\dfrac{a}{3}$.

EXERCISE 68. (Page 300.)

1. 1 : 3 : 5. 2. 3 : 4 : 2. 3. 3 : 4 : 5. 4. 2 : 8 : 5.
5. $(ab-c^2) : (cb-a^2) : (ac-b^2)$. 6. $x=7, y=10, z=8$.
7. $x=2, y=-3, z=1$. 8. $x=2, y=-4, z=6$.
9. $x=3, y=6, z=9$. 10. $x=8, y=12, z=20$.
11. $x=3, y=4, z=5$. 12. $x=3, y=2, z=1$.
13. $x=1, y=2, z=3$. 14. $x=b-c, y=c-a, z=a-b$.
15. $x=8, y=6, z=4$. 16. $x=3, y=9, z=15$.
17. $x=3, y=1\frac{1}{2}$. 18. $x=7, y=9$.
19. $x=1, y=-\frac{1}{2}$. 20. $x=3, y=2$. 21. $x=7, y=9$.
22. $x=\frac{2p^2-3}{3p+5}, y=\frac{10p+9}{3p+5}$. 23. $x=\frac{lp}{l^2+m^2}, y=\frac{mp}{l^2+m^2}$.
24. $x=\frac{1-b}{1+a}, y=\frac{b-1}{a+1}$. 25. $x=\frac{ab(ad-bc)}{a^2-b^2}, y=\frac{ab(ac-bd)}{a^2-b^2}$.
26. 5. 27. $x=-2, y=1$. 28. $x=\frac{1}{2}, y=\frac{1}{3}$.
29. $x=\frac{1}{2}, y=\frac{1}{5}$. 30. $x=-1, y=-\frac{1}{2}$. 31. $x=3, y=\frac{1}{2}$.
32. $x=\frac{1}{3}, y=\frac{1}{5}$. 33. $x=y=1$. 34. $x=\frac{1}{2}, y=1$.
35. $x=\frac{1}{15}, y=18$. 36. $x=\frac{a^2-b^2}{am-bn}, y=\frac{a^2-b^2}{an-bm}$.
37. $x=\frac{1}{a}, y=\frac{1}{b}$. 38. $x=\frac{62}{45}, y=-\frac{46}{45}$. 39. $x=-2, y=1$.
40. $x=\frac{1}{5}, y=\frac{1}{6}$. 41. $x=\frac{2}{7}, y=4$.

EXERCISE 69. (Page 304.)

1. $x=-3, y=3, z=1$. 2. $x=3, y=2, z=1$.
3. $x=-7, y=3, z=8$. 4. $x=4, y=5, z=6$.
5. $x=y=z=1$. 6. $x=1, y=\frac{1}{2}, z=\frac{1}{3}$.
7. $x=6\frac{1}{3}, y=3, z=-\frac{3}{20}$. 8. $x=10, y=1, z=9$.

SECTIONAL REVISION IV. (Page 305.)

Paper 1. 5. (i) $\frac{2c-b-a}{2ab-bc-ac}$. (ii) $2a, \frac{a}{2}$. 6. 17·49.

Paper 2. 1. $c : 8a$. 2. -2. 5. (i) $\frac{3}{5}$; (ii) 1. 6. $3y+2x=6$.

Paper 3. 1. 22. 2. $\frac{39}{38}$. 3. 2 : 1. 4. $-\frac{2}{ab}(a^2+ab+b^2)$. (i) $\frac{1}{3}$; (ii) $-\frac{23}{7}$. 6. $5x+4y=14$. 7. $x=3, y=2$.

Paper 4. 1. 4 : 1. 2. 20 : 27. 5. (i) $\frac{15}{8}$, (ii) $-\frac{47}{11}$.
7. $x=4$, $y=3$, $z=2$.

Paper 5. 2. 4. 5. (i) $\frac{4a}{5}$; (ii) 1. 6. $6x+11y=14$, $4x+3y=18$, $x+4y=11$.

Paper 6. 2. 72 and 27. 5. (i) 7 ; (ii) 7.

Paper 7. 1. $\frac{6}{5}$. 2. 0. 5. (i) -4 ; (ii) $1\frac{3}{4}$.

Paper 8. 1. 30,000, 18,000. 2. (i) 28; (ii) $22\frac{1}{2}$; (iii) $37\frac{1}{2}$.
5. (i) $-2\frac{3}{5}$, (ii) 8. 6. 20 miles. 7. (3, 2) $(-4, 5)$, $(1, -3)$,

EXERCISE 70. (Page 313.)

1. $\sqrt[5]{a^3}$. 2. $\frac{1}{\sqrt[5]{x^3}}$. 3. $\frac{6}{\sqrt[3]{x^2}\sqrt{a}}$. 4. $\frac{9}{\sqrt[4]{p^{11}}}$.

5. $\frac{\sqrt{a^3}}{\sqrt[3]{m^2}}$. 6. $4\sqrt[3]{x^2}$. 7. $\sqrt[5]{m^3x^2}$. 8. $\frac{\sqrt{a^5}}{\sqrt[2m]{x^7}}$

9. $\sqrt[2m]{x^{11}}$. 10. $x^{\frac{7}{5}}$. 11. $\frac{1}{m^{\frac{3}{2}}}$. 12. $m^{\frac{2}{3}}$. 13. $a^{\frac{4}{3}}$.

14. $x^{\frac{5}{3}}$ 15. $\frac{1}{m^{\frac{1}{2}}}$. 16. $\frac{1}{2}$. 17. 4. 18. 8. 19. 16. 20. $\frac{1}{8}$.

21. $\frac{1}{9}$. 22. 25. 23. $\frac{1}{27}$. 24. $\frac{1}{8}$. 25. $\frac{1}{49}$. 26. 9. 27. 81.

28. 36. 29. 8. 30. $\frac{1}{50}$. 31. 11. 32. 5. 33. $(25)^3$.

34. $\frac{x^2+1}{x(x-1)}$. 35. a^{m-2n}.

EXERCISE 71. (Page 316.)

1. $\frac{1}{x^4}$. 2. $\frac{y^{\frac{15}{32}}}{x^{\frac{9}{16}}}$. 3. xb^6. 4. $\frac{1}{x^8y^{\frac{5}{3}}}$. 5. m^6n^8

6. $\frac{b^3}{a^6}$. 7. $m^{\frac{21}{16}}$. 8. $\frac{y}{x^2}$ 9. $\frac{1}{a^{\frac{1}{4}}}$. 10. $\frac{b}{a^{\frac{1}{2}}}$.

11. $\frac{9a^2b^2}{16}$. 2. $\frac{9}{25a^2b^2}$. 13. 1. 14. $\frac{x^{\frac{3}{2}}z^{\frac{1}{2}}}{y^{\frac{1}{3}}}$. 15. [illegible]bc.

16. 1. 17. The latter is greater by 448.

EXERCISE 72. (Page 318.)

1. (i) $5x^{-2}+3x^{-1}+1-7x+4x^2$. (ii) $6x^{-3}+4x^{-2}+7x^{-1}+3+5x^3$.

(iii) $-x^{-\frac{1}{n}}+6x^{-\frac{1}{2n}}+4+3x^{\frac{1}{2n}}+5x^{\frac{1}{n}}+x$.

(iv) $6x^{-\frac{1}{3n}}-2+5x^{\frac{1}{2n}}+7x^{\frac{1}{n}}+x$.

2. (i) $x^{-2}y^2+2+x^2y^{-2}$; (ii) $x^{-\frac{2}{3}}y^{\frac{2}{3}}-1+x^{\frac{2}{3}}y^{-\frac{2}{3}}$;

(iii) $(7x^3+1)-5x^2y^{-1}-xy^{-2}+3y^{-3}$. 3. $a+a^{-1}$. 4. $a^{\frac{3}{2}}+b^{\frac{3}{2}}$.

5. $a^2+8ac^{\frac{1}{3}}+16c^{\frac{2}{3}}-9b$. 6. $a-2a^{\frac{1}{2}}+1$.

7. $x^{-2}+x^{-1}y^{-1}+y^{-2}$. 8. $a+b+c-3(abc)^{\frac{1}{3}}$.

9. $a^{\frac{5}{6}}-5a^{\frac{2}{3}}b^{\frac{1}{6}}+10a^{\frac{1}{2}}b^{\frac{1}{3}}-10a^{\frac{1}{3}}b^{\frac{1}{2}}+5a^{\frac{1}{6}}b^{\frac{2}{3}}-b^{\frac{5}{6}}$.

10. $x+x^{\frac{3}{4}}y^{-\frac{1}{4}}-x^{\frac{1}{4}}y^{-\frac{3}{4}}-y^{-1}$. 11. $a^{\frac{1}{3}}b^{-\frac{1}{3}}-a^{\frac{1}{3}}-b^{-\frac{1}{3}}+1$.

12. $x^{\frac{2}{3}}-x^{\frac{1}{3}}a^{\frac{1}{3}}+a^{\frac{2}{3}}$. 13. $x^{\frac{4}{3}}y^{-\frac{4}{3}}+x^{\frac{2}{3}}y^{-\frac{2}{3}}+x^{-\frac{2}{3}}y^{\frac{2}{3}}+x^{-\frac{4}{3}}y^{\frac{4}{3}}$.

14. $1-x^{-\frac{1}{4}}+x^{\frac{3}{4}}$. 15. $x^{\frac{4}{5}}-x^{\frac{3}{5}}a^{\frac{1}{5}}+x^{\frac{2}{5}}a^{\frac{3}{5}}-x^{\frac{1}{5}}a^{\frac{3}{5}}+a^{\frac{4}{5}}$.

16. $x^{-\frac{2}{3}}-x^{-\frac{1}{3}}y^{-\frac{1}{3}}+y^{-\frac{2}{3}}$. 17. $4a+b^{-1}+c^{\frac{2}{3}}-2a^{\frac{1}{2}}b^{-\frac{1}{2}}+2a^{\frac{1}{2}}c^{\frac{1}{3}}+b^{-\frac{1}{2}}c^{\frac{1}{3}}$.

18. $a^{\frac{2}{3}}+2a^{\frac{1}{3}}b^{\frac{1}{3}}+b^{\frac{2}{3}}$. 19. a^2+2+a^{-2}.

20. $a^2+2a^{\frac{3}{2}}-2a^{\frac{4}{3}}+a-2a^{\frac{5}{6}}+a^{\frac{2}{3}}$.

21. $a^{\frac{2}{3}}-4a^{\frac{5}{6}}+4a+2a^{\frac{7}{6}}-4a^{\frac{4}{3}}+a^{\frac{5}{3}}$. 22. $a^{\frac{2}{3}}+2a^{\frac{1}{3}}+3+2a^{-\frac{1}{3}}+a^{-\frac{2}{3}}$.

23. $a^{\frac{3}{2}}-2a^{\frac{5}{4}}b^{-\frac{1}{4}}+ab^{-\frac{1}{2}}+2a^{\frac{3}{4}}b^{-\frac{1}{2}}-2a^{\frac{1}{2}}b^{-\frac{3}{4}}+b^{-1}$.

25. $a^{3n}-b^{-3n}$. 26. $a^n+a^{\frac{n}{2}}b^{-\frac{n}{2}}+b^{-n}$.

27. a^3-b^3. 28. $1-x^2$. 29. $x^{2^{n+3}}$. 30. $x^{3^{n+3}}$.

31. $\frac{1}{3}$. 32. a^{m-1}. 33. $x^{2^n}-a^{2^n}$.

34. $a^{2^{n-1}}-b^{2^{n-1}}$. 35. The latter is greater by 65,280. 36. 3^{24}.

37. $\frac{2}{3}$. 38. 7. 39. $\frac{1}{2^{n-1}}$. 40. 1·6. 41. $\frac{1}{3}$. 42. $\frac{1}{9^{n+1}}$.

45. $\left(\frac{p}{q}\right)^{m+n}$ 46. $1+x^{-1}$. 47. $x^2+xy^{-1}+y^{-2}$.

48. $\frac{1}{(1-x)^n}$. 49. 1. 50. 8. 51. 1.

52. $-(b^{-1}-a^{-1})(a^{-1}-c^{-1})(c^{-1}-b^{-1})(a+b+c)$. 55. 1.

EXERCISE 73. (Page 325.)

1. 3. 2. 5. 3. 2. 4. $2\frac{1}{2}$. 5. 6. 6. -9.
7. 9. 8. 2. 9. 2. 10. $\frac{2}{3}$. 11. $\frac{4}{3}$. 12. 3.
13. $1\frac{1}{2}$. 14. $2\frac{2}{3}$. 15. $x=4$, $y=0$. 16. $x=3\frac{1}{3}$, $y=5\frac{1}{10}$.
17. $x=3$, $y=2$, $z=-2$. 18. $x=\frac{1}{2}$, $y=\frac{3}{2}$, $z=0$.
19. $x=1$, $y=2$, $z=3$. 20. $x=3$, $y=3$.
21. $x=-4$, $y=-2$. 22. $x=1$, $y=2$.

EXERCISE 75. (Page 329.)

1. $x^3+x^2-10x+8$. 2. $x^3+5x^2-2x-24$.
3. $x^3+9x^2+26x+24$. 4. $x^3-9x^2+26x-24$.
5. $x^3-x^2(a+b+c)+x(ab+bc+ca)-abc$.
6. $x^4-6x^3-5x^2+42x+40$. 7. $x^4+4x^3-19x^2-46x+120$.
8. $x^4-x^3(a+b+c+d)+x^2(ab+ac+ad+bc+bd+cd)$
$-x(abc+abd+bcd+cda)+abcd$.
9. $x^3+x^2(p+q)-x(p^2+q^2-2pq)-(p^2-q^2)(p-q)$.
10. $x^3+x^2(y+1)-x(y^2+1-2y)-(y^2-1)(y-1)$.
11. $x^5+5x^4a+10x^3a^2+10x^2a^3+5xa^4+a^5$.
12. $x^6+6x^5a+15x^4a^2+20x^3a^3+15x^2a^4+6xa^5+a^6$.
13. $x^4-4x^3a+6x^2a^2-4xa^3+a^4$.
14. $x^5-5x^4a+10x^3a^2-10x^2a^3+5xa^4-a^5$.
15. $x^5+5x^4+10x^3+10x^2+5x+1$.
16. $a^6-6a^5b+15a^4b^2-20a^3b^3+15a^2b^4-6ab^5+b^6$.
17. $p^7+7p^6q+21p^5q^2+35p^4q^3+35p^3q^4+21p^2q^5+7pq^6+q^7$.
18. $1-6x+15x^2-20x^3+15x^4-6x^5+x^6$.
19. $32+80m+80m^2+40m^3+10m^4+m^5$.
20. $81-108a+54a^2-12a^3+a^4$.
21. $32x^5-80x^4+80x^3-40x^2+10x-1$.
22. $1-18x+135x^2-540x^3+1215x^4-1458x^5+729x^6$.
23. $1-10x+45x^2-120x^3-210x^4-252x^5+210x^6-120x^7+45x^8$
$-10x^9+x^{10}$.
24. $64+576x+2160x^2+4320x^3+4860x^4+2916x^5+729x^6$.
25. $243a^5-810a^4b+1080a^3b^2-720a^2b^3+240ab^4-32b^5$.
26. $a^5+5a^3+10a+\dfrac{10}{a}+\dfrac{5}{a^3}+\dfrac{1}{a^5}$.
27. $a^4+2a^3b+\frac{3}{2}a^2b^2+\frac{1}{2}ab^3+\frac{1}{16}b^4$.
28. $x^7-7x^5+21x^3-35x+\dfrac{35}{x}-\dfrac{21}{x^3}+\dfrac{7}{x^5}-\dfrac{1}{x^7}$.
29. $8x^3+8x$. 30. $2x(x^4+10x^2+5)$.

31. $4xa(3x^4+10x^2a^2+3a^4)$. 32. $864p^3q+1536pq^3$.
33. 1114. 34. 11. 35. 249. 36. 126. 37. 8.
38. 1·0253. 39. 1·218. 40. 1·0075. 41. Rs. 1·472.

EXERCISE 77. (Page 335.)

1. $\pm(2a-3b)$. 2. $\pm(4x^2+5)$. 3. $\pm(3x^2+5y^2)$
4. $\pm(9a-b^2)$. 5. $\pm(a^3-1)$. 6. $\pm(1-6x)$.
7. $\pm(x-\frac{1}{2})$. 8. $\pm(x^2y+\frac{1}{2})$. 9. $\pm(m^2-\frac{1}{3})$.
10. $\pm\left(\frac{a}{2b}-\frac{2b}{a}\right)$. 11. $\pm\left(\frac{x^2}{y^2}-\frac{y^2}{2x^2}\right)$. 12. $\pm\left(x^2-\frac{y^2}{4}\right)$.
13. $\pm\left(\frac{1}{3}a-\frac{1}{4}b\right)$. 14. $\pm\left(\frac{1}{2x}+\frac{x}{3y}\right)$. 15. $\pm(a^2-\frac{1}{2}ab)$.
16. $\pm(2x^2+3)$. 17. $\pm(3a^2+8)$. 18. $\pm\frac{x}{y}$.
19. $\pm(5b-a)$. 20. $\pm(2a^2)$. 21. $\pm\frac{4ab}{a^2-b^2}$.
22. $\pm\{3(a+b)+x+y\}$. 23. $\pm(x+\frac{1}{x}+4)$. 24. $\pm(x+\frac{1}{x}+2)$.
25. $\pm(x^2+\frac{1}{x^2}+1)$. 26. $\pm\left(x-1-\frac{1}{x}\right)$. 27. $\pm\left(x-2-\frac{1}{x}\right)$.
28. $\pm\left(x^2-2+\frac{1}{x^2}\right)$. 29. $\pm\left(x+2+\frac{1}{x}\right)$. 30. $\pm\left(x^2-2+\frac{1}{x^2}\right)$
31. $\pm(x^2-2x+1)$. 32. $\pm(2x^2+3x-5)$. 33. $\pm(x^2+xy+y^2)$.
34. $\pm(3x^2-5xy+4y^2)$. 35. $\pm\left(\frac{x}{y}+1+\frac{y}{x}\right)$.
36. $\pm\left(x+\frac{y}{4}-z\right)$. 37. $\pm\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)$.
38. $\pm(ab+bc-ac)$. 39. $\pm(x+2y)(2x+y)(3x+y)$.
40. $\pm\frac{y+z-2x}{6}$. 41. $\pm\left(x+1+\frac{1}{x^2-1}\right)$.
42. $\pm(a^2+b^2)$. 43. $\pm(x-1)$. 46. $\pm(x^m-2x^n)$.
47. $\pm(2^{2m}+3^n)$. 48. $\pm(5x^m-2x^n)$.

EXERCISE 78. (Page 342.)

1. $4x-7y$. 2. x^2+3x+3. 3. $4x^2-3x+2$.
4. $x^2-ax+2a^2$. 5. $x^3-6x^2+12x-8$. 6. $ax^2+2x+3a$.
7. $\frac{x^2}{8}+\frac{x}{2}-1$. 8. $3x^2-3x-\frac{3}{2}$. 9. $\frac{2}{3}x^2+\frac{3}{4}x+\frac{1}{2}$.
10. $3x-4+\frac{1}{2x}$. 11. $x^2-x+\frac{1}{4}$. 12. $\frac{5}{2}-ab+3a^2b^2$.

13. $\frac{a}{2b}-1-\frac{2b}{a}$. 14. $\frac{x}{y}-\frac{1}{2}-\frac{y}{x}$. 15. $\frac{3x}{2y}-1+\frac{y}{2x}$.

16. $2x^2+3x+\frac{3}{x}$. 17. $a^3-1+\frac{1}{2a^3}$. 18. $x^{m+n}+x^{m-n}-\frac{1}{2}$.

19. $\frac{3}{2}x^{m+1}-2x^{m-1}+1$. 20. $\frac{x^2}{y^2}-2+\frac{y^2}{x^2}$.

21. $\sqrt{2}(x^2+xy+y^2)$. 22. $2x^{\frac{3}{4}}-3+4x^{-\frac{3}{4}}$. 23. $\frac{1}{2}+\frac{1}{3}x^{\frac{1}{2}}-\frac{1}{4}x^{\frac{3}{4}}$.

24. $xy^{-1}+1+x^{-1}y$. 25. $a^{\frac{2}{3}}+1+a^{-\frac{1}{3}}$. 26. $x-1$. 27. $2x-3y$.

28. $2x^2+2x+1,221$. 29. $x^2+x+3,113$. 30. $a+\frac{x^2}{2a}-\frac{x^4}{8a^3}+\frac{x^6}{16a^5}$.

31. $1+\frac{x}{2}-\frac{5}{8}x^2+\frac{5}{16}x^3$. 32. (i) $\frac{1}{4}-a$; (ii) $a-\frac{1}{4}$.

33. 2. 34. $p^2=4q$. 35. $a=12$, $b=9$.

EXERCISE 79. (Page 345.)

1. $x+4$. 2. $x-3$. 3. $2x-\frac{1}{x}$. 4. $3x-y$.

EXERCISE 80. (Page 347.)

1. $\sqrt{96}$. 2. $\sqrt[3]{500}$. 3. $\sqrt[4]{96}$. 4. $\sqrt[4]{405}$. 5. $\sqrt[5]{96}$.

6. $\sqrt[5]{972}$. 7. $\sqrt[n]{x^n y}$. 8. $\sqrt[p]{x^{2p}y}$. 9. $\sqrt[5]{x^{15}y^2}$. 10. $4\sqrt{3}$.

11. $3\sqrt{7}$. 12. $2\sqrt{11}$. 13. $5\sqrt[3]{2}$. 14. $3\sqrt[4]{7}$. 15. $2\sqrt[5]{\frac{5}{2}}$.

16. $x^2\sqrt[3]{y}$. 17. $x^3\sqrt[n]{a}$. 18. $x^4\sqrt[a]{y^2}$. 19. $-8\sqrt[3]{10}$.

20. $5x^2y\sqrt[3]{2xy}$.

EXERCISE 81. (Page 349.)

1. $7\sqrt{5}$. 2. 0. 3. $4\sqrt{3}$. 4. $9\sqrt[3]{4}$. 5. $3\sqrt[3]{5}$.

6. -34. 7. $3x(2\sqrt{5x}-15\sqrt{y})$. 8. $a(x-4a)\sqrt{x}$. 9. $4a\sqrt[3]{a}$.

10. $\sqrt[12]{46656}$, $\sqrt[12]{27}$ and $\sqrt[12]{16}$. 11. $\sqrt[6]{4}$, $\sqrt[6]{16}$, $\sqrt[6]{8}$.

12. $\sqrt[12]{x^{18}}$, $\sqrt[12]{x^8}$, $\sqrt[12]{x^3}$. 13. The first is greater.

14. The second is greater. 15. The first is greater.

16. The first is greater. 17. $\sqrt[3]{4}$, $\sqrt[3]{5}$, $\sqrt{2}$.

18. $\sqrt{3}$, $\sqrt[3]{5}$, $\sqrt[4]{6}$. 19. $3\sqrt{2}$, $2\sqrt{3}$, $\sqrt[3]{7}$. 20. $\sqrt[3]{6}$, $\sqrt[4]{8}$, $\sqrt[6]{12}$.

EXERCISE 82. (Page 351.)

1. $\sqrt[3]{63}$. 2. $\sqrt[4]{60}$. 3. $\sqrt[6]{144}$. 4. $\sqrt[5]{\frac{3}{2}}$. 5. $\sqrt[6]{\frac{4}{3}}$.

6. $\sqrt[4]{\frac{4}{3}}$. 7. $\sqrt[12]{944784}$. 8. $\sqrt[5]{486000}$. 9. $\sqrt[12]{882165816}$.

10. $\sqrt[12]{\frac{1280}{27}}$. 11. $\sqrt[30]{\frac{2187}{512}}$. 12. $\sqrt[8]{32}$. 13. $18\sqrt{6}$.

14. $16\sqrt{33}$. 15. $12\sqrt[3]{4}$. 16. $35\sqrt[3]{2}$. 17. $\frac{3}{10}$.

18. $\frac{5}{3}\sqrt[6]{\frac{9}{250}}$. 19. $\sqrt[6]{432}$. 20. $2\sqrt[12]{442368}$. 21. $\sqrt[6]{350}$.

22. $\frac{1}{3}$. 23. $\sqrt[3]{\frac{7}{6}}$ 24. $x^2y\sqrt[3]{xy^2}$. 25. $288\sqrt[12]{72}$.

EXERCISE 83. (Page 351.)

1. $\sqrt{10}+\sqrt{14}+\sqrt{15}+\sqrt{21}$. 2. $\sqrt{15}-\sqrt{10}+3\sqrt{2}-2\sqrt{3}$.
3. $2\sqrt{6}-29$. 4. $12\sqrt{35}+24\sqrt{21}-12\sqrt{15}-30$.
5. $54-22\sqrt{6}$. 6. $62-20\sqrt{6}$. 7. 63. 8. $2+2\sqrt{15}$.
9. $5+3\sqrt[3]{4}+2\sqrt[3]{9}+\sqrt[3]{18}+\sqrt[3]{12}$. 10. $3-\sqrt{2}$.
11. $9\sqrt{2}+2\sqrt{3}+\sqrt{6}-6$. 12. 8. 13. $69+12\sqrt{30}$.
14. $159-24\sqrt{42}$. 15. $43-24\sqrt{3}$. 16. $2x^2-1-2x\sqrt{x^2-1}$.
17. $a^2+a^2-b^2+2a^2\sqrt{a^2-b^2}$. 18. $2a^2-2\sqrt{a^4-4b^4}$.
19. $5-4x+12\sqrt{12x-1-32x^2}$. 20. $25a^2-7b^2+24\sqrt{a^4-b^4}$.
21. $1+6\sqrt{x}+12+8\sqrt{x^3}$.

EXERCISE 84. (Page 353.)

1. $\sqrt{2}$, $\sqrt{5}$, $\sqrt{6}$, $\sqrt{ab}$, $\sqrt[3]{a^2}$, $\sqrt{x-1}$, $\sqrt{a}\sqrt[3]{b}$. 2. $\sqrt{6}$.
3. $\sqrt[3]{16}\sqrt{3}$. 4. $\sqrt{6}\sqrt[3]{36}$. 5. $\sqrt{3}-\sqrt{2}$. 6. $\sqrt{5}+\sqrt{3}$.
7. $\sqrt{a}+\sqrt{b}$. 8. $a-\sqrt{b}$. 9. $\sqrt{5}\sqrt[3]{36}\sqrt[3]{7}$.
10. $\sqrt[3]{150}$. $\sqrt[3]{7}$. 11. $a\sqrt{x}+b\sqrt{y}$. 12. $4\sqrt{3}-2\sqrt{5}$.
13. $3\sqrt{2}+2\sqrt{3}$. 14. $\sqrt{a+x}+\sqrt{a-x}$. 15. $x-\sqrt{x^2-1}$.
16. $\sqrt{x^2+1}+\sqrt{x^2-1}$. 17. $\frac{\sqrt{15}}{7}$. 18. $\frac{7}{5}\sqrt[3]{2}$. 19. $\frac{\sqrt[3]{20}}{2}$.
20. $\frac{\sqrt[3]{135}}{9}$. 21. $\cdot 462$. 22. $1\cdot 060$. 23. $1\cdot 010$. 24. $\cdot 149$.
25. $\cdot 671$. 26. $\cdot 693$. 27. $5\cdot 828$. 28. $-3\cdot 732$.
29. $17\cdot 944$. 30. $9\cdot 898$. 31. $2\cdot 956$. 32. $5\cdot 585$.
33. $2\cdot 898$. 34. 12. 35. 1154. 36. $18+13\sqrt{2}$.
37. $\frac{1}{19}(12\sqrt{6}+3\sqrt{15}+16+2\sqrt{10})$. 38. $\frac{1}{b^2}(2a^2-b^2+2a\sqrt{a^2-b^2})$.
39. $x+\sqrt{x^2-1}$. 40. $x^2+\sqrt{x^4-1}$.
53. $\sqrt{2}(1-\sqrt{2}-\sqrt{3})$ 54. $\sqrt{2}(\sqrt{3}-\sqrt{2}+1)$.

4. $a=12, b=9$. 5. $7-2\sqrt{2}$. 6. $x=0, y=\frac{1}{3}, z=2$.

Paper 2. 1. (i) 300,200,000,000 *m.m.* per second,
(ii) 26,000,000,000,000 miles,
(iii) $30 \times 10{,}000 \times 2{\cdot}4 \times 100{,}000 = 72 \times 1{,}000{,}000{,}000$.
2. (i) 3^8 and 3^{16}; the latter is greater, (ii) $1, 1, -1, 1$.
4. $12x^5+40x^3+12x$. 5. $3\sqrt{3}-2$. 6. (i) 8. (ii) 2.

Paper 3. 1. (i) $\frac{1{\cdot}6}{(10)^{24}}$, (ii) $\frac{3}{(10)^8}$, (iii) $\frac{27}{(10)^6}$. 2. (i) $\frac{1}{6}$, (ii) $\frac{1}{5}$.
(iii) 1, (iv) $\frac{5}{6}$. 3. -48. 4. $\sqrt[4]{5}(\sqrt{3}+\sqrt{2})$. 6. 1.

Paper 4. 1. (i) $\frac{1}{8}$, (ii) $\frac{1}{9}$, (iii) 81, (iv) $\frac{81}{16}$.
2. (*a*) 0, (*b*) (i) 2·301, (ii) 1·431. 3. Rs. 1·796. 4. $\sqrt{2}$.
5. (i) 40, (ii) 0. 6. $a^2+b^2+c^2-abc=4$.

Paper 5. 1. $e-\sqrt{e^2-1}$. 2. (i) $\sqrt{2}$, (ii) $9+4\sqrt{5}$. 3. $x+3-\frac{1}{x}$.
4. x^2+x-2. 5. (i) 1, (ii) $\frac{81}{a}$. 6. $(a+b+c-4)^2=abc$.

Paper 6. 1. (i) 27, (ii) 243, (iii) 16, (iv) $\frac{1}{1000}$. 2. (i) $x+x^{\frac{1}{2}}y^{\frac{1}{2}}+y$,
(ii) $x^{\frac{3}{4}}+x^{\frac{1}{2}}y^{\frac{1}{4}}+x^{\frac{1}{4}}y^{\frac{1}{2}}+y^{\frac{3}{4}}$. 3. $\frac{1}{2}x+\frac{1}{3}y+3$. 4. (i) 3, (ii) 1.
5. (i) $\frac{4\sqrt{5}+3\sqrt{15}-10\sqrt{3}-6}{11}$, (ii) 0. 6. $x^m=nx$.

Paper 7. 1. (i) $a^{\frac{2}{3}}-a^{\frac{1}{3}}b^{\frac{1}{3}}+b^{\frac{2}{3}}$, (ii) $x-y$. 2. (i) $1+6x+15x^2+20x^3+15x^4+6x^5+x^6$, (ii) $1-6x+15x^2-20x^3+15x^4-6x^5+x^6$.
3. $x^2+\frac{x}{2}+\frac{1}{x}$. 4. (i) 2, (ii) $3\frac{3}{4}$. 5. (i) ·414, (ii) $\frac{1}{x^2nr}$. 6. $a+3b=b^3$.

Paper 8. 1. 1·4. 2. $x=-\frac{3}{2}, y=-2$. 3. (i) $x^5+5x^4a+10x^3a^2+10x^2a^3+5xa^4+a^5$,
(ii) $x^5-5x^4a+10x^3a^2-10x^2a^3+5xa^4-a^5$.
4. $2x^2-3x+5$. 5. $\frac{2n-1}{2n}$. 6. $a^2b^2(a^2+b^2+3)=1$.

EXERCISE 88. (Page 383.)

1. ± 4. 2. ± 2. 3. 10 or 0. 4. ± 2. 5. $\pm\sqrt{5}$.
6. $\pm\frac{3}{\sqrt{2}}$. 7. $\pm a$. 8. ± 2. 9. ± 1. 10. ± 3. 11. ± 2.
12. ± 2. 13. ± 9. 14. $\pm\frac{1}{\sqrt{11}}$. 15. $\pm\frac{1}{\sqrt{2}}$.
16. $\pm\sqrt{a^2-2a}$. 17. $4, -8$.

EXERCISE 89. (Page 384.)

1. $4, 6.$ 2. $-4, -6.$ 3. $5, -7.$ 4. $-8, 9.$ 5. $1, 3.$
6. $2, -5.$ 7. $a, -b.$ 8. $\frac{3}{2}, \frac{4}{3}.$ 9. $\frac{1}{6}, -\frac{2}{5}.$
10. $\frac{b}{a}, -\frac{c}{b}.$ 11. $-6, \frac{2}{9}.$ 12. $0, -5.$ 13. $-4.$
14. $\frac{7}{3}.$ 15. $-\frac{a}{3}.$ 16. $\frac{b}{a}.$ 17. $p+q.$
18. $p+q, p-q.$ 19. $7, 2.$ 20. $-6, -4.$ 21. $4, -3.$
22. $-\frac{1}{4}, -\frac{1}{3}.$ 23. $-9, -3.$ 24. $\frac{1}{5}, -1.$
25. $\frac{2}{3}, \frac{3}{2}.$ 26. $3, \frac{1}{3}.$ 27. $9, -4.$ 28. $10, -\frac{1}{16}.$
29. $\frac{3}{2}, \frac{1}{4}.$ 30. $\frac{10}{3}, -\frac{2}{5}.$ 31. $x^2-4x-21=0.$
32. $x^2-16=0.$ 33. $x^2+x(b-a)-ab=0.$
34. $x^2-3x-180=0.$ 35. $x^2-2mx+m^2-n^2=0.$

EXERCISE 90. (Page 387.)

1. $9, -5.$ 2. 11 or $-5.$ 3. $\frac{5}{3}, -6.$ 4. $\frac{3}{2}, \frac{2}{3}.$
5. $-\frac{1}{2}, \frac{9}{4}.$ 6. $\frac{4}{3}, -\frac{3}{5}.$ 7. $-\frac{3}{5}, 1.$ 8. $\frac{25}{2}, -\frac{1}{2}.$
9. $9, 8.$ 10. $\frac{b \pm \sqrt{b^2+4ac}}{2a}.$

EXERCISE 91. (Page 388.)

1. $2, -\frac{1}{3}.$ 2. $1, -\frac{1}{3}.$ 3. $2, \frac{1}{3}.$ 4. $\frac{3}{4}, \frac{2}{3}.$
5. $14, \frac{9}{4}.$ 6. $-\frac{3}{2}, \frac{2}{5}.$ 7. $2, \frac{20}{3}.$ 8. $-2, -\frac{3}{7}.$
9. $\frac{3}{5}, 4.$ 10. $\frac{2p}{3}, -\frac{4p}{5}.$ 11. $2, -p.$ 12. $\frac{p}{q}, -\frac{q}{p}.$

EXERCISE 92. (Page 390.)

1. 0 and $\frac{1}{3}.$ 2. $\frac{4}{3}$ or $\frac{3}{4}.$ 3. $\pm 13, \pm 14.$
4. $12, 14$ or $-12, -10.$ 5. $\pm 11, \pm 13.$ 6. $\pm 5, \pm 12.$
7. 17 ft., 15 ft. 8. (i) $AP=a(\pm\sqrt{3}-1)$, (ii) $AP=a(2\pm\sqrt{2})$.

(ii) $AP = a$ or $\frac{3a}{4}$, (iv) $AP = \frac{a(2 \pm \sqrt{2})}{4}$. 9. 24 ft., 10 ft.

10. 30 ft., 16 ft. 11. 50 feet, 18 feet. 12. 11 hours.
13. 15 miles per hour. 14. 4 yards. 15. 10 ft., 24 ft., 26 ft.
16. 6 hours 40 minutes. 17. 12 feet, 8 feet. 18. 6 feet and 9 feet.
19. 24 ft., 18 ft., 15 ft. 20. 1024 men.

EXERCISE 93. (Page 398.)

1. (i) 1·96, (ii) 5·29, (iii) 10·24, (iv) 1·73, (v) 2·70, (vi) 3·67. 5. (i) 2, -3, (ii) 1, 3; (iii) -3, $-\frac{1}{2}$. 6. (i) 3, 2; (ii) $\frac{1}{2}$, -3; (iii) $\frac{1}{2}$, $-\frac{3}{2}$. 7. $\frac{1}{2}$, -3. 8. $\frac{2}{3}$, -2 9. 4, $-\frac{2}{3}$.
10. 2, 144.

EXERCISE 94. (Page 403.)

1. -1. 2. -37. 3. $4a^3 - 2a^2 + 5a - 3$.
4. $-(5m^3 + m^2 + 3m + 4)$. 5. $-56a^4$.
11. $(x-1)(x^2+2x+3)$. 12. $(x-1)(x-2)(x-3)$.
13. $(x+1)(x^2+8x+15)$. 14. $(x-1)(3x^2+4x+5)$.
15. $(x+1)(4x^2-x+2)$. 16. $(x-1)(x+1)(2x^2+3x+4)$.
17. $(x-1)^2(x^2+4x+2)$. 18. $(x+1)^2(x^2-3x+5)$.
19. $(x-1)(x+1)(x+3)(x-2)$. 20. $(x-1)^2(2x^2+3x-4)$.
21. 3. 22. $a+b+c+d=0$. 23. $a=-2$, $b=5$.
24. -600. 26. $\frac{3}{2}$. 27. $(x-3)(x+2)(x+4)$.
28. $(x+2)(x-3)(x-4)$. 29. $(x+2)(3x-1)(2x-3)$.
30. $(x-5)(x+2)(x+3)$. 31. $(x-2)(x-5)(x+7)$.
32. $(x+2)(x^2-3x+6)$. 33. $(x+3)(x^2-3x+4)$.
34. $(x-4)(x-3)(x^2+5x+3)$.

EXERCISE 95. (Page 409.)

8. $4a^2 + 2ax + x^2$. 9. $a^2 - 4ab + 16b^2$.
10. $32a^5 - 16a^4x + 8a^3x^2 - 4a^2x^3 + 2ax^4 - x^5$.
11. $9m^2 + 3m + 1$. 12. $x^3 + 4x^2y + 16xy^2 + 64y^3$.
13. $1 - 2m + 4m^2 - 8m^3$. 14. $16a^4 - 8a^3b + 4a^2b^2 - 2ab^3 + b^4$.

EXERCISE 96. (Page 414.)

1. $a_0x^8 + a_1x^7 + a_2x^6 + a_3x^5 + a_4x^4 + a_5x^3 + a_6x^2 + a_7x + a_8$.
2. $a_0x^4 + a_1x^3y + a_2x^2y^2 + a_3xy^3 + a_4y^4 + a_5x^3 + a_6x^2y + a_7xy^2 + a_8y^3 + a_9x^2 + a_{10}xy + a_{11}y^2 + a_{12}x + a_{13}y + a_{14}$.
3. Fractional and irrational. 4. Fractional and rational.

5. Integral and irrational.
6. Fractional and rational.
7. Integral and rational.
8. Fractional and irrational.
9. (i) 4, (ii) 9, (iii) 12, (iv) 5, (v) $6x^2-8x+10$, (vi) $9k^2+12k+15$, (vii) $3m^2+2m+4$.
10. (i) 14, (ii) 6, (iii) $2(1-p^2)$.
11. $2(b+2ax)$.
12. 7.
13. $2x^4-5x^3-2x^2+11x-6$.
14. $x^5-5x^4+11x^3-8x^2-5x+12$.
15. $2x^5-10x^4+3x^3+16x^2-14x+3$.
16. $14x^7-32x^5-21x^4+14x^3+6x^2-12x-9$.
17. $15x^6+5x^5-6x^4-x^2-2x-1$.
18. $5x^7-13x^6+9x^5-6x^3+4x^2-3x-1$.
19. $12x^8+2x^7-18x^6-3x^5+22x^4+4x^3+3x^2-4$.
20. $\frac{2}{5}x^4-\frac{136}{75}x^3y+\frac{8}{5}x^2y^2+\frac{3}{10}xy^3-y^4$.
21. $\cdot 2x^4+\cdot 07x^3y+\cdot 04x^2y^2+\cdot 19xy^3-\cdot 1y^4$.
22. $2x^9+5x^7-2x^6-9x^5-x^4-5x^3-14x^2-3x+3$.
23. $6x^3-x^2(5+6a)+x(5a-6)+6a$, $-\frac{5}{6}$.
24. x^2-x.
25. $10x^2-3x-12$.
26. x^3+2x^2-3x+1.
27. $2x-1$.
28. x^3-x^2+x-1.
29. x^4-x^2+1.

EXERCISE 97. (Page 422.)

1. Homogeneous and symmetrical, absolute symmetry.
2. Homogeneous, but not symmetrical.
3. Neither homogeneous nor symmetrical.
4. Homogeneous, but not symmetrical.
5. Homogeneous, but not symmetrical.
6. Homogeneous, but not symmetrical.
7. Homogeneous and symmetrical, absolute symmetry.
8. Symmetrical, but not homogeneous, absolute symmetry.
9. Homogeneous, but not symmetrical.
10. Homogeneous and symmetrical, absolute symmetry.
11. Not homogeneous but symmetrical, absolute symmetry.
12. Homogeneous and symmetrical, cyclic symmetry.
13. ax^5, bx^4y, cx^3y^2.
14. ax^4, bx^3y, cx^2y^2, dx^2yz.
15. $a(x^4+y^4+z^4)+b(x^3y+x^3z+y^3z+y^3x+z^3x+z^3y)+c(x^2y^2+x^2z^2+y^2z^2)+d(x^2yz+y^2zx+z^2xy)$.
16. $2x^2+3xy+2y^2$.
17. $2(x^3+y^3)+3(x^2y+xy^2)$.
18. $(x+a)(x+b)+(x+a)(x+c)+(x+b)(x+c)$.
19. $(x-a)(b-a)(c-a)+(x-b)(c-b)(a-b)+(x-c)(a-c)(b-c)$.
20. $(x+a)(x+b)(x-c)+(x+b)(x+c)(x-a)-(x+c)(x+a)(x-b)$.
21. $\dfrac{x^2(a+b)}{ab}+\dfrac{x^2(b+c)}{bc}+\dfrac{x^2(c+a)}{ca}$.

22. $\frac{bc}{(a-b)(a-c)}+\frac{ca}{(b-c)(b-a)}+\frac{ab}{(c-a)(c-b)}$.

23. $\frac{1}{(a-b)(a-c)(x-a)}+\frac{1}{(b-c)(b-a)(x-b)}+\frac{1}{(c-a)(c-b)(x-c)}$.

24. $3x^2+2x(a+b+c)+(ab+ac+bc)$.

25. $2(a^2+b^2+c^2-ab-ac-bc)$.

26. $3x^2+2x(a+b+c)$. 27. $ab(a-b)+bc(b-c)+ca(c-a)$.

28. $\sum x^2+3\sum xy$. 29. $\sum x^2-\sum xy$.

30. $x^3y+y^3z+z^3x-x^2y^2-y^2z^2-z^2x^2$. 31. $3\sum a^2-2\sum ab$.

32. (i) $a=b=7$, and $c=d=-3$, (ii) $p=4$, and $q=-2$. (iii) $m=-n=2$, and $q=-p=7$.

EXERCISE 98. (Page 427.)

1. $a=2$, $b=8$ and $c=-3$. 2. $a=3$, $b=-9$, $c=0$, $d=15$.
3. $l=1$, $m=1$, $n=-11$, $k=-23$. 4. $a=2$, $b=0$, $c=12$, $d=0$, $e=2$.
5. $a_0=2$, $a_1=2$, $a_2=-6$, $a_3=-10$, $a_4=-14$. 6. 18.
7. $a=2$, $k=-8$. 8. $A=1$, $B=2$. 9. $A=2$, $B=-3$.
10. $A=1$, $B=8$, $C=3$. 11. $A=0$, $B=2$, $C=3$.
12. $A=1$, $B=-1$, $C=0$, $D=2$.
13. $3(x-1)^2+2(x+2)+1$. 14. $3(x+1)^3-2(x+1)+1$.
15. $(x+1)^3-4(x+1)^2+(x+1)-4$. 16. $A=1$, $B=1$, $C=0$.
17. $3\sum x^2y+6xyz$. 18. $24xyz$. 19. 0.
20. $3x^2-x+2$. 21. $3x^3-2x^2+4$.
22. $2x^3-3x+5$. 23. $a=1$, $b=-2$.
24. $p=\pm 30$, $q=19$ or 31. 25. a^2-a+2.
26. x^2-3x-2. 27. $2x^2+x-3$.
28. x^2-2x+3. 29. $p=-3$. 30. $a=7$, $b=1$.
31. $\frac{-1}{(x+1)}-\frac{1}{5(x-3)}+\frac{6}{5(x+2)}$. 32. $\frac{3}{2x+1}+\frac{7-6x}{4x^2-2x+1}$.
33. $\frac{1}{(x-2)}-\frac{x+4}{(x+1)^2}$. 34. $A=3$, $B=-1$, $C=2$.
35. $A=1$. $B=2$, $C=D=-1$.

EXERCISE 99. (Page 432.)

1. $(x-1)(x^2+2x+3)$. 2. $(x-1)(x-2)(x-3)$.
3. $(x-1)(x-3)(x+5)$. 4. $(x-3)(x+2)(x+4)$.
5. $(x+2)(x+3)(x+5)$. 6. $(x-1)(x+1)(x+5)$.
7. $(x-5)(x+2)(x+3)$. 8. $(x+2)(x^2-3x+6)$.
9. $(x-1)(x+1)(x^2-2x-2)$. 10. $(x-4)(x-3)(x^2+5x+3)$.
15. $(a-b)(b-c)(c-a)$. 16. $3(a-b)(b-c)(c-a)$.
17. $-(a+b)(b+c)(c+a)$. 18. $(a+b)(b+c)(c+a)$.

19. $24abc$.
20. $-(a-b)(b-c)(c-a)(a+b+c)$.
21. $(a-b)(b-c)(c-a)(a+b+c)$.
22. $(a+b+c)(ab+ac+bc)$.
23. $2(a-b)(b-c)(c-a)(a+b+c)$.
24. $5(x-y)(y-z)(z-x)(x^2+y^2+z^2-xy-yz-zx)$.
25. $(a-b)(b-c)(a-c)(a^2+b^2+c^2+ab+bc+ca)$.
26. $5xy(x+y)(x^2+xy+y^2)$.
27. $5(x+y)(y+z)(z+x)(x^2+y^2+z^2+xy+yz+zx)$.
28. $-(x+y)(y+z)(z+x)(x-y)(y-z)(z-x)$.
29. $12xyz(x+y+z)$.

EXERCISE 101. (Page 438.)

1. $(a-b)(b-c)(a-c)$.
2. $(a+b)(b+c)(c+a)$.
3. $(a-b)(b-c)(a-c)(a^2+b^2+c^2+ab+bc+ca)$.
4. $-(a-b)(b-c)(c-a)(ab+bc+ca)$.
5. $-(a+b)(b+c)(c+a)(a-b)(b-c)(c-a)$.
6. $(a-b)(b-c)(a-c)(a^3+b^3+c^3+a^2b+b^2a+a^2c+ac^2+b^2c+bc^2+abc)$
7. $(3x-5y)(6x-7y)$.
8. $(3a-5b)(9a+4b)$.
9. $(5x-3y)(11x+4y)$.
10. $x^2y^2(8x+7y)(15x-4y)$.
11. $(4x-5y)(11x-4y)$.
12. $(3x-7y)(7x-3y)$.
13. $(4x+y)(7x-20y)$.
14. $(a+x-2)(a+y+2)$.
15. $(x+1)(ax-x-a)$.
16. $(x+a+2)(x-a-1)$.
17. $(x-a)(x-b-1)$.
18. $(bx-x+1)(bx+x+b)$.
19. $(a-b-3)(a-b+2)$.
20. $(x+p+1)(x+p-1)$.
21. $(x+a)(x+b+c)$.
22. $(x-1)(x-3)(x+4)$.
23. $(x-4)^2(x+2)$.
24. $(x-3)(x^2+3x+6)$.
25. $(x+3)(x^2-3x+4)$.
26. $(x+2y)(x^2-5xy+10y^2)$.
27. $(2x-1)(4x^2+2x+3)$.
28. $(a-b)(2a^2+ab+b^2)$.
29. $(3x-1)(9x^2+3x+5)$.
30. $(x^2-3x-1)(x^2+2x-1)$.
31. $(x-1)^2(x^2+6x+1)$.
32. $(x^2-4x+1)(x^2-x+1)$.
33. $(x^2+x+1)(2x^2+x+2)$.
34. $(x^2-xy-y^2)(3x^2+2xy-3y^2)$.
35. $(2x^2-xy+2y^2)(x+2y)(2x+y)$.
36. $(x^2-xy-y^2)(6x^2-19xy-6y^2)$.
37. $(a-2b+3c)(2a+3b-5c)$.
38. $(x+2y-3z)(2x-y+z)$.
39. $(x+y+z)(x-2y+3z)$.
40. $(a-x+2y)(a+2x-y)$.
41. $(x+2a+b)(x-a+2b)$.
42. $(x+y+3z)(x-2y+z)$.
43. $(2x-y+z)(x+3y-2z)$.
44. $3(2a-b)(b-3c)(3c-2a)$.
45. $3(a-2b+1)(a+b-1)(b-2a)$.
46. $3(a-1)(a-2)(3-2a)$.
47. $6a(a+b+1)(b+1-a)$.
48. $3(2x+y)(x-y)(x+2y)$.
49. $3abc(a-b)(b-c)(c-a)$.

50. $-5(x-y)(y-z)(z-x)\{(x-y)(y-z)+(y-z)(z-x)+(z-x)(x-y)\}$.

51. $5(x-1)(x-2)(3-2x)(3x^2-9x+7)$.

52. $(a+b+c)(ab+bc+ca)$.

53. $-(a+b+c)(a^2+b^2+c^2-2ab-2ac-2bc)$.

54. $(a+b+c)(a^2+b^2+c^2)$.

55. $(a+b+c)(b+c-a)(c+a-b)(a+b-c)$.

56. $(a+b+c)(ab+bc+ca)$. 57. $(x^2-2x+4)(x^2-3x+4)$.

58. $(x^2+4x+10)(x^2+4x-2)$.

EXERCISE 102. (Page 445.)

1. 0. 2. 1. 3. 1. 4. $\frac{1}{abc}$. 5. 0. 6. $\frac{1}{abc}$.

7. 0. 8. 0. 9. 2. 10. $ab+bc+ca$.

11. $a^2+b^2+c^2+ab+bc+ca$. 12. $a+b+c$. 13. 0.

14. $\frac{1}{abc}$. 15. $a+b+c$. 16. 1. 17. $-\frac{(a-b)(b-c)(c-a)}{(a+b)(b+c)(c+a)}$.

18. 1. 19. 2. 20. -3. 21. x^2.

22. $\frac{1}{(x-a)(x-b)(x-c)}$. 23. $\frac{x^2}{(x-a)(x-b)(x-c)}$.

24. $\frac{1}{(x+a)(x+b)(x+c)}$. 25. $\frac{x^2}{(x+a)(x+b)(x+c)}$.

26. $\frac{x+1}{(x-a)(x-b)(x-c)}$. 27. $\frac{1+px+qx^2}{(x-a)(x-b)(x-c)}$.

28. $(a+b)(b+c)(c+a)$. 29. $ab+bc+ca$. 30. 3.

31. $\frac{3}{2}(a+b+c+d)$. 32. -1. 33. $x+y+z$.

EXERCISE 103. (Page 450.)

1. $\frac{1}{b}$. 2. $\frac{1}{m}$. 3. $\frac{a^4-10a^3b-6ab^3-b^4}{a^4+10a^3b+6ab^3-b^4}$.

4. 0. 5. $2a^2(2a^2-1)$. 6. 2. 7. 0.

EXERCISE 105. (Page 456.)

1. 2. 2. 3. 3. x^2+2x+3. 4. x^2-x-12, x^2-6x+8.

7. $(ac'-ca')^3=(cb'-bc')(ab'-ba')^2$. 8. 8. 9. $p=0$.

13. $a=4$, H.C.F. $=x+1$, $a=-14$, H.C.F. $=x-2$.

SECTIONAL REVISION VI. (Page 457.)

Paper 1. 1. $a_0x^3 + a_1x^2y + a_2xy^2 + a_3y^3 + a_4x^2 + a_5xy + a_6y^2 + a_7x + a_8y + a_9$. 2. (i) $15x^7 - x^5 + 13x^4 - 14x^3 - 3x^2 - 2x - 8$. (ii) $\frac{5}{4}$.
3. $p = q = 9$ and $a = b = -4$. 4. $A = -1, B = 2$. 5. 67.
6. $(7x + 11y)(8x - 21y)$. 7. $x^2 - x - 20 = 0$.

Paper 2. 1. (i) $-1, 1, -13$, (ii) $x + 1$.
2. $a_0(x^4 + y^4 + z^4) + a_1(x^3y + x^3z + y^3x + y^3z + z^3x + z^3y) + a_2(x^2yz + y^2xz + z^2yx)$.
3. $\dfrac{x^2(a-b)}{ab} + \dfrac{x^2(b-c)}{bc} + \dfrac{x^2(c-a)}{ca}$.
4. $a_0 = 1, a_1 = -1, a_2 = 0, a_3 = 5, a_4 = -3$.
5. $5(2x - y)(2y - z)(z - y - 2x)(4x^2 + 3y^2 + z^2 - 2xz - 3yz)$.
6. $\dfrac{x^2}{(x-a)(x-b)(x-c)}$. 7. $x^2 - x - 56 = 0$.

Paper 3. 3. $\dfrac{x^2 - y^2}{x(y-z)} + \dfrac{y^2 - z^2}{y(z-x)} + \dfrac{z^2 - x^2}{z(x-y)}$.
4. $\dfrac{15}{4(x-1)} + \dfrac{1}{20(x+3)} - \dfrac{14}{5(x-2)}$.
5. $3(2x - 3y + z)(x + 2y - z)(y - 3x)$. 6. $11, -5$.

Paper 4. 1. $-\{a^3(b-c) + b^3(c-a) + c^3(a-b)\}$.
2. $5x^2 + 3x - 1 + \dfrac{7}{x-1} - \dfrac{16}{x-2}$.
3. $\frac{1}{2}(x^2 + y^2 + z^2) + \frac{1}{3}(xy + yz + zx)$. 5. $(x-3)(x+4)(x-1)$.
6. $x = 3, y = 5, z = 1$. 7. $\frac{5}{2}, 3$.

Paper 5. 1. $A = -2$. $B = 3, p = 2$. 3. $4(a^2 + b^2 + c^2 - ab - bc - ca)$.
5. $(10x^2 - 23x - 10)(x^2 - 4x - 1)$. 6. $\frac{7}{6}, -\frac{5}{2}$.
7. $x^3 - 3x^2 + 4x - 1$.

Paper 6. 1. (i) $a = -9, b = 4$; (ii) $a = 1, b = -4$.
4. $\dfrac{ab(a+b) + bc(b+c) + ca(c+a)}{abc(a+b)(b+c)(c+a)}$. 6. $(2x+1)(4x^2 - 2x + 3)$.
7. $-\frac{7}{3}, \frac{3}{8}$.

Paper 7. 2. (i) $5ab(a+b)(a^2 + ab + b^2)$;
(ii) $(a-b)(b-c)(c-a)(ab + bc + ca)$. 3. $-\dfrac{1}{abc}$. 5. $x = -3$.
6. $3x^2 + 8xy + 4y^2,\ 2x^2 + xy - 6y^2$. 7. $\frac{4}{3}, 5$.

Paper 8. 1. (i) $(x + 2a - 3)(x - a + 2)$, (ii) $(a - 2c + b)(2a + c - 3b)$.
3. $p = 1, 2$ or 5.

4. $x=\dfrac{1}{(a-c)(a-b)}$, $y=\dfrac{1}{(b-a)(b-c)}$, $z=\dfrac{1}{(c-a)(c-b)}$.

5. 1. 6. $\dfrac{3}{x-1}+\dfrac{2(2x+1)}{x^2+x+1}$. 7. 5, $\frac{3}{2}$.

MISCELLANEOUS EXERCISES. (Page 461.)

1. (*a*) $(5370)^3$, (*b*) 180. 2. (*a*) $6x^7-19x^6+10x^5+5x^4-15x^3+20x^2+x-4$, (*b*) 24. 3. x^2+x+2. 5. (i) $(a-b)(a-ab+b)$,
(ii) $(a+b+c)(2ab+2bc+2ca-a^2-b^2-c^2)$,
(iii) $(a+b)(a-b)\left(1+\dfrac{1}{ab}\right)\left(1-\dfrac{1}{ab}\right)$. 6. 1.

7. (i) $x=5, y=3, z=7$, (ii) $a+b+c$.

9. $\dfrac{5}{x+2}-\dfrac{2}{x+3}$. 10. Length $=21$ ft., breadth $=15$ ft.

11. (*a*) $(\cdot 034)^3$, (*b*) $\frac{1}{2}$. 12. (*a*) $2x^3+5x^2-3x+1$, (*b*) 4.

13. x^2+x-4. 15. (i) $(a+1)(a-1)(b+1)(b-1)$,
(ii) $(y-x)(xy+a)$, (iii) $(18x^2+6x+1)(18x^2-6x+1)$, 1861×1741.

16. x^2. 17. (i) 5, (ii) 5. 18. $abc=1$.

19. $\dfrac{3}{x-2}-\dfrac{3x+1}{(x+1)^2}$. 20. (i) 46 ft., (ii) Length 32 ft., breadth 14 ft.

21. (i) 16,27,81, (ii) $2a+2\sqrt{a^2-b^2}$. 23. x^2+2x+3.

25. (i) $(x-37)(x+36)$, (ii) $(x^2-yz)(z-y)$. 26. $5+\sqrt{6}$.

27. (i) $\frac{79}{20}$, (ii) -2. 28. $x^4-y^4=4x^2$. 29. $A=\frac{1}{2}$, $B=-4$
$C=4\frac{1}{2}$. 30. 24 men, Rs. 27. 31. (*a*) 27, $\frac{1}{216}$. 32. (*b*) $\dfrac{5}{x}$.

33. $2x^2+3x+2$. 35. (i) $(ax-a+1)(ax+x+a)$, (ii) $(x^2+y^2-1)^2$.

36. $\dfrac{x^2+x-3}{x^2-2}$. 37. (i) $x=3, y=2$, (ii) $x=2, y=-4, z=2$.

39. $A=6, B=5, C=-6, D=-4$. 40. Rs. 2725.

42. (*a*) $8x^3$, (*b*) 8. 43. x^2-2x+1.

45. (i) $(x+4)(x^2-4x+16)$, (ii) $(x+y)(y+z)(z+x)$. 46. $\frac{1}{3}$.

47. (i) 4, (ii) $x=y=2$. 48. $abc=1$. 49. $A=4, B=-6, C=-3$.

50. 30 miles. 52. (i) 33, (ii) $\dfrac{4x+3y}{29}=\dfrac{3x-2y}{9}$.

53. $x^2+2-\dfrac{1}{x}$. 54. $x-\sqrt{x^2-1}$. 55. $(x^2+4x+8)(x^2-4x+8)$,
(ii) $(x+a)(x+b+c)$. 56. $\dfrac{1}{(x-a)(x-b)(x-c)}$.

57. (i) 2, (iii) $\left(\dfrac{2}{m}\right)^{\frac{1}{2-m}}$. 58. $\frac{7}{12}$, $4\frac{8}{15}$. 59. $p=6, q=11, r=6$.

60. 72 sheep of each kind. **61.** (a) $\left(\frac{2645}{9}\right)^3$, (b) $\frac{1}{25}$.

62. (a) 83, (b) 21.

63. x^2+x-6, x^2+2x-3. **65.** (i) $(2x^2+1)(2x^2-1)(2x^2+2x+1)$ $(2x^2-x+1)$, (ii) $6(x-y)(5x+3y)(3x+5y)$. **66.** $\frac{x(x+1}{x-1}$.

67. (i) $x=5$, $y=2$, (ii) $\frac{14}{3}$. **68.** k^3+3k.

69. $k=3$, $l=0$, $m=-1$. **70.** 63. **71.** 2.

72. (a) -1, (b) (i) $4-x$, (ii) $x-4$, (iii) 4.

73. x^3-7x-6. **76.** 1. **77.** (i) 5 or $\frac{5}{4}$, (ii) $x=4\frac{1}{2}$, $y=1\frac{1}{2}$.

79. -1, 0, $6x^2-12x+8$. **80.** $32\frac{4}{13}$ minutes past 5.

81. (a) $\frac{1}{\sqrt{a^2+b^2}}$, (b) $\frac{x^2-y^2}{\sqrt{a^2-4b^2}}$. **82.** (a) 0, (b) x^8+x^4+1.

83. $\pm\sqrt{3}$. **84.** $a^4-4a^2b+2b^2$. **85.** $\frac{x}{x-3}$. **86.** 0.

88. $5x^2+2xy+5y^2$. **89.** $12abc(a+b+c)$. **90.** 80 men

91. (a) 1. **92.** $1+x+x^2+x^3+x^4+x^5+x^6+x^7$.

93. (b) $(x-a)(x+b)(x+c)$

$$\equiv x^3+x^2(-a+b+c)+x(-ab-ac+bc)-abc.$$

94. $(3a+2b)(2a-3b)(a-b)(4a^2+6ab+9b^2)$.

95. (a) $(a-2b)(a+5b)(a-3b)$, (b) $(x-2)(x^2+2x-13)$.

96. 2. **97.** $3\sqrt{2}-2$. **100.** 600 men.